AI-assisted Theoretical Discovery

YANG-HUI HE

何楊輝

London Institute of Mathematical Sciences, Royal Institution

Merton College, University of Oxford

VaNTAGe online seminar

Where are we with AI for Mathematical Discovery?

Bottom-Up as a formal logical system

Top-Down as a creative/intuitive art

Meta-Mathematics as a language

Reviews:

YHH: Machine-Learning Mathematical Structures, 2101.06317 *IJMSDS 2021*

YHH: A Triumvirate of AI Driven Theoretical Discovery, 2405.19973 *Nature Rev. Phys, 2024*

Proof Co-Pilots

Building up from Foundations

Russell-Whitehead *Principia Mathematica* [1910s] programme (following: Euclid, Leibniz, Frege, . . .) to axiomatize mathematics

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Automated Theorem Proving (ATP) a long tradition

- Newell-Simon-Shaw [1956] Logical Theory Machine $\leadsto$ **proved some *Principia***
- Type Theory [1970s] Martin-Löf, Coquand
- Univalent Foundation / Homotopy Type Theory [2006-] Voevodsky
- Coq system: 4-color (2005); Feit-Thompson Thm (2012);

Mathlib

- Hilbert, Russel-Whitehead: Formalism ("We must know, we will know")
- `https://leanprover-community.github.io/mathlib-overview.html` [Launched 2013 by L. de Moura]
- Buzzard, Commelin, Massot: *Formalising perfectoid spaces* [2019]
- Gowers-Green-Manner-Tao: Freiman-Rusza-Marton Conj [2023]
- Buzzard: ICM 2022: XenaProject (2013 - 2023) all of undergraduate maths
- Tao: Machine-Assisted Proof 2024

Lean Community

2023- Lean F(ocused) R(esearch) O(rganization): 2023 -

towards auto-formalization & the future of formalization

- *Process-Driven Autoformalization in Lean 4*: Lu et al. [2024]; *130k Lines of Formal Topology in Two Weeks*: Urban [2026]

2024- Buzzard's Formalizing Fermat EPSRC Grant [2024 - 7]

2024- Tooby-Smith's PhysLean

2025- Buzzard-YHH-Mehta-Hill: London Lean Monthly: Imperial/UCL/LIMS

- Deepmind AlphaProof (vs. AlphaGeo & AlphaGeo2): incorporates Lean

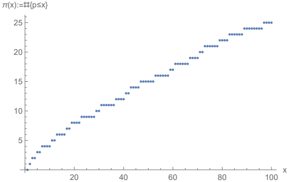
Experimental Mathematics

AI-guided intuition

- (somewhat) Brouwer: Intuitionism (mathematics is a creation of the mind; no more law of excluded middle $A \vee \neg A$)
- Theoretical physics had never been a bottom-up process
- "Mathematics is a branch of physics where the experiments are cheap" - V. Arnol'd
- Conjecture Formulation from mathematical data
- M. Douglas: Platonic Data

intuition, experience, experimentation

- C19th: Gauss



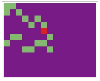
(w/o computer and before complex analysis [50 years before Hadamard-de la Vallée-Poussin's proof]): PNT $\pi(x) \sim x/\log(x)$

- C20th: BSD computer experiment of Birch & Swinnerton-Dyer [1960's] on plots of rank r & N_p on elliptic curves
- 2 / 6 remaining Millennium Prize Problems (RH, BSD) came from experimentation

- Original Motivation: Plethora (10^N possible vacuum solutions in string theory compactifications), computationally prohibitive
- A typical calculation:

$$X = \begin{pmatrix} 1&1&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&0&2&0 \\ 0&1&1&0&0&0&0&1 \\ 1&0&0&0&0&1&1&0 \\ 0&0&0&1&1&0&0&1 \end{pmatrix} \longrightarrow \text{What Bourbaki teaches us} \longrightarrow h^{2,1}(X) = 22$$

- [YHH 1706.02714] Deep-Learning the Landscape, *PLB 774, 2017*; (cf. Feature in *Science* , Aug, vol 365 issue 6452, 2019): think of it as an image processing problem



$\longrightarrow$ What Machine-Learning teaches us $\longrightarrow 22$

Thank you! Since 2017-

fantastic students

Jiakang Bao, Elli Heyes, Ed Hirst, Tejas Acharya, Daatta Aggrawal, Malik Amir, Kieran Bull, Lucille Calmon, Siqi Chen, Suvajit Majumder, Maks Manko, Toby Peterken, Juan Pérez-Ipiña, Max Sharnoff, Yan Xiao

wonderful collaborators

Physics: **Guillermo Arias-Tamargo, David Berman, Heng-Yu Chen, Andrei Constantin, Sebastián Franco, Vishnu Jejjala, Seung-Joo Lee, Andre Lukas, Shailesh Lal, Brent Nelson, Diego Rodriguez-Gomez, Zaid Zaz**

