Critical Points of Toroidal Belyĭ Maps

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Abstract

A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these values are $\{0, 1, \infty\}$. Replacing $\mathbb{P}^1$ with an elliptic curve $E : \; y^2 = x^3 + A\,x + B$, there is a similar definition of a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Since $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ is a torus, we call (E, β) a Toroidal Belyĭ pair.

There are many examples of Belyĭ maps $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ associated to elliptic curves; several can be found online at LMFDB. Given such a Toroidal Belyi map of degree N, the inverse image $G = \beta^{-1}\big(\{0, 1, \infty\}\big)$ is a set of N elements which contains the critical points of the Belyĭ map. In this project, we investigate when G is contained in $E(\mathbb{C})_{\text{tors}}$.

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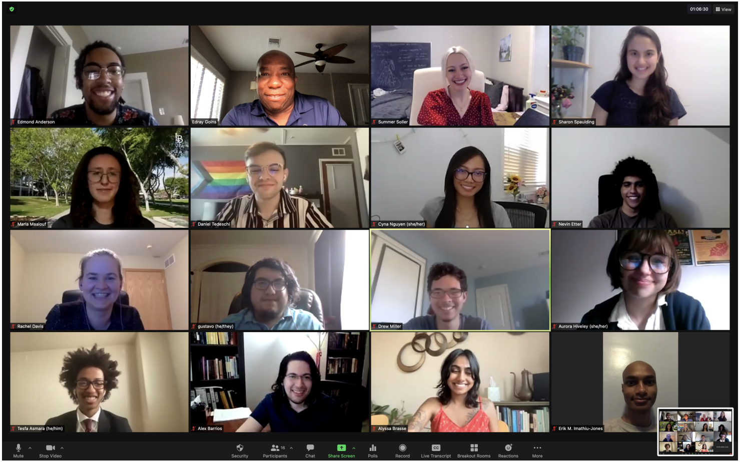
https://sites.google.com/view/vantageseminar



June 13 through August 7, 2021

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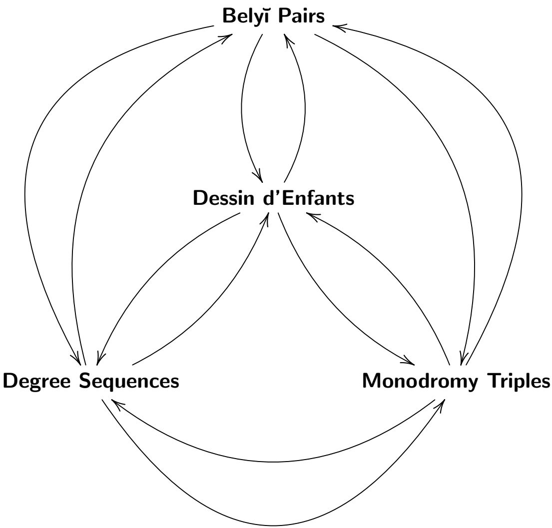
Review of Belyĭ Maps


Belyĭ Pairs
Dessin d'Enfants
Degree Sequences
Monodromy Triples

Belyĭ Pairs

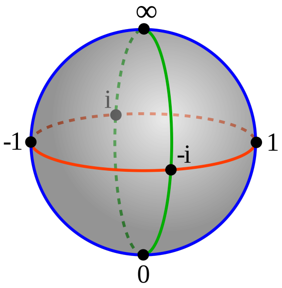
Theorem (André Weil, 1956; Gennadiĭ Vladimirovich Belyĭ, 1979)

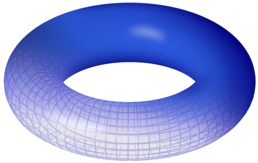
Let S be a compact, connected Riemann surface.

