Arithmetic Topology, Path integrals, and the Analogy between Function Fields and Number Fields

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I. Weil's Trichotomy

Analogy between Function Fields and Number Fields

Structural similarity:

 $F \sim k(X)$,

where F is an algebraic number field and X is a smooth projective curve over a finite field $k = \mathbb{F}_q$, $q = p^n$.

Also,

$$\operatorname{Spec}(\mathcal{O}_F) \smallsetminus T \sim X \smallsetminus S$$

S, T finite sets of closed points (possibly empty).

A better analogy is

$$\mathcal{C}^*(Spec(\mathcal{O}_F) \smallsetminus T) \sim \mathcal{C}^*(X \smallsetminus S),$$

where \mathcal{C}^\ast are suitable categories of sheaves with conditions \ast that might need adjustment.

Analogy between Function Fields and Number Fields

For example, if ${\mathcal C}$ is a category of ${\mathbb Q}_\ell\text{-sheaves}$ for $\ell\neq p$ odd, then roughly

 $\mathcal{C}(\mathcal{O}_F)^{sm} \sim \mathcal{C}(X)^{sm},$

where the superscript refers to 'lisse' \mathbb{Q}_{ℓ} -sheaves satisfying crystalline conditions on the left at places dividing ℓ .

Analogy between Function Fields and Number Fields

Weil remarks that the analogy between F and k(X) is

so strict and obvious that there is neither an argument nor a result in arithmetic that cannot be translated almost word for word to the function fields.

Substantial consequences, e.g.

- -Riemann hypothesis for varieties over finite fields;
- -Langlands correspondence for function fields;
- -The Fundamental Lemma;
- -Weight monodromy conjecture for complete intersections.

Trichotomy ('Rosetta Stone')

Weil believed k(X) to be an intermediate point in a bridge linking F and $\mathbb{C}(\Sigma)$,

the field of meromorphic functions on a compact smooth Riemann surface $\boldsymbol{\Sigma}:$

$F \sim k(X) \sim \mathbb{C}(\Sigma).$

However, his sense of the the similarity between k(X) and $\mathbb{C}(\Sigma)$ is expressed more cautiously:

The distance is not so large that a patient study would not teach us the art of passing from one to the other, and to profit in the study of the first from knowledge acquired about the second.

Of course the analogy $k(X) \sim \mathbb{C}(\Sigma)$ is not quite right.

Trichotomy: Correction

A better analogy is

$$\bar{k}(X) \sim \mathbb{C}(\Sigma),$$

where $\bar{k}(X)$ is the field of rational functions on \bar{X} , the base-change of X to the algebraic closure \bar{k} of k.

Thus, we actually have two separate analogies

$$ar{k}(X)\sim \mathbb{C}(\Sigma)$$

 $F\sim k(X)$

How to extend these to trichotomies

$$P_{i}^{i}\sim ar{k}(X)\sim \mathbb{C}(\Sigma)$$

 $F\sim k(X)\sim P_{i}^{i}$

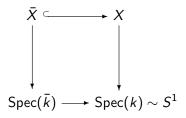
Will focus today mostly on the second.

Trichotomy: Correction

Note that

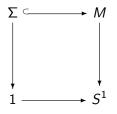
$$\bar{X} \sim \Sigma$$
,

an analogy of geometric objects and not just fields. Then we have



Trichotomy, Correction

We see that X itself is analogous to a fibered three manifold



with fibre Σ .

This is compatible with an analogy between $\text{Spec}(\mathcal{O}_F)$ and a three-manifold, not necessarily fibered, due to Mazur. In short, the original analogy between function fields and number fields was also three-dimensional in nature.

II. Quantum Mechanics and Path Integrals

Schroedinger's Equation

Time evolution in quantum mechanics is expressed by the differential equation

$$\frac{d\psi}{dt} = -iH\psi$$

Here, ψ is a time-dependent vector in a Hilbert space $\mathcal H$, while H is a self-adjoint operator called the Hamiltonian, representing energy.

When the space of classical states is $\mathcal{T}^*\Sigma$ for some manifold $\Sigma,$ often

$$\psi \in \mathcal{H} = L^2(\Sigma)$$

 \mathcal{H} is the *quantisation* of the symplectic manifold $T^*\Sigma$.

