Conjectures on Unlikely Intersections: What is known and what is open?

VaNTAGe Seminar

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An early paper on Unlikely Intersections

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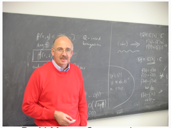
Intersecting a Curve with Algebraic Subgroups of Multiplicative Groups

E. Bombieri, D. Masser, and U. Zannier

§1 Introduction and results

The results of this paper originated from the following question: What can be said about the set of algebraic numbers $\tau \neq 0, 1$ for which τ and $1-\tau$ are multiplicatively dependent? It is easy to see that this set is infinite. On the other hand, simple arguments show that its

Enrico Bombieri, Source: IAS



David Masser, Source: private

Umberto Zannier, Source: Oberwolfach

$$\left\{ \tau \in \mathbb{C} \setminus \{0,1\} : \tau^r (1-\tau)^s = 1 \text{ for some } (r,s) \in \mathbb{Z}^2 \setminus \{0\} \right\} \text{ is infinite.}$$

But such τ seem to be sparse.

Let $\tau \in \mathbb{C} \setminus \{0,1\}$ such that $\tau^r(1-\tau)^s = 1$ for some $(r,s) \in \mathbb{Z}^2 \setminus \{0\}$.

Thus $(\tau, 1-\tau)$ lies on the **fixed** algebraic curve $X + Y = 1$ in $\mathbb{G}_m^2$ and on the **varying** algebaic subgroup of $\mathbb{G}_m^2$ given by $X^r Y^s = 1$.

By an argument of Bombieri there exists B, independent of (r,s), with $h(\tau) \le B$. Cohen and Zannier (1998) showed that $B = \log 2$ is possible.

Theorem (Bombieri–Masser–Zannier 1999)

Let $C \subset \mathbb{G}_m^n$ be an algebraic curve defined over $\overline{\mathbb{Q}}$ that is not contained in any translate of a proper algebraic subgroup of $\mathbb{G}_m^n$. Then

$$C(\overline{\mathbb{Q}}) \cap \bigcup_{\substack{H \subsetneq \mathbb{G}_m^n \\ \text{algebraic subgroup of } \mathbb{G}_m^n}} H(\overline{\mathbb{Q}}) \quad \text{is a set of bounded height.}$$

Members of $C \cap H$ are *just* likely if $\dim H = n - 1$.

Heights

Let $p, q \in \mathbb{Z}$ with $q \geq 1$ and $\gcd(p, q) = 1$. We define the height

$$h(p/q) := \log \max\{|p|, q\}.$$

For α algebraic with $\mathbb{Z}$-minimal polynomial $A = a_d T^d + \cdots + a_0$, i.e., $A(\alpha) = 0$, A is irreducible in $\mathbb{Z}[T]$, and $a_d \geq 1$, we define

$$h(\alpha) := \frac{1}{d} \log \left(a_d \prod_{z \in \mathbb{C} : A(z) = 0} \max\{1, |z|\} \right) \geq 0.$$

- $h(\sqrt{2024}) = \frac{1}{2} \log 2024$ and $h(\sqrt{2024} - \sqrt{2023}) = \frac{1}{4} \log(4047 + 68\sqrt{3542})$
- $h(2^{1/d}) = (\log 2)/d$ for all integers $d \geq 1$

Theorem (Northcott)

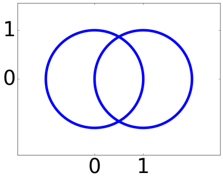
$$\forall B, D \in \mathbb{R} : \{\alpha \in \overline{\mathbb{Q}} : h(\alpha) \leq B \text{ and } [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq D\} \text{ is finite.}$$

Towards Unlikely Intersections

Consider a fixed curve $C \subset \mathbb{G}_m^n$ and $H \subset \mathbb{G}_m^n$ a varying algebraic subgroup with $\dim H \leq n-2$. We expect $C \cap H$ to be empty, but it does not need to be! A non-empty intersection is called *unlikely*.