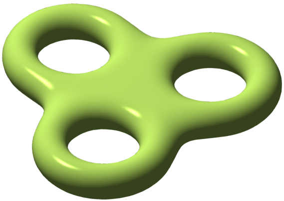
- *S is a smooth, irreducible, projective variety of dimension 1. In particular, S is an algebraic variety; that is, it can be defined by polynomial equations.*
- *If S can be defined by a polynomial equation $f(x,y) = \sum_{i,j} a_{ij}\, x^i\, y^j = 0$ where the coefficients a_{ij} are not transcendental, then there exists a rational function $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values.*
- *Conversely, if there exists rational function $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values, then S can be defined by a polynomial equation $f(x,y) = 0$ where the coefficients a_{ij} are not transcendental.*

Definition

A **Belyĭ pair** (S, β) is a compact, connected Riemann surface S and a rational function $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values $\{0,\, 1,\, \infty\}$.







https://en.wikipedia.org/wiki/Riemann_surface

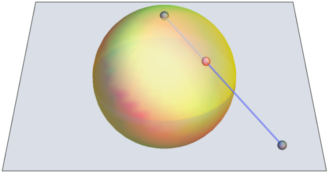
Theorem

The complex plane $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ *is the same as the unit sphere*

$$S^2(\mathbb{R}) = \left\{ (u, v, w) \in \mathbb{R}^3 \,\middle|\, u^2 + v^2 + w^2 = 1 \right\}.$$

Any graph with genus $g = 0$ *can also be drawn on the sphere without crossings.*

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \longrightarrow & S^2(\mathbb{R}) \\ z = \dfrac{u + i\,v}{1 - w} = \dfrac{1 + w}{u - i\,v} & \mapsto & (u, v, w) = \left(\dfrac{2\,\mathrm{Re}\,z}{|z|^2 + 1}, \dfrac{2\,\mathrm{Im}\,z}{|z|^2 + 1}, \dfrac{|z|^2 - 1}{|z|^2 + 1} \right) \\ 0, \quad 1, \quad \infty & \mapsto & (0, 0, -1), \quad (1, 0, 0), \quad (0, 0, 1) \end{array}$$



Theorem

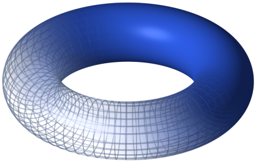
Relative to a lattice Λ, the quotient space $\mathbb{C}/\Lambda$ is the same as the torus

$$\mathbb{T}^2(\mathbb{R}) = \left\{ (u, v, w) \in \mathbb{R}^3 \;\middle|\; \left(\sqrt{u^2+v^2} - R\right)^2 + w^2 = r^2 \right\}.$$

Any graph with genus $g = 1$ can also be drawn on the torus without crossings.

$$\mathbb{C}/\Lambda \simeq (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \quad \longrightarrow \quad \mathbb{T}^2(\mathbb{R})$$

$$\theta = \arctan \frac{v}{u}, \qquad \phi = \arctan \frac{w}{\sqrt{u^2+v^2} - R} \quad \mapsto \quad \begin{aligned} u &= \left(R + r\,\cos\theta\right)\,\cos\phi \\ v &= \left(R + r\,\cos\theta\right)\,\sin\phi \\ w &= r\,\sin\theta \end{aligned}$$



Definition

Fix a Belyĭ pair $\beta : S \to \mathbb{P}^1(\mathbb{C})$, and denote the preimages

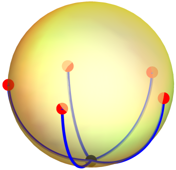
$$\begin{array}{ccccc} B = \beta^{-1}\big(\{0\}\big) & W = \beta^{-1}\big(\{1\}\big) & \beta^{-1}\big([0,1]\big) & S \xrightarrow{\beta} \mathbb{P}^1(\mathbb{C}) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \big\{\text{black vertices}\big\} & \big\{\text{red vertices}\big\} & \big\{\text{edges}\big\} & \mathbb{R}^3 \end{array}$$

The bipartite graph $\beta^{-1}\big([0,1]\big) \subseteq S \hookrightarrow \mathbb{R}^3$ is called **Dessin d'Enfant**.

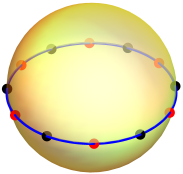
I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a *dessin* we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.

