The critical height of an endomorphism of projective space

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Critical orbits

Problem: say something about the sequence

$$z, f(z), f^2(z) = f(f(z)), f^3(z), \ldots$$

Theorem (Fatou-Julia circa 1918)

Let $f(z) \in \mathbb{C}[z]$ be a polynomial of degree at least 2. Then the Julia set $J(f)$ is connected if and only if all critical orbits of f are bounded (away from ∞).

A *critical point* is a point of ramification: $f'(z) = 0$.

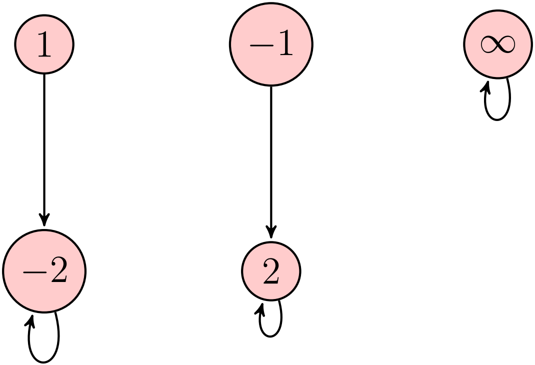
Definition

A rational function $f(z) \in \mathbb{C}(z)$ is *post-critically finite* (PCF) if and only if every critical orbit is finite.

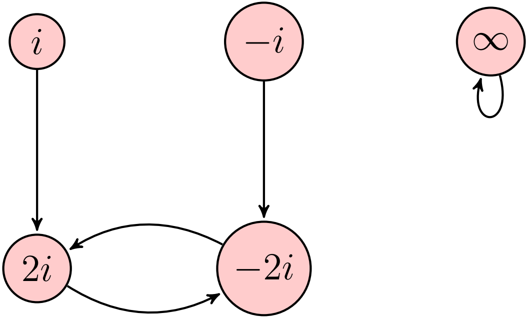
("preperiodic" = "finite orbit").

Critical orbits of $f(z) = z^3$

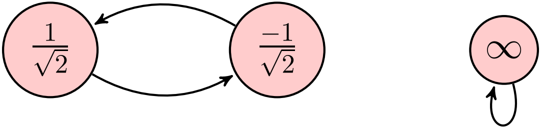
Critical orbits of $f(z) = z^3 - 3z$



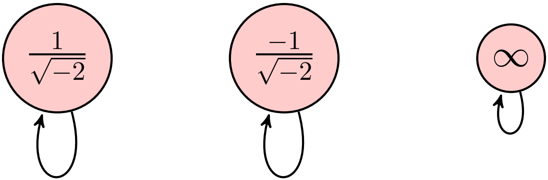
Critical orbits of $f(z) = z^3 + 3z$



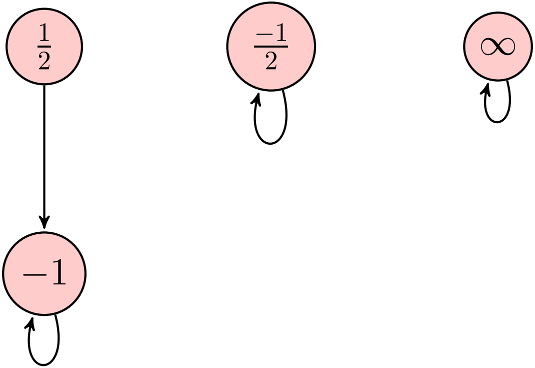
Critical orbits of $f(z) = z^3 - \frac{3}{2}z$



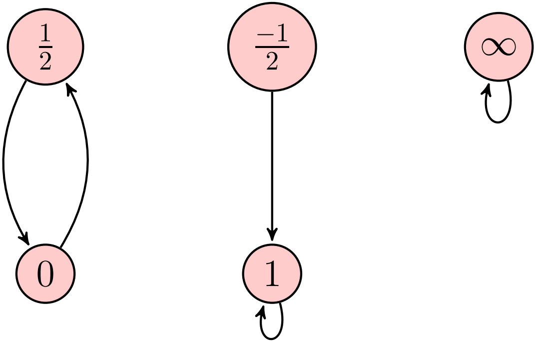
Critical orbits of $f(z) = z^3 + \frac{3}{2}z$