Source: Agmonsnir on wikipedia

$$\{\tau \in \mathbb{C} \setminus \{0,1\} : \tau^r(1-\tau)^s = 1 \text{ for } 2 \text{ linearly independent} (r,s) \in \mathbb{Z}^2\} \text{ is finite.}$$



Theorem (Bombieri–Masser–Zannier 1999)

Let $C \subset \mathbb{G}_m^n$ be an algebraic curve defined over $\overline{\mathbb{Q}}$ that is not contained in any **translate of a proper algebraic subgroup** *of $\mathbb{G}_m^n$. Then*

$$C(\overline{\mathbb{Q}}) \cap \bigcup_{\substack{H \subset \mathbb{G}_m^n \text{ algebraic subgroup} \\ \dim H \leq n-2}} H(\overline{\mathbb{Q}}) \quad \text{is finite.}$$

The proof needs:

- the height upper bound by BMZ for "$\dim H \leq n-1$" from before
- and a deep higher dimensional version of the Dobrowolski–Lehmer bound by F. Amoroso and S. David (1999).

Boris Zilber, Source: Oberwolfach

Richard Pink, Source: Oberwolfach

Conjecture (Zilber 2002, Pink 2005 on "Unlikely Intersections")

Let A be the multiplicative torus $\mathbb{G}_m^n$ or an abelian variety defined over $\mathbb{C}$. Let $X \subset A$ be a subvariety that is not contained in a **proper algebraic subgroup** *of A. Then*

$$X(\mathbb{C}) \cap \bigcup_{\substack{H \subset A \text{ algebraic subgroup} \\ \dim X + \dim H < \dim A}} H(\mathbb{C}) \quad \text{is not Zariski dense in } X.$$

This conjecture is **open**. More general versions for

- Semiabelian varieties
- Mixed Shimura varieties (Pink, 2005)
- Variations of mixed Hodge structures (Klingler, 2017)

Connections to classical results: Manin–Mumford

Theorem (Raynaud 1983, further proofs: Hrushovski, Pink–Roessler, Pila–Zannier, . . .)

Let A be an abelian variety defined over $\mathbb{C}$, let X be a subvariety of A that is not a component of an algebraic subgroup of A. Then

$$X(\mathbb{C}) \cap A_{\mathrm{tors}} \text{ is not Zariski dense in } X.$$

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If in addition X is not contained in a proper algebraic subgroup of A, then

$$\mathrm{ZP} \Longrightarrow X(\mathbb{C}) \cap \bigcup_{\substack{H \subset A \text{ algebraic subgroup} \\ \dim X + \dim H < \dim A}} H(\mathbb{C}) \quad \text{is not Zariski dense in } X.$$

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If in addition X is not contained in a proper algebraic subgroup of A, then

$$\mathrm{ZP} \Longrightarrow X(\mathbb{C}) \cap \bigcup_{\substack{H \subset A \text{ algebraic subgroup} \\ \dim H = 0}} H(\mathbb{C}) \quad \text{is not Zariski dense in } X.$$

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(If X is contained in a proper algebraic subgroup of A, then shrink A.)

Connections to classical results: Mordell I

Theorem (Faltings 1983, the Mordell Conjecture)

Let C be a smooth projective curve of genus $g \geq 2$ defined over a number field K. Then $C(K)$ is finite.

Let us see why Faltings's Theorem follows from the Zilber–Pink Conjecture.

The *Jacobian* $\mathrm{Jac}(C)$ of C is a g-dimensional abelian variety defined over K. We may assume $C \subset \mathrm{Jac}(C)$ and $K \subset \mathbb{C}$.

Let us assume for simplicity that $\mathrm{End}_{\overline{K}}(\mathrm{Jac}(C)) = \mathbb{Z}$.

