Hyperelliptic Curves mapping to Abelian Surfaces and Applications to Beilinson's Conjecture for 0-cycles

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Section 1:

Let X be a **smooth projective** variety over an **algebraically closed field** $k = \overline{k}$.

Goals of this Talk

- Discuss the structure of the **Chow group of** 0**-cycles** $\mathrm{CH}_0(X)$ when $k = \mathbb{C}, \overline{\mathbb{Q}}, \overline{\mathbb{F}}_p$.
- Focus on fascinating conjectures over $\overline{\mathbb{Q}}$.

Picard Group of a Curve

Definition

Let C be a smooth projective curve of genus g over k.

$$\mathrm{Pic}(C) := \frac{\mathrm{Div}(C)}{\langle \mathrm{div}(f) : f \in k(C)^\times \rangle} = \frac{\bigoplus_{P \in C} \mathbb{Z} \cdot (P)}{\text{rational equivalences}}.$$

Recall: If $f : C \to \mathbb{P}^1_k$ is a rational function, then f induces a divisor

$$\mathrm{div}(f) := \sum_{P \in C} \mathrm{ord}_P(f)(P).$$

Example: Hyperelliptic Curves

A smooth projective curve H is called **hyperelliptic** if affine locally $H : y^2 = f(x)$, where $f(x) \in k[x], \deg(f(x)) \geq 5$.

- Every hyperelliptic curve H has an **involution**

$$\iota : H \to H, \quad P = (x, y) \mapsto \iota(P) = \overline{P} = (x, -y).$$

A point $W \in H(k)$ is called a **Weierstrass point** if $\overline{W} = W$.

- **Fundamental rational equivalence:** For every $P \in H$,

$$[P] + [\overline{P}] - 2[W] = 0 \in \mathrm{Pic}(C).$$

Structure of Pic(C)

Degree zero subgroup

$$\mathrm{Pic}^0(C) = \left\{ \sum_{P\in C} m_P[P] : \sum_{P\in C} m_P = 0 \right\}.$$

The Abel-Jacobi isomorphism

Let $P_0 \in C$ be a fixed rational point. $\rightsquigarrow$ There exists a closed embedding $\iota_{P_0} : C \hookrightarrow J_C$, $P_0 \mapsto 0$, to the **Jacobian** of C. $\rightsquigarrow$ It extends to a homomorphism (independent of choice of basepoint)

$$\alpha_C : \quad \mathrm{Pic}^0(C) \to J_C.$$

Theorem (Abel-Jacobi): The map α_C is an isomorphism.

0-cycles in higher dimensions

Let X be a smooth projective variety over $k = \overline{k}$.

Definition

- A 0-*cycle* on X is a formal sum

$$m_1(P_1) + \cdots + m_n(P_n),$$

  where $m_i \in \mathbb{Z}$ and P_i are points in X.
- Given a closed integral curve $C \hookrightarrow X$, and a rational function f on C, we can define a 0-cycle on X:

$$\mathrm{div}(f) := \sum_{P \in C} \mathrm{ord}_P(f)(P).$$

Chow group of 0-cycles

Definition

A 0-cycle is a rational equivalence (or rationally equivalent to 0) if it can be written as a linear combination of divisors of rational functions on curves in X.
i.e. $z = \sum_i \mathrm{div}(f_i)$, for some $f_i \in k(C_i)^\times$, $C_i \hookrightarrow X$ closed curves.

Definition

The Chow group of 0-cycles

$$\mathrm{CH}_0(X) := \bigoplus_{P \in X} \mathbb{Z} \cdot (P)/(\text{rational equivalences}).$$

Note: When X is a curve, $\mathrm{CH}_0(X) = \mathrm{Pic}(X)$.

Main Question

Question

Given a 0-cycle z on X, determine whether it is a rational equivalence or not.

Issue

- When $\dim(X) \geq 2$ we get contributions from many different curves inside X: $z = 0$ if and only if $z = \sum_i \operatorname{div}(f_i)$, for some $f_i \in k(C_i)^\times$, $C_i \hookrightarrow X$ closed curves possibly not connected to each other. $\rightsquigarrow$ we can't always reduce this to a single divisor.
- Generally very hard to describe explicit relations in $\mathrm{CH}_0(X)$.