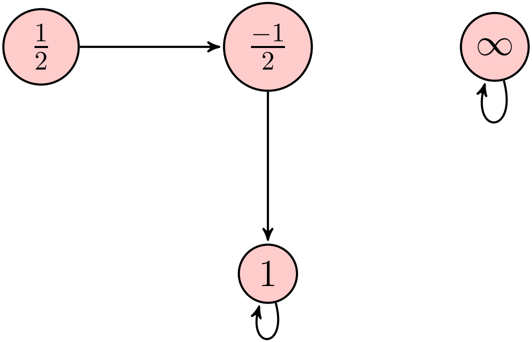
Critical orbits of $f(z) = z^3 - \frac{3}{4}z - \frac{3}{4}$



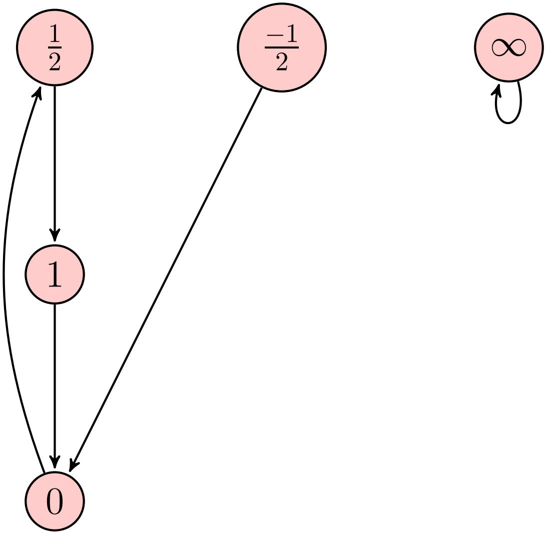
Critical orbits of $f(z) = 2z^3 - \frac{3}{2}z + \frac{1}{2}$



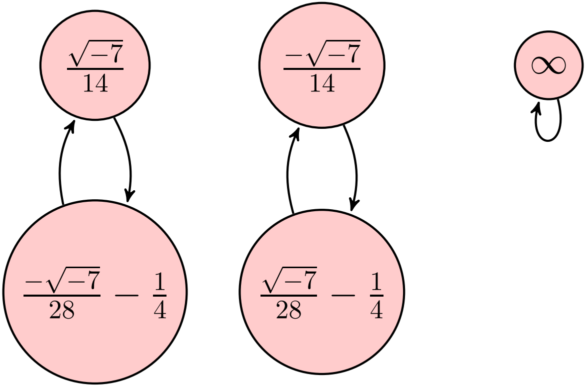
Critical orbits of $f(z) = 3z^3 - \frac{9}{4}z + \frac{1}{4}$



Critical orbits of $f(z) = -2z^3 + \frac{3}{2}z + \frac{1}{2}$



Critical orbits of $f(z) = -7z^3 - \frac{3}{4}z - \frac{1}{4}$



PCF maps and number theory

There has been some interest in *arboreal Galois representations*, i.e., the action of Galois on iterated preimage trees.

For $f \in K(z)$ (degree ≥ 2) and $\alpha \in \mathbb{P}^1(K)$, how does the tower of extensions

$$K \subseteq K(f^{-1}(\alpha)) \subseteq K(f^{-2}(\alpha)) \subseteq \cdots \subseteq K_\infty(f, \alpha) := \bigcup_{n \geq 1} K(f^{-n}(\alpha))$$

behave?

PCF maps and number theory

Theorem (Aitken–Hajir–Maire 2005, Cullinan–Hajir 2012, Bridy–I.–Jones–Juul–Levy–Manes–Rubinstein-Salzedo-Silverman 2015)

Let $f(z) \in K(z)$ be a rational function (degree ≥ 2), and let $\alpha \in \mathbb{P}^1(K)$ be non-exceptional. Then the field $K_\infty(f, \alpha)$ above is finitely ramified if and only if f is PCF.