By the **Mordell–Weil Theorem**, $\mathrm{Jac}(C)(K)$ is a finitely generated abelian group. We want to use Zilber–Pink to show that

$$C(\overline{K}) \cap \mathrm{Jac}(C)(K) \text{ is finite.}$$

So we need to relate our points to **algebraic subgroups** of some abelian variety.

Connections to classical results: Mordell II

Fix r independent points $P_1, \dots, P_r \in \mathrm{Jac}(C)(K) \cong (\text{finite}) \times \mathbb{Z}^r$.

Then $(P_1, \dots, P_r)$ is not contained in a proper algebraic subgroup of $\mathrm{Jac}(C)^r$ as the Jacobian has only trivial endomorphisms by our simplifying assumption.

The augmented curve

$$\widetilde{C} = C \times \{(P_1, \dots, P_r)\} \subset \mathrm{Jac}(C)^{r+1}$$

is not contained in a proper algebraic subgroup of $\mathrm{Jac}(C)^{r+1}$.

If $P \in C(K)$ there exist $a_0 \in \mathbb{Z}_{\geq 1}$ and $a_1, \dots, a_r \in \mathbb{Z}$ with

$$a_0 P = a_1 P_1 + \cdots + a_r P_r. \tag{$\star$}$$

So $(P, P_1, \dots, P_r) \in \widetilde{C}(\overline{K}) \cap H(\overline{K})$ where H is the (varying!) algebraic subgroup determined by $(\star)$.

Connections to classical results: Mordell III

If $P \in C(K)$ there exist $a_0 \in \mathbb{Z}_{\geq 1}$ and $a_1, \ldots, a_r \in \mathbb{Z}$ with

$$a_0 P = a_1 P_1 + \cdots + a_r P_r. \qquad (\star)$$

So $(P, P_1, \ldots, P_r) \in \widetilde{C}(\overline{K}) \cap H(\overline{K})$ where H is the (varying!) algebraic subgroup determined by $(\star)$.

Let us compute dimensions:

$$\dim C = 1 \quad \text{and} \quad \dim \mathrm{Jac}(C)^{r+1} = g(r+1) \quad \text{and} \quad \dim H = \dim \mathrm{Jac}(C)^{r+1} - g = gr.$$

$$g \geq 2 \Longrightarrow \dim C + \dim H = 1 + gr < g(r+1) = \dim \mathrm{Jac}(C)^{r+1}$$

So $(P, P_1, \ldots, P_r)$ is an unlikely intersection $\widetilde{C}(\overline{K}) \cap H(\overline{K})$. Zilber–Pink now predicts that there are only finitely many P.

What is known?

Theorem (Bombieri–Masser–Zannier 1999)

Let $C \subset \mathbb{G}_m^n$ be an algebraic curve defined over $\overline{\mathbb{Q}}$ that is not contained in any **translate of a proper algebraic subgroup** *of $\mathbb{G}_m^n$. Then*

$$C(\overline{\mathbb{Q}}) \cap \bigcup_{\substack{H \subset \mathbb{G}_m^n \text{ algebraic subgroup} \\ \dim H \leq n-2}} H(\overline{\mathbb{Q}}) \quad \text{is finite.}$$

- Proof uses bounded height, true when allowing $\dim H \leq n-1$
- ZP should hold under weaker hypothesis "**proper algebraic subgroup**".
- Difficulty: bounded height is **false** under weaker hypothesis when allowing $\dim H \leq n-1$.

Important progress by Rémond combining the Mordell–Lang aspect with algebraic subgroups led to

Theorem (Maurin 2008, Bombieri–Masser–Zannier 2008)

Zilber–Pink holds for curves in $\mathbb{G}_m^n$ defined over $\mathbb{C}$.