What we know about CH_0

- There exists a **degree map**

$$\deg : \mathrm{CH}_0(X) \twoheadrightarrow \mathbb{Z}, \sum_{x \in X} n_x[x] \mapsto \sum_{x \in X} n_x.$$

Define $F^1(X) = \ker(\deg) = \langle [x] - [y] : x, y \in X \rangle$.

- There exists **Abel-Jacobi map**

$$\alpha_X : F^1(X) \to \mathrm{Alb}_X,$$

where $\mathrm{Alb}_X =$ higher dimensional analog of the Jacobian $=$ Abelian Variety universal for morphisms from X to abelian varieties.

- **Problem:** When $\dim(X) > 1$, α_X is surjective, but often **NOT injective**. Define $F^2(X) = \ker(\alpha_X)$

$$\mathrm{CH}_0(X) \supset F^1(X) \supset F^2(X) \supset 0.$$

What about F^2?

Rojtman's Theorem ('80)

When X/k and k is algebraically closed then $F^2(X)$ is torsion-free.

In higher dimensions $F^2(X)$ often non-zero. Its structure depends heavily on the variety X

First Examples

Mumford ('68) constructed surfaces over $\mathbb{C}$ with enormous $F^2(X)$. In particular: $F^2(X)$ NOT parametrized by the points of an algebraic variety.

Key points:

- **positive geometric genus**: $p_g(X) = \dim_{\mathbb{C}}(\Gamma(X, K_X)) > 0$.
- $\mathbb{C}$ is transcendental.

When $k = \mathbb{C}$

Bloch ('75): For $X/\mathbb{C}$ of arbitrary dimension $d > 1$
$p_g(X) > 0 \Rightarrow F^2(X)$ is huge.

When $k = \overline{\mathbb{F}}_p$

Milne: $F^2(X) = 0$ unconditionally on the variety.

Slogan

Large field and $p_g(X) > 0 \Rightarrow$ large $F^2(X)$
Small field $\Rightarrow$ small $F^2(X)$.

Question

What about number fields?

Section 2:

The field $\overline{\mathbb{Q}}$

The field $\overline{\mathbb{Q}}$ of **algebraic** numbers is countable, algebraic and of characteristic 0.

Beilinson's Conjecture (mid 80's)

For X smooth projective variety over $\overline{\mathbb{Q}}$, $F^2(X) = 0$ unconditionally on $p_g(X)$.

Evidence?

CURVES!

Many examples with $p_g(X) = 0$.

NO known examples in $\dim(X) > 1$ with $p_g(X) > 0$!

Example 1: $K3$ surfaces

Definition

A $K3$ surface over $k = \overline{k}$ is a smooth projective surface such that:

- X has trivial canonical bundle
- X has Albanese variety $\mathrm{Alb}_X = 0$.

Example: $X = \{(x : y : z : w) \in \mathbb{P}^3_k : x^4 + y^4 + z^4 + w^4 = 0\}$, the Fermat quartic.

Analyzing Beilinson's Conjecture for $K3$'s

$\mathrm{Alb}_X = 0 \Rightarrow F^1(X) = F^2(X) = \langle [x] - [y] : x, y \in X \rangle$.

Thus: Beilinson's Conjecture for $X/\overline{\mathbb{Q}} \Leftrightarrow$ any two $\overline{\mathbb{Q}}$-points x, y are rationally equivalent.

CAUTION!!! $p_g(X) > 0 \Rightarrow$ this is very far from true for two general $\mathbb{C}$-points!

Theorem (Beauville-Voisin '04)

Let X be a $K3$ surface over $k = \overline{k}$. Any two points x, y that lie on some (possibly different) **rational curve** inside X are rationally equivalent.
i.e.: If there exist non-constant morphisms $f : \mathbb{P}^1 \to X$ and $g : \mathbb{P}^1 \to X$ such that $x \in \mathrm{Im}(f)$, $y \in \mathrm{Im}(g)$, then $[x] = [y]$.

