Arithmetic Topology, Path integrals, and the Analogy between Function Fields and Number Fields

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I. Weil's Trichotomy

Analogy between Function Fields and Number Fields

Structural similarity:

$$F \sim k(X),$$

where F is an algebraic number field and X is a smooth projective curve over a finite field $k = \mathbb{F}_q$, $q = p^n$.

Also,

$$\mathrm{Spec}(\mathcal{O}_F) \smallsetminus T \sim X \smallsetminus S$$

S, T finite sets of closed points (possibly empty).

A better analogy is

$$\mathcal{C}^*(Spec(\mathcal{O}_F) \smallsetminus T) \sim \mathcal{C}^*(X \smallsetminus S),$$

where $\mathcal{C}^*$ are suitable categories of sheaves with conditions $*$ that might need adjustment.

Analogy between Function Fields and Number Fields

For example, if $\mathcal{C}$ is a category of $\mathbb{Q}_\ell$-sheaves for $\ell \neq p$ odd, then roughly

$$\mathcal{C}(\mathcal{O}_F)^{sm} \sim \mathcal{C}(X)^{sm},$$

where the superscript refers to 'lisse' $\mathbb{Q}_\ell$-sheaves satisfying crystalline conditions on the left at places dividing ℓ.

Analogy between Function Fields and Number Fields

Weil remarks that the analogy between F and $k(X)$ is

> *so strict and obvious that there is neither an argument nor a result in arithmetic that cannot be translated almost word for word to the function fields.*

Substantial consequences, e.g.

–Riemann hypothesis for varieties over finite fields;
–Langlands correspondence for function fields;
–The Fundamental Lemma;
–Weight monodromy conjecture for complete intersections.

Trichotomy ('Rosetta Stone')

Weil believed $k(X)$ to be an intermediate point in a bridge linking F and

$$\mathbb{C}(\Sigma),$$

the field of meromorphic functions on a compact smooth Riemann surface Σ:

$$F \sim k(X) \sim \mathbb{C}(\Sigma).$$

However, his sense of the the similarity between $k(X)$ and $\mathbb{C}(\Sigma)$ is expressed more cautiously:

> *The distance is not so large that a patient study would not teach us the art of passing from one to the other, and to profit in the study of the first from knowledge acquired about the second.*

Of course the analogy $k(X) \sim \mathbb{C}(\Sigma)$ is not quite right.

Trichotomy: Correction

A better analogy is

$$\bar{k}(X) \sim \mathbb{C}(\Sigma),$$

where $\bar{k}(X)$ is the field of rational functions on $\bar{X}$, the base-change of X to the algebraic closure $\bar{k}$ of k.

Thus, we actually have two separate analogies

$$\bar{k}(X) \sim \mathbb{C}(\Sigma)$$

$$F \sim k(X)$$

How to extend these to trichotomies

$$? \sim \bar{k}(X) \sim \mathbb{C}(\Sigma)$$

$$F \sim k(X) \sim ?$$

Will focus today mostly on the second.

Trichotomy: Correction

Note that

$$\bar{X} \sim \Sigma,$$

an analogy of geometric objects and not just fields. Then we have

$$\begin{array}{ccc} \bar{X} & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\bar{k}) & \longrightarrow & \mathrm{Spec}(k) \sim S^1 \end{array}$$

Trichotomy, Correction

We see that X itself is analogous to a fibered three manifold

$$\begin{array}{ccc} \Sigma & \hookrightarrow & M \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & S^1 \end{array}$$

with fibre Σ.

This is compatible with an analogy between $\mathrm{Spec}(\mathcal{O}_F)$ and a three-manifold, not necessarily fibered, due to Mazur. In short, the original analogy between function fields and number fields was also three-dimensional in nature.

II. Quantum Mechanics and Path Integrals

Schroedinger's Equation

Time evolution in quantum mechanics is expressed by the differential equation

$$\frac{d\psi}{dt} = -iH\psi$$

Here, ψ is a time-dependent vector in a Hilbert space $\mathcal{H}$, while H is a self-adjoint operator called the Hamiltonian, representing energy.

When the space of classical states is $T^*\Sigma$ for some manifold Σ, often

$$\psi \in \mathcal{H} = L^2(\Sigma)$$

$\mathcal{H}$ is the *quantisation* of the symplectic manifold $T^*\Sigma$.