In fact, in the "if" direction we could replace $\mathbb{P}^1$ with any smooth, projective variety, but the converse is more subtle.

Thurston rigidity

Theorem (Thurston 1980s, over $\mathbb{C}$)

Any algebraic family of PCF rational functions is isotrivial or a (flexible) Lattès example.

Intuition

The number of degrees of freedom in specifying a rational function is equal to the number of critical points, so a naïve dimension count would indicate that any family is constant.

Lattès examples

If E is an elliptic curve, and $\alpha : E \to E$ is an affine map commuting with $[-1]$, then α induces an endomorphism of $\mathbb{P}^1$.

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & E \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow[f]{} & \mathbb{P}^1 \end{array}$$

If $\alpha(P) = [n]P + T$ for $T \in E[2]$ and $n \in \mathbb{Z}$, we get a flexible family of PCF maps.

Heights

For $a/b \in \mathbb{Q}$ in lowest terms,

$$h(a/b) = \log\max\{|a|, |b|\}.$$

More generally, every point in $\mathbb{P}^N(\mathbb{Q})$ can be written as

$$[a_0 : \cdots : a_N],$$

with $a_i \in \mathbb{Z}$ and no common factor. Then

$$h\left([a_0 : \cdots : a_N]\right) = \log\max\{|a_0|, ..., |a_N|\}.$$

There is a natural way to extend h to $\overline{\mathbb{Q}}$ and $\mathbb{P}^N(\overline{\mathbb{Q}})$, as well as to projective varieties with a specified ample divisor.

Why heights?

Theorem (Northcott-Weil)

For any N, D, and B, the set

$$\left\{P \in \mathbb{P}^N(\overline{\mathbb{Q}}) : h(P) \leq B \text{ and } [\mathbb{Q}(P) : \mathbb{Q}] \leq D\right\}$$

is finite. Consequently, same for heights with respect to ample divisors.

So sets of bounded height and algebraic degree are finite.

The statement quickly reduces to the statement that there are only finitely many polynomials in $\mathbb{Z}[x]$ with degree at most D, and coefficients bounded by B'.

The critical height

By work of Néron-Tate-Denis-Call-Silverman, for any f with $\deg(f) \geq 2$ and $P \in \mathbb{P}^1(\overline{\mathbb{Q}})$,

$$h(f^n(P)) = \deg(f)^n \hat{h}_f(P) + O_f(1)$$

as $n \to \infty$, for some $\hat{h}_f(P) \geq 0$ (with equality just if $f^n(P) = f^m(P)$ for some $n \neq m$).

Definition

Silverman proposed the *critical height*

$$\hat{h}_{\text{crit}}(f) = \sum_{\zeta \in \text{Crit}(f)} \hat{h}_f(\zeta),$$

This is well-defined on the moduli space M_d of rational functions of degree $d \geq 2$ modulo change of variables.

A conjecture about the arithmetic of critical orbits

The critical height is a candidate for a "canonical" height on M_d...

$$\hat{h}_{\mathrm{crit}}(f^n) = n\hat{h}_{\mathrm{crit}}(f)$$

$$\hat{h}_{\mathrm{crit}}(f) = 0 \Leftrightarrow f \text{ is PCF}$$

...but is it a height at all?

Conjecture (Silverman 2010)

For any ample Weil height h on M_d, we have

$$h \asymp \hat{h}_{\mathrm{crit}}$$

except on the Lattès locus.

In particular, the (non-Lattès) PCF maps are a set of bounded height.

Silverman's Conjecture

Theorem (I. 2012, Benedetto-I.-Jones-Levy 2014, I. 2017, Gauthier-Okuyama-Vigny 2018+)

Silverman's Conjecture is true. We have

$$h_{\mathsf{M}_d} \asymp \hat{h}_{\mathrm{crit}}$$

for non-Lattès maps. Consequently, non-Lattès PCF maps are a set of bounded height.

The proof for polynomials is much simpler, and also has extra corollaries.

The result for rational functions is more fundamentally dynamical, using results of Fatou, McMullen, ...

Good reduction

A rational function has *good reduction* at p if its reduction mod p is a rational function of the same degree, and *potential good reduction* if this happens after a change of variables over some extension.