– Alexander Grothendieck, *Esquisse d'un Programme* (1984)

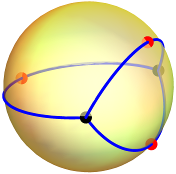
Example: Riemann Sphere



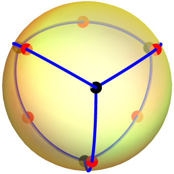
$$\beta(z) = z^n$$



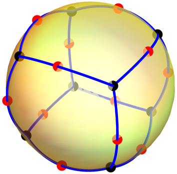
$$\beta(z) = \frac{(z^n + 1)^2}{4\, z^n}$$



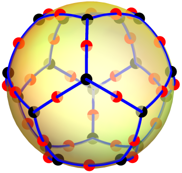
$$\beta(z) = \frac{4(z^2 - z + 1)^3}{27\, z^2\, (z - 1)^2}$$



$$\beta(z) = \frac{(z^4 + 2\sqrt{2}\, z)^3}{(2\sqrt{2}\, z^3 - 1)^3}$$



$$\beta(z) = \frac{(z^8 + 14\, z^4 + 1)^3}{108\, z^4\, (z^4 - 1)^4}$$



$$\beta(z) = \frac{(z^{20} + 228\, z^{15} + 494\, z^{10} - 228\, z^5 + 1)^3}{1728\, z^5\, (z^{10} - 11\, z^5 - 1)^5}$$

Definition

Fix a Belyĭ pair (S, β).

- Since $\{0,\, 1,\, \infty\}$ are the critical values for β, denote the **branch points** as the preimages $B = \beta^{-1}\big(\{0\}\big)$, $W = \beta^{-1}\big(\{1\}\big)$, and $F = \beta^{-1}\big(\{\infty\}\big)$.
- Let e_P denote the **ramification index** of a branch point $P \in B \cup W \cup F$.
- The **degree sequence** is a multiset

$$\mathcal{D} = \Big\{ \big\{ e_P \,\big|\, P \in B \big\},\ \big\{ e_P \,\big|\, P \in W \big\},\ \big\{ e_P \,\big|\, P \in F \big\} \Big\}$$

Proposition

- Let $\beta^{-1}\big([0,1]\big) \subseteq S \hookrightarrow \mathbb{R}^3$ be the Dessin d'Enfant. The ramification index e_P is the number of vertices adjacent to vertices $P \in B \cup W$.
- $\deg \beta = \displaystyle\sum_{P \in B} e_P = \sum_{P \in W} e_P = \sum_{P \in F} e_P = |B| + |W| + |F| + \big(2\,g(S) - 2\big)$.
- The genus $g(S)$ of S can be determined from $\beta : S \to \mathbb{P}^1(\mathbb{C})$.

Theorem (Adolf Hurwitz, 1891)

Fix a positive integer N, and say that we have a multiset

$$\mathcal{D} = \Big\{ \{e_P \,|\, P \in B\},\ \{e_P \,|\, P \in W\},\ \{e_P \,|\, P \in F\} \Big\}$$

of three partitions of N for some indexing sets B, W, and F such that

$$N = |B| + |W| + |F| + (2\,g - 2).$$

Then $\mathcal{D}$ is the degree sequence for some Belyĭ pair (S, β) with $g(S) = g$ and $\deg \beta = N$ if and only if there exist $\sigma_0,\ \sigma_1,\ \sigma_\infty \in Sym(N)$ such that

- *Each permutation $\sigma_0,\ \sigma_1,\ \sigma_\infty$ is a product of disjoint cycles with cycle type $\{e_P \,|\, P \in B\}$, $\{e_P \,|\, P \in W\}$, and $\{e_P \,|\, P \in F\}$, respectively.*
- *The composition $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$ is the trivial permutation.*
- *$G = \langle \sigma_0,\ \sigma_1,\ \sigma_\infty \rangle$ is transitive subgroup of $Sym(N)$.*

Definition

$G = \langle \sigma_0,\ \sigma_1,\ \sigma_\infty \rangle$ is called a **monodromy group** associated to $\mathcal{D}$.

Examples

- Such a group G may not be unique!