Time evolution can also be expressed via an integral kernel:

$$[e^{-iHT}\psi](x) = \int K_T(x, y)\psi(y)dy$$

for some kernel function $K_t(x, y)$ when you start with the initial condition ψ.

Schroedinger's Equation: Path Integral

Path integral interpretation:

$$K_t(x,y) = \int_{P(x,y)} e^{iA(q)} dq,$$

where $A(q)$ is the classical action defined on paths

$$P(x,y) := \{q : [0,t] \longrightarrow \Sigma \mid q(0) = y,\ q(t) = x\}$$

Schroedinger's Equation: Path Integral

For example, for a single particle, we might have

$$A(q) = \int_0^T [(m/2)q'(t)^2 - V(q(t))]dt,$$

for some potential function V on Σ.

The classical Hamiltonian in this case is

$$h(q,p) = (1/2m)p^2 + V(q),$$

while the quantum Hamiltonian is the operator

$$H = (-1/2m)\Delta + V(q)$$

acting on $L^2(\Sigma)$.

Schroedinger's Equation: Path Integral

The usual interpretation of the kernel is as matrix coefficients, so one has informally

$$\mathrm{Tr}(e^{-iHT}) = \int K_T(x,x)dx.$$

But

$$K_T(x,x) = \int_{P(x,x)} e^{iA(\gamma)} d\gamma$$

an integral over loops based as x.

Hence,

$$\mathrm{Tr}(e^{-iHT}) = \int \int_{P(x,x)} e^{iA(q)} dq dx = \int_{\Omega} e^{iA(q)} dq,$$

the last being an integral over all loops

$$S^1 \longrightarrow \Sigma.$$

Schroedinger's Equation: Path Integral

Physicists tend to assume that you can put $T = -i\beta$, leading to

$$\mathrm{Tr}(e^{-\beta H}) = \int \int_{P(x,x)} e^{-S(q)} dq dx = \int_{\Omega} e^{-S(q)} dq,$$

for a function $S(q)$ call the Euclidean action.

With this substitution, this is called the *partition function* of the system.

Path Integrals: Twists

Maps from S^1 to Σ are sections of the trivial bundle

$$S^1 \times \Sigma \longrightarrow S^1.$$

Will interpret integral over a space of sections of a non-trivial fibre bundle

$$M \longrightarrow S^1$$

In fact, write

$$M = ([0, T] \times \Sigma)/F,$$

the mapping cylinder, where the monodromy map $F : \Sigma \longrightarrow \Sigma$ is used to glue $T \times \Sigma$ to $0 \times \Sigma$.

Path Integrals: Twists

Sections can be identified with $c : [0, T] \longrightarrow \Sigma$ such that $Fc(T) = c(0)$ or $c(T) = F^{-1}c(0)$.

Integral over sections:

$$\int \int_{P(F^{-1}x,x)} e^{-iA(c)} dcdx.$$

Recall that

$$\int_{P(F^{-1}x,x)} e^{iA(c)} dc = K_T(F^{-1}x, x))$$

where $K_T(x, y)$ is the integral kernel of the operator e^{-iTH} acting on $L^2(\Sigma)$.

That is,

$$[e^{-iTH}\psi](x) = \int K_T(x, y)\psi(y)dy.$$

Path Integrals: Twists

The diffeomorphism F acts on functions as

$$F\psi(x) = \psi(F^{-1}x).$$

Hence,

$$[Fe^{-iTH}\psi](x) = [e^{-iTH}\psi](F^{-1}x) = \int K_T(F^{-1}x, y)\psi(y)dy.$$

That is, $K_T(F^{-1}x, y)$ is the integral kernel for the operator

$$Fe^{-iTH}.$$

Path Integrals: Twists

Therefore,

$$\mathrm{Tr}(Fe^{-iTH}) = \int K_T(F^{-1}x, x)dx = \int \int_{P(F^{-1}x,x)} e^{iA(c)} dcdx,$$

$$= \int_{\mathcal{F}} e^{iA(s)} ds$$

an integral over the space $\mathcal{F}$ of sections of $M \longrightarrow S^1$.

When the theory is *topological* so that the Hamiltonian is zero, we get

$$\mathrm{Tr}(F) = \int_{\mathcal{F}} e^{iA(s)} ds.$$

More on Quantum Hilbert Spaces

When the space of classical states is a Kaehler manifold X (symplectic, Riemannian, complex in a compatible way), often have *geometric quantisation*, where

$$\mathcal{H} = \Gamma(X, \mathcal{L})$$

for some holomorphic line bundle $\mathcal{L}$.