Corollary (I. 2012)

Let f be a PCF polynomial of degree $d \geq 2$. Then f has potential good reduction at any prime $p > d$.

Example (I. unpublished)

For any $d \geq 2$ and finite set S of primes, there exists a PCF rational function f of degree d and a $p \notin S$ such that f does not have potential good reduction at p.

PCF on $\mathbb{P}^1$ beyond Silverman's Conjecture

It is a consequence of Silverman's Conjecture that if f_t is a family of rational functions over a curve C, then

$$\mathrm{PCF}(f) := \{t \in C : f_t \text{ is PCF}\}$$

is a set of bounded height (exclude Lattès, isotrivial).

But in fact $\mathrm{PCF}(f)$ is vastly over-constrained.

Generally, one should expect this set to be finite.

PCF on $\mathbb{P}^1$ beyond Silverman's Conjecture

Conjecture (Baker-DeMarco; dynamical André-Oort)

Any algebraic family with a Zariski dense set of PCF specializations is defined by critical orbit relations.

Theorem (Baker-DeMarco 2013)

The curve $\mathrm{Per}_1(\lambda)$ *of cubic polynomials with a fixed point of multiplier* λ *contains infinitely many PCF points if and only if* $\lambda = 0$.

Theorem (DeMarco-Wang-Ye 2014)

The curve $\mathrm{Per}_1(\lambda)$ *of quadratic rational functions with a fixed point of multiplier* λ *contains infinitely many PCF points if and only if* $\lambda = 0$.

PCF on $\mathbb{P}^1$ beyond Silverman's Conjecture

Theorem (Favre-Gauthier 2018, Ghioca-Ye 2018)

The curve $\mathrm{Per}_n(\lambda)$ *in the space of cubic polynomials contains infinitely many PCF points (over* $\mathbb{C}$*) if and only if* $\lambda = 0$.

Theorem (Ghioca-Krieger-Nguyen-Ye 2017)

Let C *be a plane curve defined over* $\mathbb{C}$*, and fix* $d \geq 2$*. Then there are infinitely many points* $(a, b) \in C$ *such that* $z^d + a$ *and* $z^d + b$ *are both PCF only if* C *is a vertical or horizontal line, or a line of slope* ζ *with* $\zeta^{d-1} = 1$.

PCF on $\mathbb{P}^1$ beyond Silverman's Conjecture

Theorem (Favre-Gauthier 2020)

For any one-parameter family f of polynomials with infinitely many PCF specializations, either:

1. $f = \varphi \circ g^n$ *for some symmetry* φ *of* f, $n \geq 2$, *and* g *a family with infinitely many PCF specializations OR*
2. f *has one generically free critical orbit, up to symmetries.*

Theorem (I. 2019)

Any one-parameter family of polynomials contained in $\mathrm{Per}_n(\lambda)$, *for* $\lambda \neq 0$ *fixed, has at least two generically free critical orbits (up to symmetries).*

Higher-dimensional dynamics

We know much less about dynamics in more than one variable, for morphisms $f : \mathbb{P}^N \to \mathbb{P}^N$.

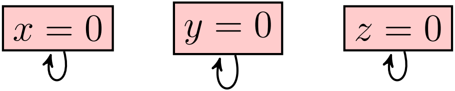
Definition

We will say that f is PCF iff the critical locus C_f is (setwise) preperiodic, i.e., iff

$$\bigcup_{n\geq 1} f^n(C_f) \subseteq \mathbb{P}^N \text{ is algebraic.}$$

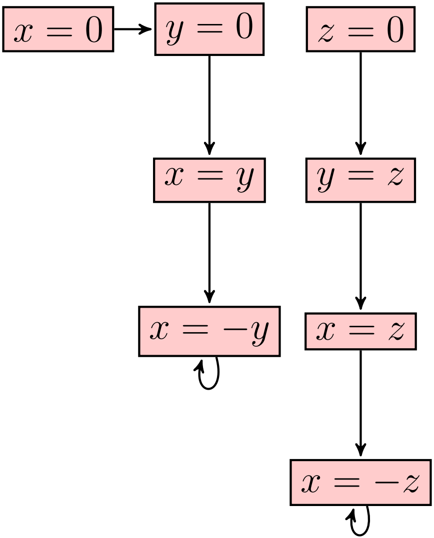
C_f is, as usual, defined by the vanishing of the Jacobian determinant; i.e., C_f is the ramification divisor.