Algebraic Geometry: **Anthony Ashmore, Challenger Mishra, Rehan Deen, Burt Ovrut**

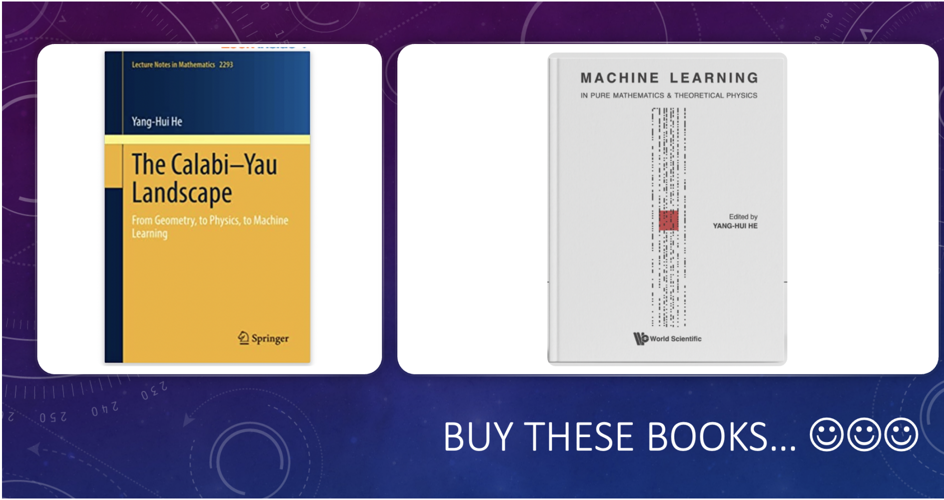
Number Theory: **Laura Alessandretti, Andrea Baronchelli, Kyu-Hwan Lee, Tom Oliver, Alexey Pozdnyakov, Drew Sutherland, Eldar Sultanow**

Representation Theory: **Mandy Cheung, Pierre Dechant, Minhyong Kim, Jianrong Li, Gregg Musiker**

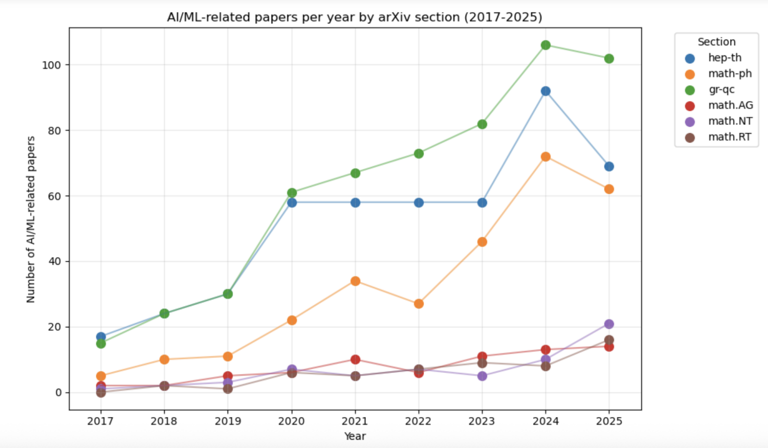
Combinatorics: **Johannes Hofscheier, Alexander Kasprzyk, Shiing-Tung Yau**

2018: Springer, Lecture Notes in Maths 2293

2020: World Scientific, editorial



A Growing Field



AI4Maths Annual Series

2017 - String_Data (w/ Krippendorf, Mishra, Halverson, Ruehle, Grimm, Hashimoto, Gukov, et al)

2020 - DANGER (w/ Kaspryzyk, Hirst, Heyes, Oliver, Lee)

2021- Math-AI @ NeurIPS

2024 - ICMS, Harvard (Douglas et al)

2024 - AI for Math Workshop @ ICML

2025 - AI4x @ NUS Singapore

The Birch Test: "AI + N" @ "AI"

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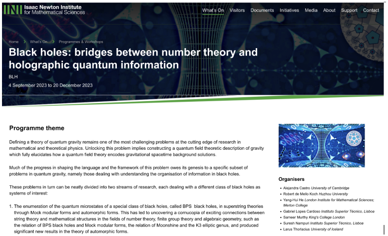

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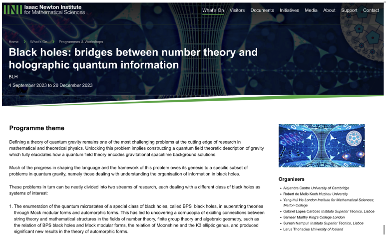

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Black holes: bridges between number theory and holographic quantum information

BLH

4 September 2023 to 20 December 2023

Programme theme

Defining a theory of quantum gravity remains one of the most challenging problems at the cutting edge of research in mathematical and theoretical physics. Unlocking this problem implies constructing a quantum field theoretic description of gravity which fully elucidates how a quantum field theory encodes gravitational spacetime background solutions.