For example, when $X = \mathbb{R}^2 = \mathbb{C}$, could take

$$\mathcal{H} = L^2(\mathbb{R}, dx)$$

or

$$\mathcal{H} = L^2_{hol}(\mathbb{C}, e^{-|z|^2} idz d\bar{z}).$$

III. Return to Trichotomy

Return to Trichotomy

$X/\mathbb{F}_q$, a smooth projective curve over a finite field.

$$X \sim M \simeq (\Sigma \times [0,1])/f,$$

where $f : \Sigma \simeq \Sigma$ is a monodromy diffeomorphism.

The analogy is that

$$X \sim (\bar{X} \times [0,1])/Fr_q.$$

Similarly, for the Jacobian,

$$J \sim (\bar{J} \times [0,1])/Fr_q.$$

The Jacobian $\bar{J}$ arises as the state space for many kinds of quantum field theories.

Trichotomy: Hilbert Space

Let Y be a lift of X to $W = W(k)$.

Let $L \longrightarrow J_Y$ a theta line bundle on the Jacobian of Y, giving a principal polarisation.

Let N be an odd prime such that $q \equiv 1 \mod N$ and

$$\mathcal{H} := \Gamma(J_Y, L^N) \otimes \mathbb{C}$$

Trichotomy: Hilbert Space

The vector space $\mathcal{H}$ is acted on by the finite Heisenberg group with centre μ_N:

$$\mathrm{Heis}_N = \mu_N \times J[N]$$

with group structure given by

$$(\lambda, a) \circ (\mu, b) = (\langle a, b \rangle^{1/2} \lambda\mu, a + b).$$

It is the unique irreducible representation of Heis_N with μ_N acting as scalar multiplication via identity character.

Thus, $\mathcal{H}$ is the quantisation of the symplectic vector space $J[N]$.

Trichotomy: Hilbert Space

There is also an action of the finite symplectic group of $J[N]$ (Gurevich and Hadani).

The Frobenius Fr_q acts on $J[N]$ by symplectic transformations, so that Fr_q acts on $\mathcal{H}$.

Formula (with Y. Cheng, A. Venkatesh):

Assume either (i) the Frobenius action on $J[N]$ is semi-simple and $\sqrt{-1} \in \mathbb{F}_q$, or (ii) there is a Lagrangian subspace in $J[N]$ stabilised by Fr_q. Then

$$Tr(Fr_q|\mathcal{H}) = \pm\sqrt{|Cl(X)[N]|}$$

Trichotomy: Application

Proof of Formula:

Assume there is a Lagrangian subspace $L \subset J[N]$.

$\mathcal{H}$ is the unique (up to almost unique isomorphism) irreducible representation of Heis_N with identity central character.

Thus,

$$\mathcal{H} \simeq C_{L^o} = \mathrm{Fun}(J[N]/L, \mathbb{C}),$$

where L^o denotes L with some fixed basis of $\wedge^{top} L$. Hadani and Gurevich show that there are canonical isomorphisms

$$T_{L^o,(L')^o} : C_{L^o} \simeq C_{(L')^o},$$

for any pair of oriented Lagrangians.

This is used to define the action of the symplectic group: Given $g \in Sp(J[N])$,

$$C_{L^o} \simeq^{\circ g^{-1}} C_{gL^o} \simeq^{T_{(gL^o),L^o}} C_{L^o}.$$

Trichotomy: Application

Proof of Formula (continued):

When $gL = L$, then $T_{g(L^o),L^o} = \pm 1$. Thus,

$$Tr(Fr_q|\mathcal{H})) = \pm Tr(Fr_q|C_L) = \pm Tr(Fr_q|\mathrm{Fun}(L', \mathbb{C})),$$

where $L' \subset J[N]$ is a complementary subspace.