Critical orbits of $f([x:y:z]) = [x^2 : y^2 : z^2]$



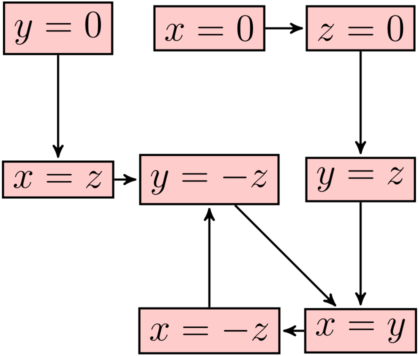
Critical orbits of $f([x:y:z]) = [x^2 - 2y^2 : x^2 : x^2 - 2z^2]$

This example is due to Fornæss and Sibony.

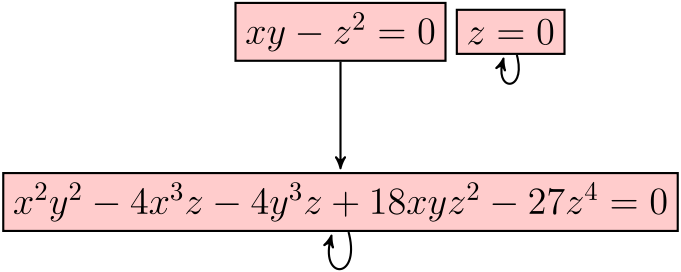


Critical orbits of $f([x : y : z]) = [x^2 - 2y^2 : x^2 - 2z^2 : x^2]$

Another example due to Fornæss and Sibony.



Critical orbits of $f([x:y:z]) = [x^2 - 2yz : y^2 - 2xz : z^2]$



PCF on $\mathbb{P}^N$

How common are these things in higher dimension?

Theorem (Folklore)

Endomorphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$ with periodic critical locus are Zariski dense in the appropriate moduli space.

Theorem (I.-Ramadas-Silverman 2019+)

*For $N \geq 2$, endomorphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \geq 3$ with periodic critical locus (or pre-periodic with tail length ≤ 2) are **not** Zariski dense in the moduli space.*

Critical height on $\mathbb{P}^N$

Zhang extends the canonical height to subvarieties.

Define $\hat{h}_{\text{crit}}(f) = \hat{h}_f(C_f)$, which again is coordinate invariant.

This can be defined concretely. If divisor D is defined by $F = 0$, set

$$h(D) = h(\text{tuple of coefficients of } F),$$

and then

$$\hat{h}_f(D) = \lim_{k \to \infty} d^{-Nk} h(f_*^k D).$$

Critical height on $\mathbb{P}^N$

Theorem

For $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \geq 2$, we have

1. $\hat{h}_{\text{crit}}(f^n) = n\hat{h}_{\text{crit}}(f)$
2. $\hat{h}_{\text{crit}}(f) = 0$ *if f is PCF*
3. $\hat{h}_{\text{crit}}(f) \ll h(f)$ *for any ample height h on moduli space.*

Conjecture (The critical height is a moduli height)

There is a proper Zariski-closed $Z \subseteq \mathsf{M}_d^N$ such that

$$h_{\mathsf{M}_d^N} \asymp \hat{h}_{\text{crit}}$$

off of Z. (Maybe Z is just the Lattès locus?)

One-parameter families

Theorem (I. 2019)

*Let f_t be a one-parameter family $f_t : \mathbb{P}^N \to \mathbb{P}^N$ over the base **curve** C. Then*

$$\hat{h}_{\mathrm{crit}}(f_t) = (\hat{h}_{\mathrm{crit}}(f) + o(1))h_C(t)$$

with $o(1) \to 0$ as $h(t) \to \infty$, and where $\hat{h}_{\mathrm{crit}}(f)$ is the critical height on the generic fibre.