Much of the progress in shaping the language and the framework of this problem owes its genesis to a specific subset of problems in quantum gravity, namely those dealing with understanding the organisation of information in black holes.

These problems in turn can be neatly divided into two streams of research, each dealing with a different class of black holes as systems of interest:

1. The enumeration of the quantum microstates of a special class of black holes, called BPS black holes, in superstring theories through Mock modular forms and automorphic forms. This has led to uncovering a cornucopia of exciting connections between string theory and mathematical structures in the fields of number theory, finite group theory and algebraic geometry, such as the relation of BPS black holes and Mock modular forms, the relation of Moonshine and the K3 elliptic genus, and produced significant new results in the theory of automorphic forms.

Organisers

- Alejandra Castro *University of Cambridge*
- Robert de Mello Koch *Huzhou University*
- Yang-Hui He *London Institute for Mathematical Sciences; Merton College*
- Gabriel Lopes Cardoso *Instituto Superior Técnico, Lisboa*
- Sameer Murthy *King's College London*
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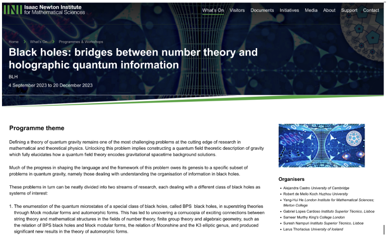

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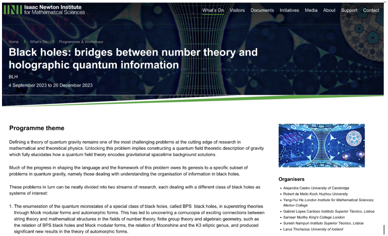

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- (Interpretability) concrete enough to be a conjecture
- (Non-Triviality) for the community to work on it

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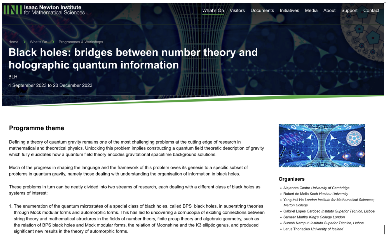


- (Automaticity) be generated by AI
- (Interpretability) concrete enough to be a conjecture
- (Non-Triviality) for the community to work on it
- make Birch happy

Four Closest Examples

2022 new knot invariants relations from saliency [Davies et al] Deepmind (fails N)

2022 Murmuration Conjectures (fails A) [YHH-Lee-Oliver-Podznyakov, 2022, YHH-Lee-Oliver-Podznyakov-Sutherland, tbc] A patten in L-functions reflecting Chebyshev's bias in the primes

2025 Discovering Singularities [Wang et al, DeepMind] (fails A)
Aristole@HarmonicMath: Erdos problem #124, 481: zero intervention (A, I, almost-N, no conjecture auto-raised)

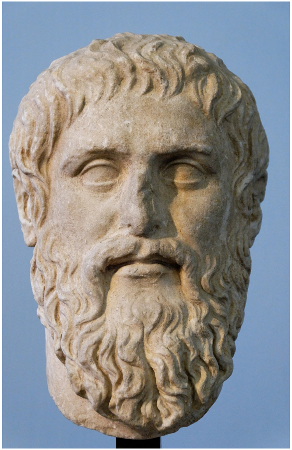
Rmk: Graffiti (1990s), TxGraffiti (2017) have been raising conjectures from tables for graph theory, but no AI yet

2022 - YHH, Kyu-Hwan Lee, Tom Oliver, Alexey Pozdnyakov (2204.10140),

Experimental Mathematics, Volume 34, 2025 - Issue 3

Quanta Feature 2024:

Platonic Data



- ~ 1 Terabyte free online
- Finite Groups: GAP
- Algebraic Geometry: GrDB , CY Databases
- Number Theory: lmfDB
- Knot Theory: SnaPY
- ...

Patterns

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas... *G. Hardy, A Mathematician's Apology*

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One (only?) sure thing that AI can do better than humans is pattern detection.

Pattern Recognition: Human Eye

- $[0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots]$

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- $[0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots]$

  multiple of 3 or not.

- **[1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0 . . .]**

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  Prime or Not for odd integers.