The degree sequence $\mathcal{D} = \big\{\{1,\ 4\},\ \{1,\ 4\},\ \{5\}\big\}$ for $N = 5$ corresponds to

$$\begin{aligned} \sigma_0 &= (2)\ (1\ 3\ 5\ 4) & \qquad \sigma_0 &= (3)\ (1\ 2\ 5\ 4) \\ \sigma_1 &= (4)\ (1\ 3\ 5\ 2) & \qquad \sigma_1 &= (5)\ (1\ 2\ 4\ 3) \\ \sigma_\infty &= (1\ 2\ 3\ 4\ 5) & \qquad \sigma_\infty &= (1\ 2\ 3\ 4\ 5) \\ \Longrightarrow \quad & G \simeq S_5 & \qquad \Longrightarrow \quad & G \simeq F_{20} \simeq Z_5 \rtimes Z_4 \end{aligned}$$

In particular, there are at least two Belyĭ pairs (S, β) associated with this degree sequence.

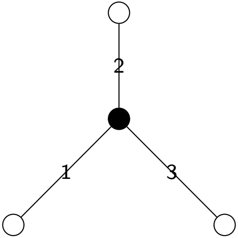
- Such a group may not exist!

The degree sequence $\mathcal{D} = \big\{\{1,\ 1,\ 2,\ 2\}, \{6\}, \{6\}\big\}$ for $N = 6$ has no such group. In particular, there are no Belyĭ pairs (S, β) associated with this degree sequence.

$$\sigma_0 = (1\ 2)$$
$$\sigma_1 = (1)\,(2)$$
$$\sigma_\infty = (1\ 2)$$

https://beta.lmfdb.org/Belyi/2T1/2/2/1.1/a

$$\sigma_0 = (1\ 3\ 2)$$
$$\sigma_1 = (1)\,(2)\,(3)$$
$$\sigma_\infty = (1\ 2\ 3)$$



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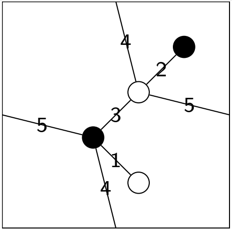
$$\sigma_0 = (2)\ (1\ 3\ 5\ 4)$$
$$\sigma_1 = (1)\,(2\ 4\ 3\ 5)$$
$$\sigma_\infty = (1\ 2\ 5\ 4\ 3)$$



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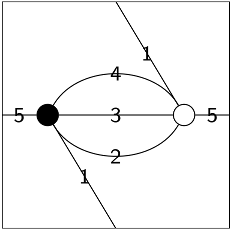
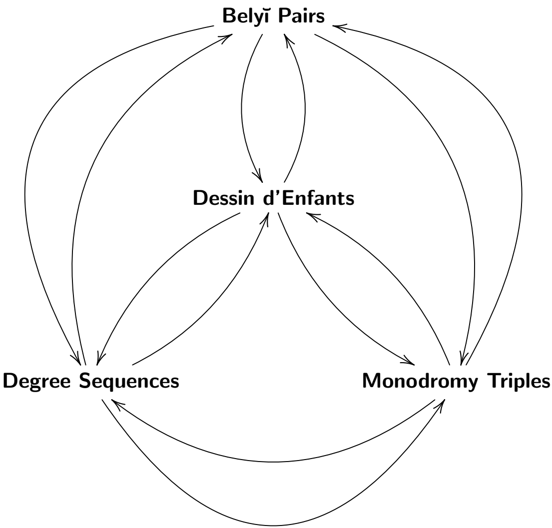
$$\sigma_0 = (1\ 2\ 3\ 4\ 5)$$
$$\sigma_1 = (1\ 4\ 3\ 2\ 5)$$
$$\sigma_\infty = (2)\,(3)\,(1\ 4\ 5)$$



https://beta.lmfdb.org/Belyi/5T4/5/5/3.1.1/a
or
https://beta.lmfdb.org/Belyi/5T4/5/5/3.1.1/b?


Belyĭ Pairs
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What about the torus?

Consider an equation in the form

$$y^2 + a_1\,x\,y + a_3\,y = x^3 + a_2\,x^2 + a_4\,x + a_6$$

where $a_1,\ a_2,\ a_3,\ a_4,\ a_6 \in \mathbb{C}$. We can make a substitution to find $Y^2 = X^3 + A\,X + B$.