Roughly, the proof splits up into three steps:

- Maurin using work of Rémond: Bound the height from above when $C/\overline{\mathbb{Q}}$ satisfies the weak hypothesis. Later: new proof by Bombieri–H.–Masser–Zannier
- Prove finiteness when $C/\overline{\mathbb{Q}}$ using height lower bounds à la Amoroso–David.
- A specialization argument done by Bombieri–Masser–Zannier yields

$$\text{ZP for curves}/\overline{\mathbb{Q}} \text{ in } \mathbb{G}_m^n \Longrightarrow \text{ZP for curves}/\mathbb{C} \text{ in } \mathbb{G}_m^n$$

What about abelian varieties? What about C replaced by a subvariety of any dimension?

Curves in abelian varieties

A polarized abelian variety $A/\overline{\mathbb{Q}}$ carries a canonical height $\hat{h}\colon A(\overline{\mathbb{Q}}) \to [0,\infty)$.

Theorem (Rémond 2007)

Let $C \subset A$ be a curve that is not contained in proper algebraic subgroup of A. Then $\hat{h}$ is bounded from above on

$$C(\overline{\mathbb{Q}}) \cap \bigcup_{\substack{H \subset A \text{ algebraic subgroup} \\ \dim H \leq \dim A - 2}} H(\overline{\mathbb{Q}}).$$

Abelian version of Amoroso–David's height lower bounds are more delicate and remain open in general. Finiteness due to

- Ratazzi (2007), Carrizosa (2008): A with complex multiplication
- Galateau (2010), Viada (2010): $A = E^n$, E an elliptic curve

Pila–Zannier Strategy

Jonathan Pila, Source: Oxford

Alex Wilkie, Source: Academia Europaea

In 2008, Pila and Zannier introduced **o-minimal geometry** and the **Pila–Wilkie Counting Theorem** to diophantine geometry and transcendence theory. They reproved the Manin–Mumford Conjecture for abelian varieties over $\overline{\mathbb{Q}}$. Pila then used o-minimal geometry to prove new unconditional cases of the André–Oort Conjecture around 2011.

Using further new ideas, the André–Oort was proved in 2021 by Pila, Shankar, Tsimerman, after much work by Andreatta, André, Daw, Edixhoven, Esnault, Gao, Goren, Groechenig, Howard, Klingler, Madapusi Pera, Masser, Orr, Pila, Shankar, Tsimerman, Ullmo, Wüstholz, Yafaev, Yuan, Zhang . . . .

The Pila–Zannier strategy allowed us to work with a height lower bound by Masser.

Theorem (H.–Pila 2016)

Zilber–Pink holds for curves in an abelian variety of $\overline{\mathbb{Q}}$.

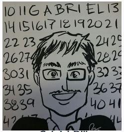
Semi-abelian case by Barroero, Kühne, Schmidt (2021).

Theorem (Barroero–Dill 2019)

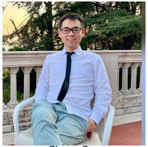
ZP holds for curves $C/\mathbb{C}$ in an abelian variety $A/\mathbb{C}$. I.e., if C is not contained in a proper algebraic subgroup of A, then

$$C(\mathbb{C}) \cap \bigcup_{\substack{H \subset A \\ \dim H \leq \dim A - 2}} H(\mathbb{C}) \quad \textit{is finite.}$$


Fabrizio Barroero



Gabriel Dill


Ziyang Gao

A key ingredient is Gao's mixed Ax–Schanuel Theorem for the universal family of abelian varieties.

What about subvarieties of higher dimension?

Suppose $A/\mathbb{C}$ is an abelian variety and $X \subset A$ a subvariety. We recall a concept due to Bombieri–Masser–Zannier:

A point $P \in X(\mathbb{C})$ is called **anomalous** (for X) if there exists an abelian subvariety $B \subset A$ with

$$\dim_P X \cap (P + B) > \max\{0, \dim X + \dim B - \dim A\}.$$

We set $X^{\mathrm{oa}} = X \smallsetminus \{\text{anomalous points}\}$.