Wishful Hope

Maybe $X(\overline{\mathbb{Q}})$ can be covered by rational curves.
Conjecture: (Bogomolov '81) predicts exactly this!
Nowadays: This might be too strong.

Conjecture (Bogomolov/Hassett/Tshinkel 2010)

Every $K3$ surface $X/\overline{k}$ contains infinitely many rational curves.
Evidence: Known for many classes of $K3$'s over $\mathbb{C}$, some over $\overline{\mathbb{Q}}$.

Example 2: Abelian Surfaces

Abelian Surfaces

Let A be an abelian surface over $k = \overline{k}$ with zero element 0.

- **Fact 1:** $p_g(A) > 0$.
- **Fact 2:** $\mathrm{Alb}_A = A$ and the Abel-Jacobi map is:

$$\alpha_A : \quad F^1(A) \to A$$
$$\sum_{x \in A} n_x[x] \mapsto \sum n_x x$$

Lemma: The kernel $F^2(A)$ is generated by 0-cycles of the form:

$$z_{a,b} := [a+b] - [a] - [b] + [0], \text{ with } a, b \in A.$$

Analyzing Beilinson's Conjecture for Abelian Surfaces

Beilinson's Conjecture for $A/\overline{\mathbb{Q}} \Leftrightarrow z_{a,b} = 0$, for all $\overline{\mathbb{Q}}$-points $a, b \in A$.

CAUTION!!! $p_g(A) > 0 \Rightarrow$ this is very far from true for two general $\mathbb{C}$-points! $\rightsquigarrow$ extremely hard to construct examples.

Summary

To attack Beilinson's Conjecture for a smooth projective variety $X/\overline{\mathbb{Q}}$ we need:

- To find many "special" curves $C \hookrightarrow X$ defined over $\overline{\mathbb{Q}}$ that produce many rational equivalences.
- To use the special properties of $\overline{\mathbb{Q}}$ (**algebraicity**) that distinguish it from $\mathbb{C}$ in an essential manner.

Special Curves

- For $K3$ surfaces: Beauville-Voisin result suggests that maybe **rational curves** are enough.
- Analog for abelian surfaces????
  Remark: Abelian surfaces don't contain any rational curves: Any morphism $\mathbb{P}^1 \to A$ factors through $J_{\mathbb{P}^1} = 0$.
  Our Idea: Replace rational curves with hyperelliptic curves.

Example: A product of two Elliptic Curves

Finding generators

Let $A = E_1 \times E_2$ over $\overline{\mathbb{Q}}$ with zero element $O = (0,0)$. Then:

- $F^1(A) = \langle [p,q] - [0,0] : p \in E_1, q \in E_2 \rangle$.
- $F^2(A)$ is generated by the fewer elements $[p,q] - [p,0] - [0,q] + [0,0] = z_{(p,0),(0,q)}$.
- In fact, there is a surjection

$$\varepsilon : E_1(\overline{\mathbb{Q}}) \otimes E_2(\overline{\mathbb{Q}}) \twoheadrightarrow F^2(A)$$
$$p \otimes q \mapsto [p,q] - [p,0] - [0,q] + [0,0]$$

The Bad News

The group $E_1(\overline{\mathbb{Q}}) \otimes E_2(\overline{\mathbb{Q}})$ is very large.
Mordell-Weil Theorem:
$E_1(L) \otimes E_2(L) \simeq \mathbb{Z}^{r_1(L)} \otimes \mathbb{Z}^{r_2(L)} \oplus (\text{torsion})$. $\rightsquigarrow$ we need to kill the images of $E_1(L) \otimes E_2(L)$ with $E_i(L)$ of increasingly large rank.

The Good News

The surjection $\varepsilon : E_1(\overline{\mathbb{Q}}) \otimes E_2(\overline{\mathbb{Q}}) \twoheadrightarrow F^2(A)$ implies that the 0-cycle $z_{(p,0),(0,q)}$ is **bilinear** on p, q.
Take-away: Bilinearity + Mordell-Weil reduce the number of cancellations we need!

A weaker Question

Suppose E_1, E_2 are defined over $\mathbb{Q}$ and $\mathrm{rk}(E_1(\mathbb{Q})) = 1$, $\mathrm{rk}(E_2(\mathbb{Q})) = 1$. Let $A = E_1 \times E_2$. What do we need in order to show that $z_{a,b} = 0$ for all $a, b \in A(\mathbb{Q})$?

