Division Fields of Superelliptic Jacobians

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Basic Data

k : algebraically closed field of characteristic $p \geq 0$

m : positive integer

$K = k(t_0, \ldots, t_m)$: function field of $\mathbb{A}^{1+m}$

r : prime distinct from p

$\boldsymbol{\mu}_r \subset k^\times \subset K^\times$: rth roots of unity

$0 < e_0, \ldots, e_m < r$, $e_\infty := -\sum_{i=0}^m e_i$: exponents (assume $e_\infty \notin r\mathbb{Z}$)

U/K : affine curve $y^r = \prod_{i=0}^m (x - t_i)^{e_i}$

C/K : smooth completion of U

- say C is *elliptic* or *hyperelliptic* when $r = 2$
- say C is *superelliptic* curve when $r > 2$

$K(U) = K(C) = K(x, y)$: function fields

Kummer Theory

U/K : affine curve $y^r = \prod_{i=0}^m (x - t_i)^{e_i}$

Proposition

1. *$K(C)$ is Galois over $K(x)$ with group $\boldsymbol{\mu}_r$*
2. *Galois action induces faithful action of $\boldsymbol{\mu}_r$ on C*
3. *the fibres of $x\colon C \to \mathbb{P}^1$ are the orbits of $\boldsymbol{\mu}_r$*

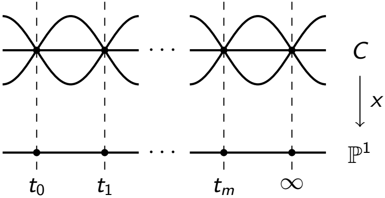
Proposition

1. *$x\colon C \to \mathbb{P}^1$ is totally ramified over $x = t_0, \ldots, t_m$ and $x = \infty$*
2. *$x\colon C \to \mathbb{P}^1$ is unramified everywhere else*

Proof Sketch.

- model of U about $x = t_i$ has form $y^r = (\text{unit}) \cdot (\text{uniformizer})^{e_i}$ and $r \nmid e_i$
- Zariski closure of U in $\mathbb{P}^2$ has similar model about $x = \infty$ and $r \nmid e_\infty$
- model of U about all other x has form $y^r = (\text{unit})$ □

Genus Calculation



(= picture for $r = 3$)

Proposition

$2 \cdot \mathrm{genus}(C) = m \cdot (r-1) > 0$

Proof Sketch.

$x\colon C \to \mathbb{P}^1$ is tamely ramified since $\deg(x) = |\boldsymbol{\mu}_r| = r$ is invertible in K

$\Rightarrow$ can apply Riemann-Hurwitz formula

$\Rightarrow$ $2 \cdot \mathrm{genus}(C) - 2 = r \cdot (2 \cdot \mathrm{genus}(\mathbb{P}^1) - 2) + \sum_{i=0}^{m}(r-1) + (r-1)$ □

Division Fields

g : genus of C

J/K : Jacobian of C (= g-dimensional ppav)

$\ell \neq 2, p, r$: rational prime

$J[\ell] \subset J$: subgroup of ℓ-torsion (= $2g$-dimensional $\mathbb{F}_\ell$-vector space)

$\langle\ \rangle \colon J[\ell] \times J[\ell] \to \boldsymbol{\mu}_\ell$: Weil pairing (= $\mathbb{F}_\ell$-bilinear and alternating)

$\mathrm{Sp}(J[\ell]) \subseteq \mathrm{GL}(J[\ell])$: pairing isometries and linear automorphisms

$K_\ell = K(J[\ell])$: ℓth division field (= Galois extension of K)

$G_\ell \subseteq \mathrm{Sp}(J[\ell]) \simeq \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$: Galois group of K_ℓ/K (recall $\boldsymbol{\mu}_\ell \subset K$)

Question

How does G_ℓ vary with $(r, m$ and$)$ ℓ?

Symplectic Lower Bound

$r = 2$: slide hypothesis (recall $\ell \neq 2, p$)

Theorem (Yu,Achter-Pries,H.)

$G_\ell = \mathrm{Sp}(J[\ell])$

Theorem (H.)