See a similar result of Favre for Lyapunov exponents of families over a complex disk.

So, the critical height is a moduli height for one-parameter families with $\hat{h}_{\mathrm{crit}}(f) \neq 0$ on the generic fibre.

Minimally critical endomorphisms of $\mathbb{P}^N$

Milnor conducted a nice study of *bicritical maps*, that is, rational functions $f(z) \in \mathbb{C}(z)$ with exactly two critical points.

The generalization of this (I claim) is endomorphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ ramified along exactly $N + 1$ hyperplanes.

Such a morphism always has the form $f(\mathbf{X}) = A\mathbf{X}^d$, for $A \in \mathrm{PGL}_{N+1}$ and $d \geq 2$, after a change of variables.

Some examples

For $N = d = 2$, the following matrices define PCF maps:

Fornæss-Sibony

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}$$

Ueda via Dupont

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Belk-Koch

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

Minimally critical maps

Theorem (I. 2020+)

The critical height is a moduli height for minimally critical maps of degree $d > N^2 + N + 1$ on $\mathbb{P}^N$. I.e.,

$$h_{\mathsf{M}_d^N} \asymp \hat{h}_{\mathrm{crit}}$$

on this subset of moduli space for $d > N^2 + N + 1$.

Here, the bounds are completely explicit if one uses the height on PGL_{N+1} as a proxy for $h(f)$.

The proof is also relatively elementary, but doesn't have any strong consequences for local dynamics.

Regular polynomial endomorphisms

Regular polynomial endomorphisms are endomorphisms f with a totally invariant hyperplane $f^*H = dH$.

Following Bedford and Jonsson, there is a natural approach here of understanding dynamics by understanding the restriction $f|_H$, and then the “relative” behaviour given this.

Concretely, we can approach $\hat{h}_{\mathrm{crit}}(f)$ by understanding $\hat{h}_{\mathrm{crit}}(f|_H)$ and $\hat{h}_{\mathrm{crit}}(f) - \hat{h}_{\mathrm{crit}}(f|_H)$ separately.

This is particularly interesting for endomorphisms of $\mathbb{P}^2$, where $f|_H$ is an endomorphism of $\mathbb{P}^1$.

Minimally critical *affine* maps

Consider minimally critical endomorphisms for which one ramified hyperplane is fixed:

$$f(\mathbf{X}) = \begin{pmatrix} A_\infty & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} \mathbf{X}^d = A_\infty \mathbf{X}^d + \mathbf{b}$$

away from that hyperplane.

These naturally generalize the *unicritical* polynomials $z^d + c$.

What can we say about the variation in behaviour relative to the variation at infinity?

Relative critical height

Theorem (I. 2019+)

For $f(\mathbf{X}) = A_\infty \mathbf{X}^d + \mathbf{b}$, *with* $d \geq 2$,

$$\hat{h}_{\mathrm{crit}}(f) - \hat{h}_{\mathrm{crit}}(f|_H) \asymp h(\mathbf{b}) + O(h(A_\infty)),$$

In particular,

$$\hat{h}_{\mathrm{crit}}(f) \asymp h(f)$$

on fibres of the restriction to the invariant hyperplane.

These results can be combined with the previous result when $d > N^2 - N + 1$, but it's not pretty.

In this context, there are also corresponding results on good reduction (integrality) and pluripotential theory (asymptotics for Lyapunov exponents).

Next steps

There is a lot of room for progress on understanding regular polynomial endomorphisms relative to their behaviour at infinity.

Conjecture

For $f \in \mathsf{M}_d^N$ *with* $f^*H = dH$,

$$\hat{h}_{\mathrm{crit}}(f) \asymp h_{\mathsf{M}_d^N}(f) + O_{f|_H}(1).$$

In particular, the critical height is a moduli height on fibres of the restriction to infinity (which is a map to M_d^{N-1}*).*

Conjecture (In characteristic 0, say)

If f *is an algebraic family of PCF maps with* $f^*H = dH$*, and* $f|_H$ *is isotrivial, then* f *is isotrivial.*

Thank you.