Lemma

Let $p \in E_1(\mathbb{Q}), q \in E_2(\mathbb{Q})$ be points of infinite order. Then

$$[p, q] - [p, 0] - [0, q] + [0, 0] = 0 \Rightarrow [a + b] - [a] - [b] + [0] = 0,$$

for all $a, b \in A(\mathbb{Q})$.

Note: The elements of $F^2(E_1 \times E_2)$ that are defined over $\mathbb{Q}$ are generated by $z_{(a,0),(0,b)} = [a, b] - [a, 0] - [0, b] + [0, 0]$ with $a \in E_1(\mathbb{Q}), b \in E_2(\mathbb{Q})$. $\rightsquigarrow$ Enough to show these vanish.

Proof of Lemma

Let $a \in E_1(\mathbb{Q}), b \in E_2(\mathbb{Q})$. $\mathrm{rk}(E_1(\mathbb{Q})) = 1$ and $p \in E_1(\mathbb{Q})$ has infinite order $\Rightarrow$ the points $a, p \in E_1(\mathbb{Q})$ are $\mathbb{Z}$-linearly dependent and the same is true for b, q. $\rightsquigarrow$ There exist $n, m, l, r \in \mathbb{Z}$ such that $na + mp = 0 = lb + rq$.
Bilinearity gives

$$z_{(na,0),(0,lb)} = nlz_{(a,0),(0,b)} = mrz_{(p,0),(0,q)} = 0.$$

Thus: $z_{(a,0),(0,b)}$ is a **torsion** element of $F^2(E_1 \times E_2)$.
Rojtman's Theorem: F^2 is torsion-free. $\Rightarrow z_{(a,0),(0,b)} = 0$.

Conclusions

- When $\mathrm{rk}(E_1(\mathbb{Q})) = \mathrm{rk}(E_2(\mathbb{Q})) = 1$, we only need **ONE** relation to be able to show $z_{a,b} = 0$ for all points $a, b \in A(\mathbb{Q})$.
- When E_1, E_2 are isogenous: easy.
- When E_1, E_2 non-isogenous: very **nontrivial**!

Section 3:

Abelian Surfaces

Definition

Let A be an abelian surface over $k = \overline{k}$. A point $x \in A$ is called **hyperelliptic** if some **nonzero multiple** of x lies in the image of a morphism $\phi : H \to A$, where H is a hyperelliptic curve over k such that **the hyperelliptic involution on H commutes with negation on A**, i.e. $\phi(\iota(p)) = -\phi(p)$, for all $p \in H$.

Rational Equivalences from Hyperelliptic Curves

Theorem 1 (G., Love '23)

Let $a, b \in A$. Suppose there exist nonzero integers m, n such that each of the points a, b, and $ma + nb$ is hyperelliptic. Then $z_{a,b} = 0$. In fact,

$$z_{c,d} := [c + d] - [c] - [d] + [0] = 0, \text{ for all } c, d \in B_{a,b},$$

where $B_{a,b}$ is the **divisible hull** of the subgroup $\langle a, b \rangle$,

$$B_{a,b} := \{x \in A : \exists N \neq 0 \text{ such that } N \cdot x \in \langle a, b \rangle\}.$$

Key points in the Proof

- **Pushing forward fundamental rational equivalences from hyperelliptics:**
  Let $\phi : H \to A$ with H hyperelliptic and $\phi(\iota(p)) = -\phi(p)$. Let $a = \phi(p)$.

$$[p] + [\iota(p)] - 2[w] = 0 \in \mathrm{CH}_0(H) \Rightarrow z_{a,-a} = 0 \in \mathrm{CH}_0(A).$$

- **Bilinearity:** The 0-cycle $z_{a,b}$ is bilinear on a, b.

Remark

The points $a, b, na + mb$ may lie in the images of morphisms from 3 distinct hyperelliptic curves $\rightsquigarrow$ Theorem 1 is an analog for abelian surfaces of the Beauville-Voisin result for $K3$'s.