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  Even/Odd of number of prime factors (Liouville Lambda)

Machine-Learning of Platonic Data: Since 2017

2017 (YHH) Binary Classification of a Binary Vector (sliding window of, say, length 100); supervised learning: predict next one, e.g., Prime/Not becomes:

$$
\begin{array}{ccc}
\{0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,\ldots,0\} & \longrightarrow & 1 \\
\{1,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,\ldots,1\} & \longrightarrow & 0 \\
\{0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,0,0,1,1,0,0,\ldots,0\} & \longrightarrow & 1 \\
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\ldots & \ldots & \ldots
\end{array}$$

- pass to standard classifiers: SVW, Bayes, Nearest Neighbour; NN of the form $\mathbb{R}^{100} \xrightarrow{\text{linear}} \mathbb{R}^{20} \xrightarrow{\tanh} \mathbb{R}^{20} \xrightarrow{\textbf{Round}\sum} \mathbb{Z}$, your kitchen sink, ...
- take 50,000 samples, 20-80 cross-validation, record (precision, MCC)
- similar performance for most: Mod3: (1.0, 1.0); PrimeQ, after balancing: (0.8, 0.6); Liouville Λ: (0.5, 0.001)

LLMs @ Math

Rapid Progress

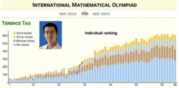
2022 ChatGPT passes the Turing Test

2023 DeepMind's FunSearch, Meta-AI's LLama

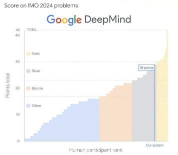
2024 DeepMind's AlphaGeometry, AlphaGeometry2,

2024-5 AlphaProof (incoporates Lean)

July 2025: AI $\simeq$ 12 year old Terry Tao



Year	Country	P1	P2	P3	P4	P5	P6	Total	Rank Abs.	Rank Rel.	Award
1988	Australia	7	5	7	7	7	1	34	13	95.51%	Gold medal
1987	Australia	7	7	7	7	7	5	40	28	88.56%	Silver medal
1986	Australia	7	7	3	1	0	1	19	87	58.85%	Bronze medal



Graph showing performance of our AI system relative to human competitors at IMO 2024. We earned 28 out of 42 total points, achieving the same level as a silver medalist in the competition.

Terence Tao, 28 points - Silver Medal; AlphaGeo2, 28 points - Silver Medal (2024), 35 points - Gold (2025)

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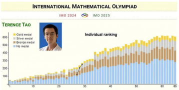

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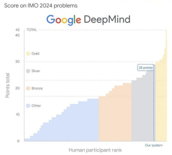

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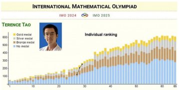

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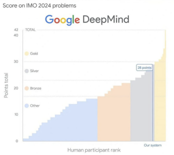

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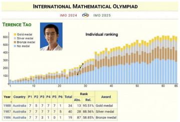
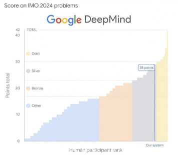
- but ... Mar 2025, Tier 1-3, 10-25% of $125 + 75 + 100$ graduate-level problems solved `https://epoch.ai/frontiermath/`

July 2025: AI $\simeq$ 12 year old Terry Tao





Graph showing performance of our AI system relative to human competitors at IMO 2024. We earned 28 out of 42 total points, achieving the same level as a silver medalist in the competition.

Terence Tao, 28 points - Silver Medal; AlphaGeo2, 28 points - Silver Medal (2024), 35 points - Gold (2025)

- but ... FontierMath from EpochAI: only 2% on advanced problems, Dec 2024
- but ... Mar 2025, Tier 1-3, 10-25% of $125 + 75 + 100$ graduate-level problems solved `https://epoch.ai/frontiermath/`
- and May 2025, 30 of us $\to$ Berkeley HQ for Tier 4 (Oct 2025, 10%))

Scientific American

Financial Times

Elliptic curves and higher Galois Theory

Tier 4

Problem Solution

Define functions $F(z)$ and $G(z)$ on the upper-half of the complex plane by

$$F(z) := 1 + \sum_{n=1}^{\infty} \frac{e^{2\pi i n^2 z}}{\prod_{j=1}^{n}(1+e^{2\pi i j z})^2} \quad \text{and}$$

$$G(z) := \prod_{n=1}^{\infty} \frac{(1-e^{2\pi i n z})(1-e^{2\pi i (2n-1)z})}{(1+e^{2\pi i n z})}.$$

Let ℓ_1 be the smallest prime for which all of the following hold:

1. $D_{\ell_1} := -\ell_1$ is the discriminant of the ring of integers of $\mathbb{Q}(\sqrt{-\ell_1})$.
2. The class number $h(D_{\ell_1})$ of this quadratic field is a prime number $\ell_2 \geq 5$.
3. The class $\ell_2 \pmod{\ell_1}$ is a primitive root of the multiplicative group $(\mathbb{Z}/\ell_1\mathbb{Z})^\times$.
4. The Mordell-Weil group over $\mathbb{Q}$ of the elliptic curve

$$E(\mathbb{Q}) \ : \ Y^2 = X^3 - \ell_1^2 X$$

is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

With this pair of primes (ℓ_1, ℓ_2), we define the algebraic number α by the limit

$$\alpha := \lim_{y \to 0^+} \left(F\left(\frac{\ell_1}{4\ell_2} + iy\right) - G\left(\frac{\ell_1}{4\ell_2} + iy\right) + \frac{F\left(\frac{\ell_1}{4\ell_2} + iy\right)}{G\left(\frac{\ell_1}{4\ell_2} + iy\right)} - \frac{G\left(\frac{\ell_1}{4\ell_2} + iy\right)}{F\left(\frac{\ell_1}{4\ell_2} + iy\right)} \right).$$

If $P_\alpha(X) \in \mathbb{Q}[X]$ is its *minimal polynomial* (i.e the monic irreducible polynomial with $P_\alpha(\alpha) = 0$) and K_α is its *splitting field* (a.k.a. the *normal closure* of $P_\alpha(X)$), then compute

$$\Omega := \frac{1}{[K_\alpha : \mathbb{Q}]} \cdot (P_\alpha(\ell_1) + P_\alpha(\ell_2))^{\ell_2 - 1}.$$

Subject: Analytic and algebraic number theory, quantum modular forms

Technique: Class numbers, elliptic curves, harmonic Maass forms, cyclotomic fields

Answer: 739484920973016917654640307149389211803529797747913492121698299966157004 82.

SAMPLE

Problem

Methods of Proof

- STANDARD: Proof by contradiction, induction, syllogism

Methods of Proof

- STANDARD: Proof by contradiction, induction, syllogism
- IN PRACTICE: Proof by intimidation, authority, citation, obfuscation, vibe

. . .

Methods of Proof

- STANDARD: Proof by contradiction, induction, syllogism
- IN PRACTICE: Proof by intimidation, authority, citation, obfuscation, vibe

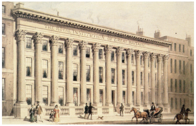
  . . .

- LLMs: Proof by vibe, by citation, by authority

The London Institute for Mathematical Sciences

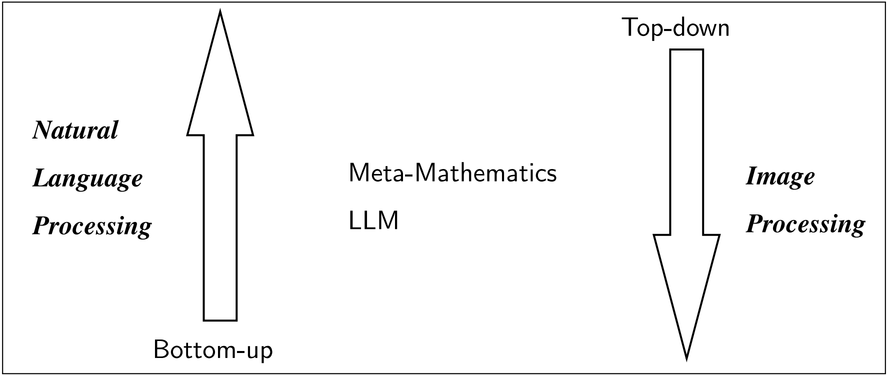
- UK's only indep research institute for maths; after IAS, Princeton est. 2011 Dr. Thomas Fink
- Housed in Faraday Suites of Royal Institution of GB
- 1 of 4 themes: AI for Maths Discovery

  https://lims.ac.uk/event/ai-assisted-maths-discovery/
- Nebius AI for Theory series: launch 2026
- Apply to our Cognia AI/Math JRF
- In process: LIMS-India Fellowship supported by Indian High Commission to the UK



AI-guided theoretical/mathematical Discovery

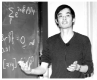
YHH: 2405.19973 *Nature Rev. Phys, 2024*:



The Future of Maths

The Future of Mathematical Research

Papers ⇄



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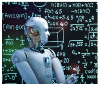
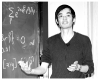
Human *AI*

Top-Down:

Conjecture Formulation from Platonic Data

Meta: Auto-Formalization

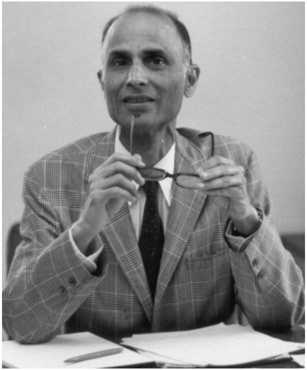




Bottom-Up:

Find proof pathways in *Mathlib*/*Lean*

THANK YOU



"I have often wondered over the roles of knowledge or experience, on the one hand, and imagination or intuition, on the other, in the process of discovery. I believe that there is a certain fundamental conflict between the two, and knowledge, by advocating caution, tends to inhibit the flight of imagination. Therefore, a certain naivete, unburdened by conventional wisdom, can sometimes be a positive asset."