Example

If $X \subset P + B$ with $B \neq A$ and $\dim X \geq 1$, then $X^{\mathrm{oa}} = \emptyset$.

Conversely, if X is a curve with $X^{\mathrm{oa}} = \emptyset$ then X is in the translate of a proper abelian subvariety of A.

Theorem (BMZ for $\mathbb{G}_m^n$ 2007, Rémond for abelian varieties 2009)

$X^{\mathrm{oa}} = X \setminus \{anomalous\ points\}$ *is Zariski open in* X.

Anomalous points coming from B satisfy

$$\dim_P \varphi|_X^{-1}(\varphi(P)) > \max\{0, \dim X + \dim B - \dim A\} \quad \text{for} \quad \varphi\colon A \to A/B.$$

Key step: Although there can be infinitely many B, at most finitely many are needed.

All fibers have local dimension $\geq \dim X - \dim \varphi(X)$. So

$$\dim \varphi(X) < \min\{\dim X, \dim A/B\} \implies X^{\mathrm{oa}} = \emptyset.$$

Example

If $X = C \times C \subset A \times A$ with $1 = \dim C < \dim A$, we take $\varphi\colon A \times A \to A$ to be the first projection. Then

$$\dim \varphi(X) = 1 < 2 = \min\{\dim X, \dim A\}.$$

This shows that $(C \times C)^{\mathrm{oa}} = \emptyset$.

Theorem

Let A be a polarized abelian variety over $\overline{\mathbb{Q}}$ and let $X \subset A$ be a subvariety.

- *(H.) The height $\hat{h}$ is bounded from above on*

$$X^{\mathrm{oa}}(\overline{\mathbb{Q}}) \cap \bigcup_{\substack{H \subset A \\ \dim H \leq \dim A - \dim X}} H(\mathbb{C}).$$

- *(H.–Pila) If $X^{\mathrm{oa}} \neq \emptyset$, then ZP holds for X.*

This theorem **does not** cover ZP for the surface $C \times C \subset A \times A$.

The first part uses the Ax–Schanuel function transcendence result. The second part uses the Pila–Zannier counting strategy.
The latter was implemented in the $\mathbb{G}_m^n$ case by Capuano, Masser, Pila, and Zannier

Laura Capuano

The modular side

The Zilber–Pink Conjecture was stated for

- Semiabelian varieties
- Mixed Shimura varieties (Pink, 2005)
- Variations of mixed Hodge structures (Klingler, 2017)

The Shimura variety of the Manin–Mumford Conjecture is called the André–Oort Conjecture.

Let us formulate one case of ZP in the (pure) Shimura setting.

Let $Y(1) = \mathbb{A}^1$ be the j-line, the j-invariant of elliptic curves induces a bijection

$$j\colon \{\text{Isomorphism class of elliptic curves}/\mathbb{C}\} \to Y(1)(\mathbb{C}) = \mathbb{C}.$$

Special subvarieties of $Y(1)^n$ play the role of the algebraic subgroups of $\mathbb{G}_m^n$.

After permuting coordinates, a special subvariety S of $Y(1)^n$ is a product $S_0 \times S_1 \times \cdots \times S_r$ with

- S_0 a singleton in $Y(1)^{n_0}$ consisting of a complex multiplication point
- each $S_i \subset Y(1)^{n_i}$, $i \geq 1$, is an irreducible component of

$$\Phi_{N_{i,1}}(X_2, X_1) = \cdots = \Phi_{N_{i,n_i-1}}(X_{n_i}, X_1) = 0$$

  where $\Phi_N \in \mathbb{Z}[x, y]$ are classical modular polynomials of level $N \in \mathbb{Z}_{\geq 1}$.

Modular interpretation:

$\Phi_N(j_1, j_2) = 0 \Longleftrightarrow$ the elliptic curves represented by the j-invariants j_1 and j_2 are linked by an isogeny whose kernel is cyclic of order N.