Products of Elliptic Curves

For $A = E_1 \times E_2$ we have

$$F^2(A) = \langle [p,q] - [p,0] - [0,q] + [0,0],\ \ p \in E_1, q \in E_2 \rangle.$$

The points $(p, 0)$ and $(0, q)$ are always hyperelliptic!

Example

Suppose E_1, E_2 are defined over $\mathbb{Q}$ and $\mathrm{rk}(E_1(\mathbb{Q})) = 1$, $\mathrm{rk}(E_2(\mathbb{Q})) = 1$. To show that $z_{a,b} = 0$ for all $a, b \in A(\mathbb{Q})$ enough to find:

- $\phi = (\phi_1, \phi_2) : H \to A$ with H hyperelliptic and ϕ commuting with negation on A,
- a point $p \in H(\mathbb{Q})$ such that $\phi_1(p), \phi_2(p)$ both have infinite order.

More generally: For a number field $L/\mathbb{Q}$, showing $z_{a,b} = 0$ for all $a, b \in A(L)$ can be reduced to finding **finitely many hyperelliptic points** in $A(L)$. $\rightsquigarrow$ Theorem 1 has the potential of taking advantage of the Mordell-Weil Theorem (algebraicity of $\overline{\mathbb{Q}}$).

Conclusions

1. **Theorem 1** $\Rightarrow$ Hyperelliptic curves in abelian surfaces are **special curves** that produce many rational equivalences.
2. **Theorem 1** $\Rightarrow$ working with the divisible hull reduces the question of showing that every point in $A(L)$ is hyperelliptic to only finding finitely many hyperelliptic points in $A(L)$.
3. **Question:** Can we find any such curves?

Why hyperelliptic curves?

Goal

Look for special curves in an abelian surface that produce lots of rational equivalences.

Approach 1

Look for curves with extra symmetries.
Hyperelliptic curves have an involution ι such that $H/\iota \simeq \mathbb{P}^1$. $\rightsquigarrow$ this gives many easy rational equivalences.

Approach 2: Small genus

- Genus 0: Abelian surfaces don't contain any $g = 0$ curves.
- Genus 1 curves = Elliptic curves. $\rightsquigarrow$ Not many of these.
  Example: $A = E_1 \times E_2$ with E_1, E_2 **non-isogenous**. Then every $E \xrightarrow{f} E_1 \times E_2$ must be constant in one of the factors.
- Genus 2: All of them are hyperelliptic!

Producing many Hyperelliptic Curves

A $K3$ Surface associated to an Abelian Surface A

Let $X_0 := \dfrac{A}{\langle -1 \rangle}$ be the quotient of A by the negation involution. $\rightsquigarrow$ This is a singular $K3$ surface with 16 singularities corresponding to the 16 2-torsion points of A. $\rightsquigarrow$ It admits a $2:1$ map

$$\pi : A \to X_0.$$

By **blowing-up the singularities** we get a smooth projective $K3$ surface $X = \mathrm{Kum}(A)$ called **the Kummer surface** associated to A.

Pulling back rational curves from the Kummer surface

Expectation: X being a $K3$ should contain many rational curves $\rightsquigarrow$ these will pull back to hyperelliptic curves in A.

Theorem 2 (G.-Love '23)

Suppose A is **isogenous to a product of two elliptic curves**. Then for **infinitely many values** of $g \geq 2$, there exist **infinitely many** pairwise non-isomorphic genus g hyperelliptic curves H mapping birationally into A with the hyperelliptic involution on H commuting with the negation on A.

Remarks

1. For A isogenous to $E_1 \times E_2$ the hyperelliptic points are plentiful!
2. The birationality onto their image guarantees that any new curve we produce gives genuinely new rational equivalences.