1. *$J[\ell]$ is a simple $\mathbb{F}_\ell[G_\ell]$-module*
2. *G_ℓ is generated by transvections τ, i.e.,* $\mathrm{rank}(\tau - 1) = 1$ *and* $\det(\tau) = 1$

Corollary (Zalesskiĭ-Serežkin)

$G_\ell \supseteq \mathrm{Sp}(J[\ell])$

[compare irreducible subgroups of S_n generated by 2-cycles]

Proposition

1. *$\boldsymbol{\mu}_r$ acts canonically and faithfully on J and $J[\ell]$*
2. $\langle \zeta P, Q \rangle = \langle P, \zeta^{-1} Q \rangle$ *when* $\zeta \in \boldsymbol{\mu}_r$ *and* $P, Q \in J[\ell]$

$\Gamma_\ell \subseteq \mathrm{Sp}(J[\ell])$: centralizer of $\boldsymbol{\mu}_r$ $\qquad$ $(= \mathrm{Sp}(J[\ell]))$ when $r = 2$)

Proposition

$G_\ell \subseteq \Gamma_\ell$

$\mathcal{D}\Gamma_\ell = [\Gamma_\ell, \Gamma_\ell]$: derived subgroup (= product of special linear/unitary groups[1])

Theorem (Achter-Pries)

G_ℓ is big (i.e., $\mathcal{D}\Gamma_\ell \subseteq G_\ell$) when $m \geq r = 3$ and $\ell \neq 2, r$

Theorem (H.)

G_ℓ is big and $G_\ell / \mathcal{D}\Gamma_\ell = \boldsymbol{\mu}_r$ when $m, r \geq 3$ and $\ell \neq 2, 3, r$

[1] $r > 2$

Module Structure

$\mathbb{F}_\ell[\boldsymbol{\mu}_r]$: group ring (= commutative semisimple $\mathbb{F}_\ell$-algebra)

$R \subset \mathbb{Q}(\boldsymbol{\mu}_r)$: ring of integers (= $\mathbb{Z}$-submodule and $\boldsymbol{\mu}_r$-module)

$R_\ell = R/\ell R$: quotient ring (= $\mathbb{F}_\ell[\boldsymbol{\mu}_r]$-module)

Proposition

$J[\ell] \simeq R_\ell^{\oplus m}$ as $\mathbb{F}_\ell[\boldsymbol{\mu}_r]$-modules

Proof Sketch.

let $\rho_r \colon \boldsymbol{\mu}_r \to \mathbb{F}_\ell$ be the character of $\mathbb{F}_\ell[\boldsymbol{\mu}_r]$, that is, $\rho_r(\zeta) = \mathrm{Tr}(\zeta \mid \mathbb{F}_\ell[\boldsymbol{\mu}_r])$

$\Rightarrow$ character of R_ℓ is $\rho_r - 1$

let $\psi_\ell \colon \boldsymbol{\mu}_r \to \mathbb{F}_\ell$ be the character of $J[\ell]$

apply equivariant Riemann-Hurwitz

$\Rightarrow$ $\psi_\ell - 2 = r \cdot (0-2) + \sum_{i=0}^{m}(\rho_r - 1) + (\rho_r - 1) = m \cdot (\rho_r - 1) - 2$

$\Rightarrow$ $J[\ell]$ and $R_\ell^{\oplus m}$ have the same characters as $\mathbb{F}_\ell[\boldsymbol{\mu}_r]$-modules $\square$

λ-division Fields

$\lambda \subset R$: prime over ℓ (= $\mathbb{Z}[\boldsymbol{\mu}_r]$-submodule)

$\mathbb{F}_\lambda = R/\lambda R$: residue field (= $\mathbb{F}_\ell[\boldsymbol{\mu}_r]$-module)

$J[\lambda] \subseteq J[\ell]$: module of λ-torsion (= submodule annihilated by λ)

Proposition

1. $R_\ell = \oplus_{\lambda|\ell}\mathbb{F}_\lambda$ *as rings and modules*
2. $J[\ell] = \oplus_{\lambda|\ell} J[\lambda]$ *and* $J[\lambda] \simeq \mathbb{F}_\lambda^{\oplus m}$ *as modules*

$\mathrm{GL}(J[\lambda]) \simeq \mathrm{GL}_m(\mathbb{F}_\lambda)$: $\mathbb{F}_\lambda$-linear automorphisms

$K_\lambda = K(J[\lambda])$: λ-division field (= Galois subextension of K_ℓ/K)

$G_\lambda \subseteq \mathrm{GL}(J[\lambda])$: Galois group of K_λ/K (= quotient of G_ℓ)

Proposition

G_ℓ is a subdirect product of $\prod_{\lambda|\ell} G_\lambda$

Unitary Lower Bound

$(m > 2) \wedge (r > 2) \wedge (\ell > 3)$: slide hypothesis (recall $\ell \neq 2, p, r$)

Theorem (H.)