- Harish-Chandra Mehrotra

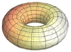
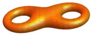
+++++

- Trichtomy classification of (connected compact orientable) surfaces Σ

  Euler: topological classification of $\dim_{\mathbb{R}} = 2$ Euler number $\chi(\Sigma)$, genus $g(\Sigma)$

  Gauss: relates topology to metric geometry

  Riemann: complexify $\rightsquigarrow$ Riemann surfaces or complex curves: $\dim_{\mathbb{C}} = 1$

		   ...
$g(\Sigma) = 0$	$g(\Sigma) = 1$	$g(\Sigma) > 1$
$\chi(\Sigma) = 2$	$\chi(\Sigma) = 0$	$\chi(\Sigma) < 0$
Spherical	Ricci-Flat	Hyperbolic
$+$ curvature	0 curvature	$-$ curvature

Classical Results for Riemann Surface Σ

$\chi(\Sigma) = 2 - 2g(\Sigma) =$	$= [c_1(\Sigma)] \cdot [\Sigma] =$	$= \frac{1}{2\pi} \int_\Sigma R =$	$= \sum_{i=0}^{2} (-1)^i h^i(\Sigma)$
Topology	Algebraic Geometry	Differential Geometry	Index Theorem (co-)Homology
Invariants	Characteristic classes	Curvature	Betti Numbers

Going up in Complex Dimension

- $\dim_{\mathbb{R}} > 2$ manifolds extremely complicated
- Luckily, for a special class of complex manifolds called **Kähler**

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K(z, \bar{z})$$

all Σ in $\dim_{\mathbb{C}} = 1$ automatically Kähler

- CONJECTURE [E. Calabi, 1954, 1957]: M compact Kähler manifold (g, ω) and $([R] = [c_1(M)])_{H^{1,1}(M)}$.

  Then $\exists!(\tilde{g}, \tilde{\omega})$ such that $([\omega] = [\tilde{\omega}])_{H^2(M;\mathbb{R})}$ and $Ricci(\tilde{\omega}) = R$.

Rmk: $c_1(M) = 0 \Leftrightarrow$ Ricci-flat (rmk: Ricci-flat familiar to physicists through GR)

- THEOREM [S-T Yau, 1977-8; Fields 1982] Existence Proof

Calabi-Yaus in Physics

Complex, Ricci-flat (vacuum Einstein) manifolds with $g_{\mu\bar{\nu}}$ coming from a potential

- String Pheno [Candelas-Horowitz-Strominger-Witten] Nuc Phys B 1985

$$10 = 4 + 6$$

Coined the term "Calabi-Yau", and began the perhaps the largest dialogue between theoretical physics and algebraic geometry for the past 1/2 century

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- elsewhere
  - Space of vacua of electro-weak sector of MSSM is Calabi-Yau cone ([He-Jejjala-Matti-Nelson] PLB 2014)
  - Ordinary ϕ^4, L-loop Feynman integral $\rightsquigarrow$ integration over Calab-Yau $(L-1)$-fold [Bourjaily-He-McLeod-von Hippel-Wilhelm] PRL 2018

Explicit Examples of Calabi-Yau Spaces

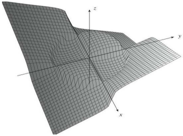
An interesting sequence: 1,2, ??? ...

$\dim_{\mathbb{C}} = 1$ Torus $T^2 = S^1 \times S^1$

$\dim_{\mathbb{C}} = 2$

(1) 4-Torus $T^4 = S^1 \times S^1 \times S^1 \times S^1$

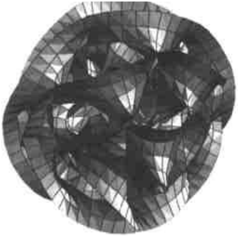
(2) K3 surface



$\dim_{\mathbb{C}} = 3$

Billions Known (Candelas, de la Ossa, Lutken, Krezuer, Skarke ...)

???



The String Landscape

Perhaps the biggest theoretical challenge to string theory:

selection criterion??? metric on the landscape???