Definition

We call the expression $E : Y^2 = X^3 + A\,X + B$ an **elliptic curve** if its **discriminant** $\Delta(E) = -16\,(4\,A^3 + 27\,B^2)$ is nonzero.

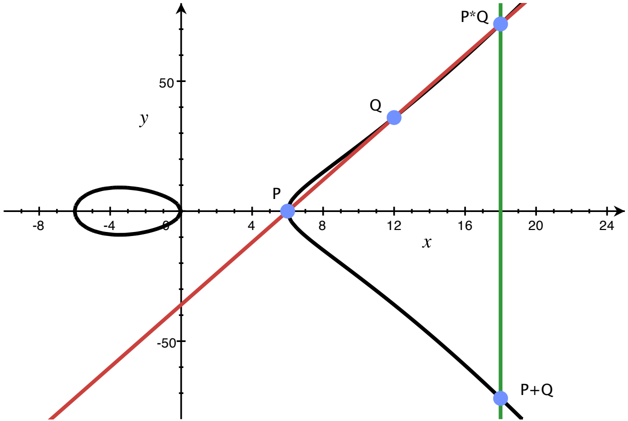
Proposition

- There is a well-defined tangent line $Y = \lambda\,X + \nu$ at every point $P = (X_0, Y_0)$ on the curve $Y^2 = X^3 + A\,X + B$ if and only if the quantity $4\,A^3 + 27\,B^2 \neq 0$.
- Given an elliptic curve $E : y^2 + a_1\,x\,y + a_3\,y = x^3 + a_2\,x^2 + a_4\,x + a_6$, the collection $S = E(\mathbb{C}) \simeq (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \simeq \mathbb{T}^2(\mathbb{R})$ of complex points is a **torus**.
- If $P * Q$ denotes the third point of intersection of a line through P and Q, then $P \oplus Q = \big(P * Q\big) * O_E$ turns $\big(E(\mathbb{C}),\ \oplus\big)$ into an abelian group.

Example: $y^2 = x^3 - 36\,x$

Consider the two rational points

$$P = (6, 0) \qquad \text{and} \qquad Q = (12, 36).$$



$$P \oplus Q = (18, -72)$$

Question

Say $E(\mathbb{C}) = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\} \cup \{O_E\}$ in terms of

$$f(x,y) = \left(y^2 + a_1\,x\,y + a_3\,y\right) - \left(x^3 + a_2\,x^2 + a_4\,x + a_6\right).$$

How do we check whether $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a Toroidal Belyĭ map?

#1. Compute the critical points as the set

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2 \;\middle|\; f(x,y) = 0, \quad \frac{\partial \beta}{\partial x}\,\frac{\partial f}{\partial y} - \frac{\partial \beta}{\partial y}\,\frac{\partial f}{\partial x} = 0 \right\}.$$

#2. Compute the critical values as the set

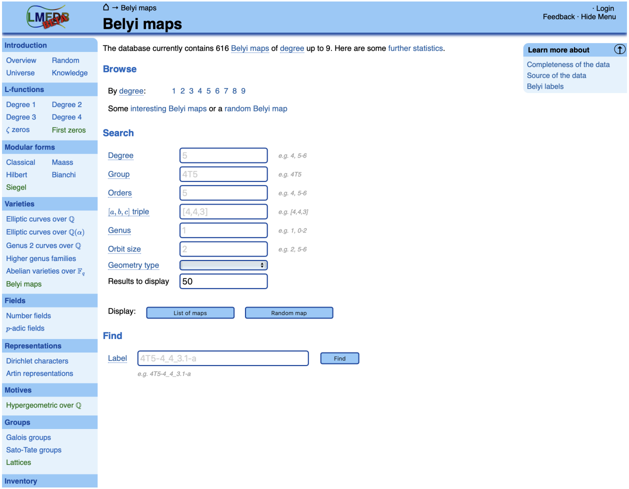
$$\left\{ q \in \mathbb{P}^1(\mathbb{C}) \;\middle|\; q = \beta(P) \text{ for some critical point } P = \begin{bmatrix} x \\ y \end{bmatrix} \right\}$$