Kummer Surface as an Elliptic Fibration

Elliptic Curves in Leibniz form

Let E_1, E_2 elliptic curves over $\overline{\mathbb{Q}}$ and $A = E_1 \times E_2$. $\rightsquigarrow$

$$E_1 : y_1^2 = f(x_1) = x_1(x_1 - 1)(x_1 - \lambda)$$
$$E_2 : y_2^2 = f(x_2) = x_2(x_2 - 1)(x_2 - \mu)$$

The Kummer surface

$X = \mathrm{Kum}(A)$ has an affine chart
$U = \{(x_1, x_2, r) \in \overline{\mathbb{Q}}^3 : f(x_1)r^2 = f(x_2)\}$

$$\pi : \quad E_1 \times E_2 \dashrightarrow U$$
$$(x_1, y_1, x_2, y_2) \mapsto \left(x_1, x_2, \frac{y_1}{y_2}\right)$$

Inose's Pencil

- The equation $f(x_1)r^2 = f(x_2)$ becomes an elliptic curve $\mathcal{E}$ over the function field $\overline{\mathbb{Q}}(r)$ by taking $(0,0)$ as the point at infinity.
- Formally: The map $U \to \mathbb{A}^1$, $(x_1, x_2, r) \mapsto r$ gives $X = \mathrm{Kum}(A)$ the structure of an **elliptic fibration** known as Inose's pencil.
- The Mordell-Weil group of $\mathcal{E}$ has rank at least 4.
- Every $\mathbb{Z}$-linear combination of the 4 generators gives a section $\mathbb{P}^1 \to X$, which pulls-back to a hyperelliptic $H \to A$.
- This process produces hyperelliptics of larger and larger genus.

- To see repetition of the genus: we repeat the process for any isogenous pair $E_1' \times E_2'$. $\rightsquigarrow$ Over $\overline{\mathbb{Q}}$ we get plenty of such pairs.
- Heuristically: we expect all genera $g \equiv 2 \mod 4$ to appear infinitely often.
- The curves we produce can be made very **explicit**. We can write down Weierstrass equations for them and compute the maps $H \to E_1 \times E_2$.

Examples

Suppose we have Weierstrass equations

$$E_1 : y_1^2 = x_1(x_1 - 1)(x_1 - a), \qquad E_2 : y_2^2 = x_2(x_2 - 1)(x_2 - b).$$

- Genus 2 Curve:

$$\begin{aligned} y^2 = {} & ((a-1)^3 r^2 - (b-1)^3) \\ & ((a-b)(b-1)^2 - (a-1)^3 r^2 + (b-1)^3) \\ & ((a-b)(b-1)^2 - a(a-1)^3 r^2 + a(b-1)^3)) \end{aligned}$$

- Genus 6 Curve: Too long to fit in the slide!

Some computations

Let $A = E_1 \times E_2$ for $E_1, E_2/\mathbb{Q}$. How often can we prove that $z_{c,d} = 0$ for all $c, d \in A(\mathbb{Q})$?

Sample curves taken from LMFDB

rk $E_1(\mathbb{Q})$	rk $E_2(\mathbb{Q})$	# pairs checked	# s.t. $z_{c,d}=0$ for all $c,d \in A(\mathbb{Q})$
1	1	4950	2602
1	2	10000	3311
1	3	10000	955
2	2	4950	995
2	3	10000	615
3	3	190	17

Table not complete: only uses $\mathbb{Q}$-points on ≤ 6 hyperelliptic curves/$\mathbb{Q}$ of genus 2 mapping into A.

Some Future Directions

- **More Computational Experimentation** using hyperelliptic points in higher degree extensions and recent breakthroughs on density of degree d points on curves.
- Find "special curves" that produce rational equivalences for $E \times C$ and $(E \times C)/\iota$, for E elliptic curve and C of genus 2.
- Use the hyperelliptic curves in $E_1 \times E_2$ to produce **"Ceresa-like"** cohomologically trivial 1-cycles on triple products $E_1 \times E_2 \times E_3$.
- **Towards Bass-Bloch-Beilinson Conjectures:** Construct **indecomposable cycles** on a product of **non-isogenous** E_1, E_2 over $\mathbb{Q}$ to show that the kernel $\Sigma := \ker(\mathrm{CH}^2(\mathcal{E}_1 \times \mathcal{E}_2) \to \mathrm{CH}^2(E_1 \times E_2))$ is torsion, where $\mathcal{E}_1 \times \mathcal{E}_2$ is a smooth model over $\mathrm{Spec}(\mathbb{Z}[1/N])$.

Thank you!