Time evolution can also be expressed via an integral kernel:

$$[e^{-iHT}\psi](x) = \int K_T(x,y)\psi(y)dy$$

for some kernel function $K_t(x, y)$ when you start with the initial condition ψ .

Path integral interpretation:

$$K_t(x,y) = \int_{P(x,y)} e^{iA(q)} dq,$$

where A(q) is the classical action defined on paths

$$P(x,y) := \{q: [0,t] \longrightarrow \Sigma \mid q(0) = y, q(t) = x\}$$

For example, for a single particle, we might have

$$A(q) = \int_0^T [(m/2)q'(t)^2 - V(q(t))]dt,$$

for some potential function V on Σ .

The classical Hamiltonian in this case is

$$h(q,p) = (1/2m)p^2 + V(q),$$

while the quantum Hamiltonian is the operator

$$H = (-1/2m)\Delta + V(q)$$

acting on $L^2(\Sigma)$.

The usual interpretation of the kernel is as matrix coefficients, so one has informally

$$\operatorname{Tr}(e^{-iHT}) = \int K_T(x,x) dx.$$

But

$$K_T(x,x) = \int_{P(x,x)} e^{iA(\gamma)} d\gamma$$

an integral over loops based as x.

Hence,

$$\operatorname{Tr}(e^{-iHT}) = \int \int_{P(x,x)} e^{iA(q)} dq dx = \int_{\Omega} e^{iA(q)} dq,$$

the last being an integral over all loops

$$S^1 \longrightarrow \Sigma.$$

Physicists tend to assume that you can put $T = -i\beta$, leading to

$$\operatorname{Tr}(e^{-\beta H}) = \int \int_{P(x,x)} e^{-S(q)} dq dx = \int_{\Omega} e^{-S(q)} dq,$$

for a function S(q) call the Euclidean action.

With this substitution, this is called the *partition function* of the system.

Maps from S^1 to Σ are sections of the trivial bundle

$$S^1 \times \Sigma \longrightarrow S^1.$$

Will interpret integral over a space of sections of a non-trivial fibre bundle

$$M \longrightarrow S^1$$

In fact, write

$$M = ([0, T] \times \Sigma)/F,$$

the mapping cylinder, where the monodromy map $F : \Sigma \longrightarrow \Sigma$ is used to glue $T \times \Sigma$ to $0 \times \Sigma$.

Sections can be identified with $c : [0, T] \longrightarrow \Sigma$ such that Fc(T) = c(0) or $c(T) = F^{-1}c(0)$.

Integral over sections:

$$\int \int_{P(F^{-1}x,x)} e^{-iA(c)} dc dx.$$

Recall that

$$\int_{P(F^{-1}x,x)} e^{iA(c)} dc = K_T(F^{-1}x,x))$$

where $K_T(x, y)$ is the integral kernel of the operator e^{-iTH} acting on $L^2(\Sigma)$.

That is,

$$[e^{-iTH}\psi](x) = \int K_T(x,y)\psi(y)dy.$$

The diffeomorphism F acts on functions as

$$F\psi(x)=\psi(F^{-1}x).$$

Hence,

$$[Fe^{-iTH}\psi](x) = [e^{-iTH}\psi](F^{-1}x) = \int K_T(F^{-1}x,y)\psi(y)dy.$$

That is, $K_T(F^{-1}x, y)$ is the integral kernel for the operator

 Fe^{-iTH} .

Therefore,

$$Tr(Fe^{-iTH}) = \int K_T(F^{-1}x, x) dx = \int \int_{P(F^{-1}x, x)} e^{iA(c)} dc dx,$$
$$= \int_{\mathcal{F}} e^{iA(s)} ds$$

an integral over the space \mathcal{F} of sections of $M \longrightarrow S^1$.

When the theory is *topological* so that the Hamiltonian is zero, we get

$$\operatorname{Tr}(F) = \int_{\mathcal{F}} e^{iA(s)} ds$$

More on Quantum Hilbert Spaces

When the space of classical states is a Kaehler manifold X (symplectic, Riemannian, complex in a compatible way), often have *geometric quantisation*, where

$$\mathcal{H} = \Gamma(X, \mathcal{L})$$

for some holomorphic line bundle \mathcal{L} .