#3. If there are just three critical values q_0, q_1, and q_∞, then compose

$$\begin{array}{ccccc} S = E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R}) & \longrightarrow & \mathbb{P}^1(\mathbb{C}) & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \\ P = \begin{bmatrix} x \\ y \end{bmatrix} & \longmapsto & q = \beta(P) & \longmapsto & z = \dfrac{q_1 - q_\infty}{q_1 - q_0}\,\dfrac{q - q_0}{q - q_\infty} \end{array}$$

Definition

A **Toroidal Belyĭ Pair** (S, β) consists of $S = E(\mathbb{C}) \simeq (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \simeq \mathbb{T}^2(\mathbb{R})$ associated to an elliptic curve E and a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$.



https://beta.lmfdb.org/Belyi/

Is there anything special about $P \in \beta^{-1}\big(\{0, 1, \infty\}\big)$?

Main Research Question

Denote $G = \beta^{-1}\big(\{0,\, 1,\, \infty\}\big)$. Under what conditions is $(G, \oplus)$ a group under the Group Law?

Example #1

Consider the Toroidal Belyĭ map $\beta : (x, y) \mapsto (y+1)/2$ for the elliptic curve $E : y^2 = x^3 + 1$. This map has degree $N = 3$, sequence $\mathcal{D} = \big\{\{3\},\, \{3\},\, \{3\}\big\}$. We have the critical points $G = \big\{(0, -1),\ (0, +1),\ O_E\big\} \simeq \mathbb{Z}/3\mathbb{Z}$.

Example #2

Consider the Toroidal Belyĭ map $\beta : (x, y) \mapsto x^2$ for the elliptic curve $E : y^2 = x^3 - x$. This map has degree $N = 4$, sequence $\mathcal{D} = \big\{\{4\},\, \{2,\, 2\},\, \{4\}\big\}$. We have the critical points $G = \big\{(-1, 0),\ (0, 0),\ (+1, 0),\ O_E\big\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Example #3

Consider the Toroidal Belyĭ map $\beta : (x, y) \mapsto (4\,y + x^2 + 56)/108$ for the elliptic curve $E : y^2 = x^3 + x^2 + 16\,x + 180$. This map has degree $N = 4$, sequence $\mathcal{D} = \big\{\{4\},\, \{1,\, 3\},\, \{4\}\big\}$. We have the critical points

$$
\begin{aligned}
G &= \big\{(4, -18),\ (-2, 12),\ (22, -108),\ O_E\big\} \\
&\subseteq \big\{(-5, 0),\ (4, \pm 18),\ (-2, \pm 12),\ (22, \pm 108),\ O_E\big\} \simeq \mathbb{Z}/8\mathbb{Z}.
\end{aligned}
$$

Known Toroidal Belyĭ Pairs with Torsion Critical Points (as of 7/22/2021)

LMFDB Label	Elliptic Curve X	Belyĭ Map ϕ	Group $\langle G \rangle \subseteq X(\mathbb{C})_{\text{tors}}$
3T1-3_3_3-a	$y^2 = x^3 + 1$	$\frac{1-y}{2}$	Z_3
4T1-4_4_2.2-a	$y^2 = x^3 - x$	$1 - x^2$	$Z_2 \times Z_2$
4T5-4_4_3.1-a	$y^2 = x^3 + x^2 + 16x + 180$	$\frac{4y + x^2 + 56}{108}$	Z_8
5T4-5_5_3.1.1-a	$y^2 + xy = x^3 - 28x + 272$	$\frac{(x+13)y + 3x^2 + 4x + 220}{432}$	$Z_2 \times Z_{10}$
6T1-6_2.2.2_3.3-a	$y^2 = x^3 + 1$	$-x^3$	$Z_2 \times Z_6$
6T4-3.3_3.3_3.3-a	$y^2 = x^3 - 15x + 22$	$\frac{8(x-2)^2 - (x^2 - 4x + 7)y}{16(x-2)^2}$	Z_6
6T5-6_6_3.1.1.1-a	$y^2 = x^3 + 1$	$\frac{(1-y)(3+y)}{4}$	$Z_2 \times Z_6$
6T6-6_6_2.2.1.1-a	$y^2 = x^3 + 6x - 7$	$\frac{(x-1)^3}{27}$	$Z_2 \times Z_4$
6T7-4.2_4.2_3.3-a	$y^2 = x^3 - 10731x + 408170$	$\frac{11907(x-49)}{(x-7)^3}$	$Z_2 \times Z_4$
6T12-5.1_5.1_3.3-b	$y^2 + xy + y = x^3 + x^2 - 10x - 10$	$27\,\frac{(x+4)(2x^2 - 2x - 13) - (x+1)^2 y}{(x^2 - x - 11)^3}$	$Z_2 \times Z_8$
6T12-5.1_5.1_5.1-a	$y^2 = x^3 + x^2 + 4x + 4$	$-16\,\frac{(x^2 - 2x - 4)y + 8(x+1)}{(x-4)x^5}$	Z_6
8T2-4.4_4.4_2.2.2.2-a	$y^2 = x^3 + x$	$\frac{(x+1)^4}{8x(x^2+1)}$	$Z_2 \times Z_4$
8T7-8_8_2.2.1.1.1.1-a	$y^2 = x^3 - x$	x^4	$Z_2 \times Z_4$