Easy to see that

$$Tr(Fr_q|\mathrm{Fun}(L', \mathbb{C})) = |(L')^{Fr_q}|.$$

Via duality given by the Weil pairing

$$|(L')^{Fr_q}| = |L^{Fr_q}|,$$

so that

$$|(L')^{Fr_q}| = \sqrt{|(L \times L')^{Fr_q}|} = \sqrt{|J[N]^{Fr_q}|} = \sqrt{|Cl(X)[N]|}.$$

IV. An Arithmetic Path Integral

Arithmetic Path Integral

Once again,

$$\begin{array}{c} X \\ \downarrow \\ \mathrm{Spec}(\mathbb{F}_q) \end{array}$$

is a smooth projective curve and

$$\begin{array}{c} J \\ \downarrow \\ \mathrm{Spec}(\mathbb{F}_q) \end{array}$$

is its Jacobian.

Arithmetic Path Integral

Once again,

$$\begin{array}{c} X \\ \downarrow \\ \mathrm{Spec}(\mathbb{F}_q) \end{array}$$

is a smooth projective curve and

$$\begin{array}{c} J[N] \\ \downarrow \\ \mathrm{Spec}(\mathbb{F}_q) \end{array}$$

is a finite group scheme over $\mathbb{F}_q$.

Arithmetic Path Integral

A rational point $\gamma \in J[N](\mathbb{F}_q)$ is a section of the 'fibre bundle over S^1'

$$\begin{array}{c} J[N] \\ \big\downarrow \big\uparrow \gamma \\ \mathrm{Spec}(\mathbb{F}_q) \end{array}$$

Will define an 'arithmetic action'

$$A : J[N](\mathbb{F}_q) \longrightarrow \frac{1}{N}\mathbb{Z}/\mathbb{Z}$$

Recall that $\mu_N \subset \mathbb{F}_q$. Choose an isomorphism $\mu_N \simeq \frac{1}{N}\mathbb{Z}/\mathbb{Z}$.

Arithmetic Path Integral

There is the reciprocity map

$$rec : J(\mathbb{F}_q) = CH_0(X)^0 \longrightarrow \pi_1^{ab}(X)^0,$$

where the target corresponds to

$$0 \longrightarrow \pi_1^{ab}(X)^0 \longrightarrow \pi_1^{ab}(X) \longrightarrow \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 0.$$

Hence, also

$$rec : J(\mathbb{F}_q)[N] = CH_0(X)^0[N] \longrightarrow \pi_1^{ab}(X)^0[N].$$

Arithmetic Path Integral

However, $\gamma \in J(\mathbb{F}_q)[N]$ also defines a class in

$$c_\gamma \in H^1(X, \frac{1}{N}\mathbb{Z}/\mathbb{Z})/H^1(\mathrm{Spec}(\mathbb{F}_q), \frac{1}{N}\mathbb{Z}/\mathbb{Z}).$$

This is because γ determines a line bundle L_γ on X such that

$$L_\gamma^N \simeq^f \mathcal{O}_X.$$

generating a $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$-torsor

$$c_\gamma = \{y \in L_\gamma \mid f(y^N) = 1.\}.$$

$$H^1(X, \frac{1}{N}\mathbb{Z}/\mathbb{Z}) = \mathrm{Hom}(\pi_1(X)^{ab}, \frac{1}{N}\mathbb{Z}/\mathbb{Z})$$

Arithmetic Path Integral

Thus, we can define the quadratic function

$$A(\gamma) := c_\gamma(rec(\gamma))$$

and 'path integral'

$$\sum_{\gamma \in J[N](\mathbb{F}_q)} e^{2\pi i A(\gamma)}$$

Theorem (with H. Chung, D. Kim G. Pappas, J. Park, H. Yoo supplemented by Y. Cheng)

$$\sum_{\gamma \in J[N](\mathbb{F}_q)} e^{2\pi i A(\gamma)} = \sqrt{|J(\mathbb{F}_q)[N]|} \left(\frac{det(A)}{N}\right) [i^{(N-1)^2/4}]^{dim(J(\mathbb{F}_q)[N])}.$$

Arithmetic Path Integral: Comments

Note that

$$\mathrm{Tr}(F|\mathcal{H}) = \pm \sum_{\gamma \in J[N](\mathbb{F}_q)} e^{iA(\gamma)}$$

Proof of formula is a simple consequence of result of Neretin on Gaussian integrals over finite fields.

Compare to

$$\int e^{-\underline{x}^T A \underline{x}} d^n \underline{x} = \frac{\pi^{n/2}}{\sqrt{\det(A)}}.$$

Finite field case reduces to

$$\sum_{k=1}^{p-1} \left(\frac{k}{p}\right) e^{2\pi i/k} = \sqrt{\left(\frac{-1}{p}\right) p}.$$