For example, when $X = \mathbb{R}^2 = \mathbb{C}$, could take

$$\mathcal{H} = L^2(\mathbb{R}, dx)$$

or

$$\mathcal{H} = L^2_{hol}(\mathbb{C}, e^{-|z|^2} i dz d\bar{z}).$$

III. Return to Trichotomy

Return to Trichotomy

 X/\mathbb{F}_q , a smooth projective curve over a finite field.

 $X \sim M \simeq (\Sigma \times [0,1])/f$

where $f : \Sigma \simeq \Sigma$ is a monodromy diffeomorphism. The analogy is that

$$X \sim (\bar{X} \times [0,1])/Fr_q.$$

Similarly, for the Jacobian,

$$J \sim (\bar{J} \times [0,1])/Fr_q.$$

The Jacobian \overline{J} arises as the state space for many kinds of quantum field theories.

Trichotomy: Hilbert Space

Let Y be a lift of X to W = W(k).

Let $L \longrightarrow J_Y$ a theta line bundle on the Jacobian of Y, giving a principal polarisation.

Let N be an odd prime such that $q \equiv 1 \mod N$ and

$$\mathcal{H} := \Gamma(J_Y, L^N) \otimes \mathbb{C}$$

Trichotomy: Hilbert Space

The vector space ${\mathcal H}$ is acted on by the finite Heisenberg group with centre $\mu_{\it N}$:

$$\mathsf{Heis}_{N} = \mu_{N} \times J[N]$$

with group structure given by

$$(\lambda, \mathbf{a}) \circ (\mu, \mathbf{b}) = (\langle \mathbf{a}, \mathbf{b} \rangle^{1/2} \lambda \mu, \mathbf{a} + \mathbf{b}).$$

It is the unique irreducible representation of Heis_N with μ_N acting as scalar multiplication via identity character.

Thus, \mathcal{H} is the quantisation of the symplectic vector space J[N].

Trichotomy: Hilbert Space

There is also an action of the finite symplectic group of J[N] (Gurevich and Hadani).

The Frobenius Fr_q acts on J[N] by symplectic transformations, so that Fr_q acts on \mathcal{H} .

Formula (with Y. Cheng, A. Venkatesh):

Assume either (i) the Frobenius action on J[N] is semi-simple and $\sqrt{-1} \in \mathbb{F}_q$, or (ii) there is a Lagrangian subspace in J[N] stabilised by Fr_q . Then

$$Tr(Fr_q|\mathcal{H}) = \pm \sqrt{|CI(X)[N]|}$$

Trichotomy: Application *Proof of Formula:*

Assume there is a Lagrangian subspace $L \subset J[N]$.

 \mathcal{H} is the unique (up to almost unique isomorphism) irreducible representation of Heis_N with identity central character.

Thus,

$$\mathcal{H} \simeq C_{L^o} = \operatorname{Fun}(J[N]/L, \mathbb{C}),$$

where L^o denotes L with some fixed basis of $\wedge^{top}L$. Hadani and Gurevich show that there are canonical isomorphisms

$$T_{L^{\circ},(L')^{\circ}}: C_{L^{\circ}} \simeq C_{(L')^{\circ}},$$

for any pair of oriented Lagrangians.

This is used to define the action of the symplectic group: Given $g \in Sp(J[N])$,

$$C_{L^o}\simeq^{\circ g^{-1}}C_{gL^o}\simeq^{T_{(gL^o),L^o}}C_{L^o}.$$

Trichotomy: Application Proof of Formula (continued):