Imprimitive Pairs with Torsion Critical Points (as of 7/22/2021)

LMFDB Label	Elliptic Curve E	Belyĭ Map β	$\langle G \rangle \subseteq X(\mathbb{C})_{\text{tors}}$
4T1-4_4_2.2-a	$y^2 = x^3 - x$	$1 - x^2$	$Z_2 \times Z_2$
6T1-6_2.2.2_3.3-a	$y^2 = x^3 + 1$	$-x^3$	$Z_2 \times Z_6$
6T4-3.3_3.3_3.3-a	$y^2 = x^3 - 15\,x + 22$	$\dfrac{8\,(x-2)^2 - (x^2 - 4\,x + 7)\,y}{16\,(x-2)^2}$	Z_6
6T5-6_6_3.1.1.1-a	$y^2 = x^3 + 1$	$\dfrac{(1-y)\,(3+y)}{4}$	$Z_2 \times Z_6$
6T6-6_6_2.2.1.1-a	$y^2 = x^3 + 6\,x - 7$	$\dfrac{(x-1)^3}{27}$	$Z_2 \times Z_4$
6T7-4.2_4.2_3.3-a	$y^2 = x^3 - 10731\,x + 408170$	$\dfrac{11907\,(x-49)}{(x-7)^3}$	$Z_2 \times Z_4$
8T2-4.4_4.4_2.2.2.2-a	$y^2 = x^3 + x$	$\dfrac{(x+1)^4}{8\,x\,(x^2+1)}$	$Z_2 \times Z_4$
8T7-8_8_2.2.1.1.1.1-a	$y^2 = x^3 - x$	x^4	$Z_2 \times Z_4$

Imprimitive Pairs with Torsion Critical Points (as of 7/22/2021)

LMFDB Label	Elliptic Curve E	Toroidal Belyĭ $\beta(x, y)$	Belyĭ $\gamma(z)$	Meromorphic $\phi(x, y)$
4T1-4_4_2.2-a	$y^2 = x^3 - x$	$1 - x^2$	$4\, z\,(1 - z)$	$\frac{x + 1}{2}$
6T1-6_2.2.2_3.3-a	$y^2 = x^3 + 1$	$-x^3$	$4\, z\,(1 - z)$	$\frac{1 - y}{2}$
6T5-6_6_3.1.1.1-a	$y^2 = x^3 + 1$	$\frac{(1 - y)\,(3 + y)}{4}$	$4\, z\,(1 - z)$	$\frac{3 + y}{4}$
6T6-6_6_2.2.1.1-a	$y^2 = x^3 + 6\,x - 7$	$\frac{(x - 1)^3}{27}$	z^3	$\frac{x - 1}{3}$
6T7-4.2_4.2_3.3-a	$y^2 = x^3 - 10731\,x + 408170$	$\frac{11907\,(x - 49)}{(x - 7)^3}$	$z^2\,(3 - 2\,z)$	$\frac{63}{x - 7}$
8T2-4.4_4.4_2.2.2.2-a	$y^2 = x^3 + x$	$\frac{(x + 1)^4}{8\,x\,(x^2 + 1)}$	$\frac{(z^2 + 1)^2}{4\,z^2}$	$\frac{\sqrt{2}\,x}{y}$
8T7-8_8_2.2.1.1.1.1-a	$y^2 = x^3 - x$	x^4	z^4	x