- Douglas (2003): Statistics of String vacua
- Kachru-Kallosh-Linde-Trivedi (2003): type II/CY estimates of 10^{500}
- Taylor-YN Wang (2015-7): F-theory estimates 10^{3000} to 10^{10^5}
- Basic Reason: Algebraic Geometry $\rightsquigarrow$ Combinatorial Geometry $\rightsquigarrow$ Exp Growth in dim

Even in the CY landscape:

- compact + hyper-surface in toric variety: $1, 16, 4319, 473800776,$???
- Conjecture [Yau]: finite topological type for CYn Back to ML for Maths

Computing Hodge Numbers: Sketch

- Recall Hodge decomposition $H^{p,q}(X) \simeq H^q(X, \wedge^p T^\star X) \rightsquigarrow$

$$H^{1,1}(X) = H^1(X, T_X^\star), \qquad H^{2,1}(X) \simeq H^{1,2} = H^2(X, T_X^\star) \simeq H^1(X, T_X)$$

- Euler Sequence for subvariety $X \subset A$ is short exact:

$0 \to T_X \to T_M|_X \to N_X \to 0$

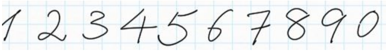
- Induces long exact sequence in cohomology:

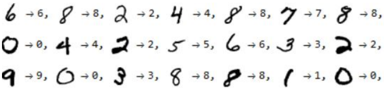
$$\begin{array}{cccccccc} 0 & \to & H^0(X,T_X) \; (\text{struck out} \nearrow 0) & \to & H^0(X,T_A|_X) & \to & H^0(X,N_X) & \to \\ & \to & \boxed{H^1(X,T_X)} & \stackrel{d}{\to} & H^1(X,T_A|_X) & \to & H^1(X,N_X) & \to \\ & \to & H^2(X,T_X) & \to & \ldots & & & \end{array}$$

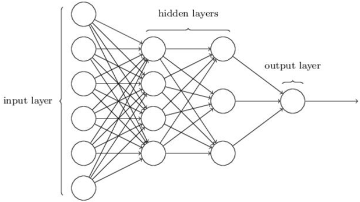
- Need to compute Rk(d), cohomology and $H^i(X, T_A|_X)$ (Cf. Hübsch)

Back to AG

The Neural Network Approach

- Bijection from  to $\{1, 2, \ldots, 9, 0\}$?
- Take large sample, take a few hundred thousand (e.g. NIST database)





- Data = Training Data $\sqcup$ Validation Data

  Test trained NN on validations data to see accuracy performance

Universal Approximation Theorems

Large Depth Thm: (Cybenko-Hornik) For every continuous function $f : \mathbb{R}^d \to \mathbb{R}^D$, every compact subset $K \subset \mathbb{R}^d$, and every $\epsilon > 0$, there exists a continuous function $f_\epsilon : \mathbb{R}^d \to \mathbb{R}^D$ such that $f_\epsilon = W_2(\sigma(W_1))$, where σ is a fixed continuous function, $W_{1,2}$ affine transformations and composition appropriately defined, so that $\sup\limits_{x \in K} |f(x) - f_\epsilon(x)| < \epsilon$.

Large Width Thm: (Kidger-Lyons) Consider a feed-forward NN with n input neurons, m output neuron and an arbitrary number of hidden layers each with $n + m + 2$ neurons, such that every hidden neuron has activation function φ and every output neuron has activation function the identity. Then, given any vector-valued function f from a compact subset $K \subset \mathbb{R}^m$, and any $\epsilon > 0$, one can find an F, a NN of the above type, so that $|F(x) - f(x)| < \epsilon$ for all $x \in K$.

ReLU Thm: (Hanin) For any Lebesgue-integral function $f : \mathbb{R}^n \to \mathbb{R}$ and any $\epsilon > 0$, there exists a fully connected ReLU NN F with width of all layers less than $n + 4$ such that $\int_{\mathbb{R}^n} |f(x) - F(x)| dx < \epsilon$.

Back to NN@Alg Geo

A minute waltz on BSD

Diophatine Equations (rational $\mathbb{Q}$ solutions to polynomials)

- quadratic (Pythagoras) $x^2 + y^2 = 1$, many e.g. $(x, y) = (\frac{3}{5}, \frac{4}{5})$

A minute waltz on BSD

Diophatine Equations (rational $\mathbb{Q}$ solutions to polynomials)

- quadratic (Pythagoras) $x^2 + y^2 = 1$, many e.g. $(x, y) = (\frac{3}{5}, \frac{4}{5})$
- cubic (Elliptic Curve) HARD!!!!
  - e.g., $E : x^3 + y^2 = 1$ (Thm: Weierstrass form $y^2 = x^3 - g_2 x - g_4$)
  - has enormous implications: e.g. Fermat's Last Theorem
  - Thm: $E(\mathbb{Q}) \simeq \mathbb{Z}^r \times Z_{tor}$ (r rank = # infinite families of solutions)
  - Work modulo prime p, e.g., $2^3 + 4^2 \equiv 1 (\text{mod} 23)$, $3^3 + 2^2 \equiv 1 (\text{mod} 5)$
  - Euler Coeffcients $a_p = p + 1 - \#E(\mathbb{F}_p)$

ML on BSD

- E an elliptic curve, local zeta-function & L-function:

$$Z(E/\mathbb{F}_p;T) = \exp\left(\sum_{k=1}^{\infty} \frac{\#E(\mathbb{F}_{p^k})T^k}{k}\right) = \frac{L_p(E,T)}{(1-T)(1-pT)} \ ;$$
$$L_p(E,T) = 1 - a_p T + pT^2; \quad a_p = p + 1 - \#E(\mathbb{F}_p) \, .$$

Fix N and define vector $v_L(E) = (a_{p_1}, \ldots, a_{p_N}) \in \mathbb{Z}^N$;

$\sim 10^5$ balanced data from `www.lmfdb.org`; 50-50 cross validation.