When
$$gL=L$$
, then $T_{g(L^o),L^o}=\pm 1$. Thus,

$$Tr(Fr_q|\mathcal{H})) = \pm Tr(Fr_q|C_L) = \pm Tr(Fr_q|Fun(L', \mathbb{C})),$$

where $L' \subset J[N]$ is a complementary subspace. Easy to see that

$$Tr(Fr_q|Fun(L',\mathbb{C})) = |(L')^{Fr_q}|.$$

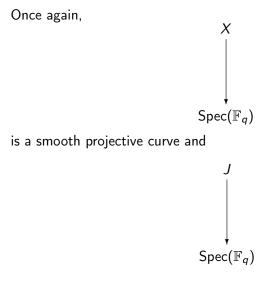
Via duality given by the Weil pairing

$$|(L')^{Fr_q}|=|L^{Fr_q}|,$$

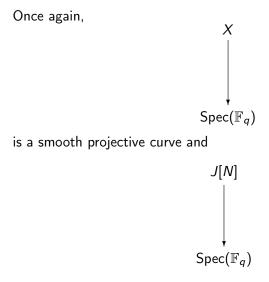
so that

$$|(L')^{Fr_q}| = \sqrt{|(L \times L')^{Fr_q}|} = \sqrt{|J[N]^{Fr_q}|} = \sqrt{|CI(X)[N]|}.$$

IV. An Arithmetic Path Integral

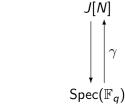


is its Jacobian.



is a finite group scheme over \mathbb{F}_q .

A rational point $\gamma \in J[N](\mathbb{F}_q)$ is a section of the 'fibre bundle over S^1 '



Will define an 'arithmetic action'

$$A: J[N](\mathbb{F}_q) \longrightarrow \frac{1}{N}\mathbb{Z}/\mathbb{Z}$$

Recall that $\mu_N \subset \mathbb{F}_q$. Choose an isomorphism $\mu_N \simeq \frac{1}{N} \mathbb{Z}/\mathbb{Z}$.

There is the reciprocity map

$$\operatorname{rec}: J(\mathbb{F}_q) = CH_0(X)^0 \longrightarrow \pi_1^{ab}(X)^0,$$

where the target corresponds to

$$0 \longrightarrow \pi_1^{ab}(X)^0 \longrightarrow \pi_1^{ab}(X) \longrightarrow \operatorname{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 0.$$

Hence, also

$$\operatorname{rec}: J(\mathbb{F}_q)[N] = CH_0(X)^0[N] \longrightarrow \pi_1^{ab}(X)^0[N].$$

However, $\gamma \in J(\mathbb{F}_q)[N]$ also defines a class in

$$c_\gamma \in H^1(X, rac{1}{N}\mathbb{Z}/\mathbb{Z})/H^1(\operatorname{Spec}(\mathbb{F}_q), rac{1}{N}\mathbb{Z}/\mathbb{Z}).$$

This is because γ determines a line bundle L_{γ} on X such that

$$L^N_{\gamma} \simeq^f \mathcal{O}_X.$$

generating a $\frac{1}{N}\mathbb{Z}/\mathbb{Z}\text{-torsor}$

$$c_{\gamma} = \{ y \in L_{\gamma} \mid f(y^N) = 1. \}.$$

$$H^1(X, rac{1}{N}\mathbb{Z}/\mathbb{Z}) = \operatorname{Hom}(\pi_1(X)^{ab}, rac{1}{N}\mathbb{Z}/\mathbb{Z})$$

Thus, we can define the quadratic function

 $A(\gamma) := c_{\gamma}(rec(\gamma))$

and 'path integral'

$$\sum_{\gamma \in J[N](\mathbb{F}_q)} e^{2\pi i A(\gamma)}$$

Theorem (with H. Chung, D. Kim G. Pappas, J. Park, H. Yoo supplemented by Y. Cheng)

$$\sum_{\gamma \in J[N](\mathbb{F}_q)} e^{2\pi i A(\gamma)} = \sqrt{|J(\mathbb{F}_q)[N]|} \left(\frac{\det(A)}{N}\right) [i^{(N-1)^2/4}]^{\dim(J(\mathbb{F}_q)[N])}.$$

Arithmetic Path Integral: Comments

Note that

$$\mathsf{Tr}(F|\mathcal{H}) = \pm \sum_{\gamma \in J[N](\mathbb{F}_q)} e^{i \mathcal{A}(\gamma)}$$

Proof of formula is a simple consequence of result of Neretin on Gaussian integrals over finite fields.

Compare to

$$\int e^{-\underline{x}^T A \underline{x}} d^n \underline{x} = \frac{\pi^{n/2}}{\sqrt{\det(A)}}.$$

Finite field case reduces to

$$\sum_{k=1}^{p-1} \left(\frac{k}{p}\right) e^{2\pi i/k} = \sqrt{\left(\frac{-1}{p}\right)p}.$$