LMFDB Label	Elliptic Curve E	Toroidal Belyĭ $\beta(x, y)$	Belyĭ $\phi(x, y)$	Isogeny $\psi(x, y)$
6T4-3.3_3.3_3.3-a	$y^2 = x^3 - 15\,x + 22$	$\frac{8\,(x - 2)^2 - (x^2 - 4\,x + 7)\,y}{16\,(x - 2)^2}$	$\frac{1 - y}{2}$	$\left(\frac{x^2 - 2\,x - 3}{4\,(x - 2)}, \frac{(x^2 - 4\,x + 7)\,y}{8\,(x - 2)^2} \right)$

LMFDB Label	Elliptic Curve X	Belyĭ Map ϕ	$\langle G \rangle \subseteq X(\mathbb{C})_{\text{tors}}$
3T1-3_3_3-a	$y^2 = x^3 + 1$	$(1 - y)/2$	Z_3
6T12-5.1_5.1_5.1-a	$y^2 = x^3 + x^2 + 4x + 4$	$-16\left((x^2 - 2x - 4)\,y + 8\,(x+1)\right)/\left((x-4)\,x^5\right)$	Z_6
4T1-4_4_2.2-a	$y^2 = x^3 - x$	$1 - x^2$	$Z_2 \times Z_2$
6T4-3.3_3.3_3.3-a	$y^2 = x^3 - 15x + 22$	$\left(8\,(x-2)^2 - (x^2 - 4x + 7)\,y\right)/\left(16\,(x-2)^2\right)$	Z_6
8T2-4.4_4.4_2.2.2.2-a	$y^2 = x^3 + x$	$(x+1)^4/\left(8\,x\,(x^2+1)\right)$	$Z_2 \times Z_4$
8T7-8_8_2.2.1.1.1.1-a	$y^2 = x^3 - x$	x^4	$Z_2 \times Z_4$

Theorem (PRiME 2021)

Say (X, ϕ) is a Toroidal Belyĭ pair, and denote $G = \phi^{-1}\left(\{0,\, 1,\, \infty\}\right)$.

- $\beta = \phi \circ \psi$ yields a Toroidal Belyĭ pair (E, β) for any isogeny $\psi : E(\mathbb{C}) \to X(\mathbb{C})$.
- $\Gamma = \beta^{-1}\left(\{0,\, 1,\, \infty\}\right)$ is contained in $E(\mathbb{C})_{\text{tors}}$ whenever $G \subseteq X(\mathbb{C})_{\text{tors}}$.
- Γ is a subgroup of $\left(E(\mathbb{C}),\, \oplus\right)$ whenever G is a subgroup of $\left(X(\mathbb{C}),\, \oplus\right)$.
- There are infinitely many **imprimitive** (E, β) for which $\Gamma \subseteq E(\mathbb{C})_{\text{tors}}$ is a group.

$P \in E(\mathbb{C})$ is a critical point if and only if $\psi(P) \in X(\mathbb{C})$ is a critical point because $e_\psi(P) = 1$ and $e_\beta(P) = e_\phi\left(\psi(P)\right)$. A point $P \in \Gamma$ is torsion whenever $\psi(P) \in G$ is torsion. $\Gamma = \psi^{-1}(G)$ is a group whenever G is a group by the Subgroup Criteria.

Question

Are there infinitely many **primitive** Toroidal Belyĭ pairs (X, ϕ) for which G is a group?

Future Work / Acknowledgements

We would eventually like to do the following:

- Make graphing software freely available.
- Extend graphing capability to Riemann surfaces S with $g(S) \geq 2$.
- Upload all graphics and movies to LMFDB.

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Thank You! Questions?