Theorem (Daw–Orr 2021)

Let $C \subset Y(1)^n$ be a curve defined over $\overline{\mathbb{Q}}$. Suppose

- *that C is not contained in a proper special subvariety of $Y(1)^n$*
- *that the Zariski closure of C in $(\mathbb{P}^1)^n$ intersects $(\infty, \ldots, \infty)$.*

Then

$$C(\overline{\mathbb{Q}}) \cap \bigcup_{\substack{S \subset Y(1)^n \text{ special} \\ \dim S \leq n-2}} S(\overline{\mathbb{Q}}) \quad \text{is finite.}$$

- Bounded height is false when unioning over $\dim S \leq n-1$.
- Daw and Orr use André's method involving G-functions to obtain a weak height upper bound.

The mixed side: Relative Manin–Mumford

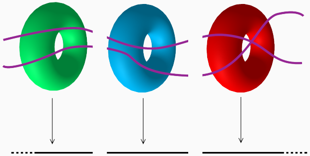
Consider a family $\mathcal{A}$ of abelian varieties parametrized by a smooth variety S.

Example (Legendre family of elliptic curves)

$$y^2 = x(x-1)(x-\lambda) \quad \text{where} \quad S = Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

A subvariety $X \subset \mathcal{A}$ is unlikely to meet the torsion multi-section $\ker[N]$ if

$$\dim X + \dim \ker[N] < \dim \mathcal{A} \iff \dim X < g = \dim \mathcal{A}/S.$$



The full torsion set is $\mathcal{A}_{\mathrm{tors}} = \bigcup_{N \in \mathbb{Z}} \ker[N]$.

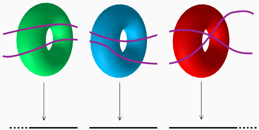
Theorem (Masser–Zannier + Corvaja (2008 - 2020))

Suppose $X \subset \mathcal{A}$ is a curve and $\bigcup_{N \in \mathbb{Z}} N(X)$ is Zariski dense in $\mathcal{A}$. If $g \geq 2$, then $X(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tors}}$ is finite.

Using 2-adic methods, Stoll obtained

Theorem (Stoll 2014)

For all $\lambda \in \mathbb{C} \setminus \{0,1\}$, the points $(2, \sqrt{2-\lambda})$ and $(3, \sqrt{6(3-\lambda)})$ are not simultaneously torsion in the Legendre family represented by $y^2 = x(x-1)(x-\lambda)$.



Where does a subvariety meet the union of all torsion sections $\bigcup_{N\in\mathbb{Z}} \ker[N]$?

Theorem (Corvaja–Tsimerman–Zannier 2023)

Relative Manin–Mumford Conjecture holds for many surfaces $X \subset \mathcal{A}$ defined over $\mathbb{C}$.

Theorem (Gao–H. 2023)

Suppose $X \subset \mathcal{A}$ such that $\bigcup_{N\in\mathbb{Z}} N(X)$ is Zariski dense in $\mathcal{A}$. If $g \geq 1 + \dim X$, then $X(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tors}}$ is not Zariski dense in X.

Some open problems

Question (Aaron Levin, related to Zilber–Pink for surfaces)

Let $C_1, C_2 \subset \mathbb{G}_m^n$ be curves that are not contained in a proper algebraic subgroup of $\mathbb{G}_m^n$. Suppose $n \geq 3$, what can we say about

$$\left\{P \in C_1(\overline{\mathbb{Q}}) : \text{ there exists } k \geq 1 \text{ with } P^k \in C_2(\overline{\mathbb{Q}})\right\}?$$

Some evidence by Bays–H. (2012).

Question

When are results of Zilber–Pink type uniform in geometric qualitites? When can we hope for effective results?

Evidence in work of Aslanyan, Bilu, Fowler, Kühne, Luca, Tron, ...

Thank you for your attenion.