1. $J[\lambda]$ *is a simple and primitive* $\mathbb{F}_\lambda[G_\lambda]$*-module*
2. G_λ *is generated by pseudoreflections* σ*, i.e.,* $\mathrm{rank}(\sigma - 1) = 1$
3. $\det(G_\lambda) = \boldsymbol{\mu}_r \subseteq \mathbb{F}_\lambda^\times$

$\bar{\lambda} \subset R$: complex conjugate prime

Corollary (Wagner)

$\mathcal{D}G_\lambda \simeq \mathrm{SU}_m(\mathbb{F}_\lambda)$ *when* $\lambda = \bar{\lambda}$ *and* $\mathcal{D}G_\lambda \simeq \mathsf{SL}_m(\mathbb{F}_\lambda)$ *when* $\lambda \neq \bar{\lambda}$

Corollary (Goursat-Ribet)

$G_\ell \supseteq \mathcal{D}\Gamma_\ell$ *and* $\det(G_\ell) = \boldsymbol{\mu}_r$

Monodromy Groups of Lisse Sheaves

$\Delta \subset \mathbb{A}^{1+m}$: thick diagonal (= union of hyperplanes $t_i = t_j$)

$B = \mathbb{A}^{1+m} \smallsetminus \Delta$: open complement (= points with distinct coordinates)

$\mathcal{C} \to B$: proper smooth model of C/K (= family of curves)

$\mathcal{J} \to B$: Néron model of J/K (= family of abelian varieties)

Proposition

1. *the actions of $\boldsymbol{\mu}_r$ on C and J extend canonically to $\mathcal{C}$ and $\mathcal{J}$*
2. *the group schemes $\mathcal{J}[\ell] \to B$ and $\mathcal{J}[\lambda] \to B$ are lisse sheaves*

$b = (b_0, \ldots, b_m) \in B$: base point

$\pi_1(B) = \pi_1(B, b)$: étale fundamental group

Proposition

G_ℓ and G_λ are quotients of $\pi_1(B)$

One-Parameter Family

$\mathbb{A}^{1+m} \to \mathbb{A}^m$: projection $(t_1, \ldots, t_m)$ $\quad$ $(=$ forget $t_0)$

$T \subset B$: fibre over $(b_1, \ldots, b_m)$ $\quad$ $(= \mathbb{P}^1 \smallsetminus \{b_1, \ldots, b_m, \infty\})$

$\mathcal{D} \to T$: pullback of $\mathcal{C} \to B$ $\quad$ $(=$ 1-parameter family of curves$)$

Proposition

generic fibre of $\mathcal{D} \to T$ is birational to $y^r = (x-t)^{e_0} \prod_{i=1}^{m} (x-b_i)^{e_i}$

$\pi_1(T) \to \pi_1(B)$: functorial morphism $\quad$ $(= \pi_1((T, b_0) \to (B, b)))$

$H_\lambda \subseteq G_\lambda$: image of $\pi_1(T)$ $\quad$ $(=$ specialized Galois group$)$

$\chi \colon \pi_1(\mathbb{G}_m) \to R_\ell^\times$: Kummer character of order r $\quad$ (image is $\boldsymbol{\mu}_r$)

Theorem (H.)

1. *$\mathcal{J}[\lambda]$ is the middle convolution of $\mathcal{L}_{\chi^{e_0}(x)}[\lambda]$ and $\otimes_{i=1}^{m} \mathcal{L}_{\chi^{e_i}(x-b_i)}[\lambda]$*
2. *$J[\lambda]$ is a simple and primitive $\mathbb{F}_\lambda[H_\lambda]$-module*
3. *H_λ is big when $e_0 + e_i \notin r\mathbb{Z}$ for some $i \in \{1, \ldots, m\}$ when $m > 2$*

Super Legendre Curve

$(m = 2) \wedge (r > 2)$: slide hypothesis

$L = k(T)$: function field

U/L : affine curve $y^r = x^{r-1}(x-1)(x-t)$ (= super Legendre curve)

C/L : smooth completion

J/L : Jacobian of C

$L_\ell = L(J[\ell])$: division field

$H_\ell = \mathrm{Gal}(L_\ell/L)$: specialized Galois group (= subgroup of G_ℓ)

Theorem (Berger-H.-Pannekoek-Park-Pries-Sharif-Silverberg-Ulmer)

1. *J admits an extra involution over a $\boldsymbol{\mu}_r$-extension of L*
2. *J is geometrically isogenous to the square of an abelian variety*
3. *H_ℓ and G_ℓ are not big*