- Labeled data: $v_L(E) \longrightarrow$ rank, torsion, ... ([Birch-Swinnerton-Dyer:])

$$L(E,s) := \prod_p L^{-1}(E, T := p^{-s}); \quad \frac{L^{(r)}(E,1)}{r!} \stackrel{???}{=} \frac{|Ш|\Omega \mathrm{Reg} \prod_p c_p}{(\#E(\mathbb{Q})_{\mathrm{tors}})^2},$$

r=rank; Ш=Shafarevich group; Reg=regulator; c_p=Tamagawa; tors=Torsion

A simple PCA!

Q: YHH, Lee, Oliver, Pozdnyakov on HLOP results from 2020 - 22: WHY is ML so good at telling ranks apart by looking at a_p coefficients?? e.g., PCA:

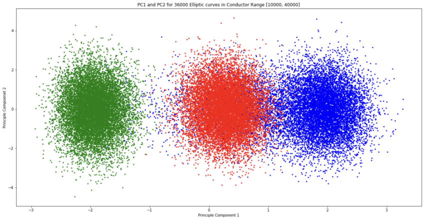


Figure 2: A plot of PC1 (x-axis) against PC2 (y-axis) for elliptic curves in the balanced dataset of $36,000$ randomly chosen elliptic curves with rank $r_E \in \{0, 1, 2\}$ and conductor $N_E \in [10000, 40000]$. The blue (resp. red, green) points are the images of the vectors $v_L(E)$ corresponding to the elliptic curves in our dataset with rank 0 (resp. 1, 2) under a map $\mathbb{R}^{1000} \to \mathbb{R}^2$ constructed using PCA.

Murmuration function

construct a vertical average (rank r, conductor range $[N_1, N_2]$, n-th prime p_n)

$$f_r(n) := \frac{1}{\#\mathcal{E}_r[N_1, N_2]} \sum_{E \in \mathcal{E}_r[N_1, N_2]} a_{p_n}(E)$$

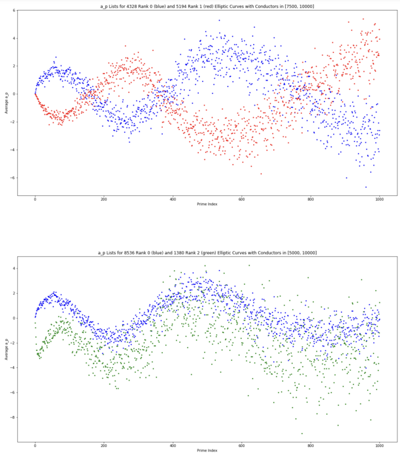


Figure 1: (Top) Plots of the functions $f_0(n)$ (blue) and $f_1(n)$ (red) for $1 \le n \le 1000$ and $[N_1, N_2] = [7500, 10000]$. (Bottom) Plots of the functions $f_0(n)$ (blue) and $f_2(n)$ (green) for $1 \le n \le 1000$ and $[N_1, N_2] = [5000, 10000]$. Further details are given in Example 1.

Murmurations: An interesting Phenomenon

- To appear [HLOP + Sutherland]
  - Does not work if ordered by height (Weierstrass coef)
  - Take dyadic conductor range: $[N^x, N^{x+1}]$: scale invariant (indep of x)
  - Taking more data $(10^{7\sim8})$ at high N: converges to oscillatory curve
- A General Phenomenon that reflects biases in distribution of primes
  - L-function for Dirichlet characters (Lee-Oliver-Podznyakov 2023)
  - Zubrilina, Cowan: for weight 2 modular forms (2023)
  - conference at ICERM in July

    `https://icerm.brown.edu/events/htw-23-ma/`

Important Lesson: HYBRID human-AI math

Importance of Representation

(Alessandretti-Baronchelli-YHH 1911.02008, *New Scientist* feature 2019 used Weierstrass coefficients of elliptic curves: useless in predicting any of the BSD quantities

needed insights from Oliver+Lee to use a_p coefficients

Importance of Human Interpretation

Murmurations of elliptic curves: YHH, Lee, Oliver, Pozdnyakov, 2204.10140

A new mathematical phenomenon Back to AI for Maths

Q: is deviation from random distribution a measure of ML of mathematical structure?