Supersingular Loci of Unitary Shimura Varieties

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Motivation: Modular Curves over $\overline{\mathbb{F}}_p$

Seek to understand

$$\mathcal{Y}_p(\overline{\mathbb{F}}_p) = \{\text{elliptic curves } E \text{ over } \overline{\mathbb{F}}_p\}/\cong$$

j-invariant (Dedekind or Klein, 1800's)

There is a bijection:

$$\begin{array}{ccc} \mathcal{Y}_p(\overline{\mathbb{F}}_p) & \leftrightarrow & \overline{\mathbb{F}}_p \\ E : y^2 = x^3 + Ax + B & \mapsto & j(E) = \frac{1728(4A)^3}{16(4A^3+27B^2)} \\ E_j : y^2 = x^3 - \frac{1}{4}x^2 - \frac{36}{j-1728}x - \frac{1}{j-1728} & \leftarrow\!\shortmid & j \end{array}$$

$\mathrm{GU}(a, b)$ Shimura Variety

Fix a quad. im. field K and $p \neq 2$ inert in K

The $\mathrm{GU}(a, b)$ Shimura variety $\mathcal{M}(a, b)$

parametrizes $(A, \iota, \lambda, \eta)$:

- A an A.V. of dim a+b
- ι an action of $\mathcal{O} \subseteq K$

Meeting the signature (a, b) condition:

$$\det(T - \iota(k); \mathrm{Lie}(A)) = (T - \varphi_1(k))^a (T - \varphi_2(k))^b.$$

Example: Let $E : y^2 = x^3 - x$. $\mathbb{Z}[i]$ acts on E, where:

$$\mathbf{i} : E \to E$$

$$(x, y) \mapsto (-x, iy)$$

Define $\mathbb{Z}[i]$-action on $A = E \times E \times E$ as:

Need signature to define a nice moduli space!

$\mathrm{GU}(a,b)$ Shimura Variety

The $\mathrm{GU}(a,b)$ Shimura variety $\mathcal{M}(a,b)$

parametrizes (A,ι,λ,η):

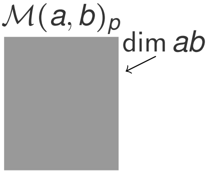
- A an A.V. of dim a+b
- ι an action of $\mathcal{O} \subseteq K$

Meeting the signature (a,b) condition:

$$\det(T-\iota(k);\mathrm{Lie}(A)) = (T-\varphi_1(k))^a(T-\varphi_2(k))^b.$$

$$\mathcal{M}(a,b)_p(\overline{\mathbb{F}}_p) = \{(A,\iota,\lambda,\eta) \text{ over } \overline{\mathbb{F}}_p\}/\cong.$$

This is a moduli space of dimension ab over $\overline{\mathbb{F}}_p$:



Objectives

1. Study the Newton stratification of $\mathcal{M}(a, b)_p$
   - Describe the geometry of the supersingular locus in several examples
   - See how how the general behavior differs from “low dimensional” cases
2. Discuss the Ekedahl-Oort stratification of $\mathcal{M}(a, b)_p$
3. See how the Ekedahl-Oort stratification informs (or fails to inform!) the geometry of the supersingular loci

Newton Stratification of Modular Curve

An e.c. E over $\overline{\mathbb{F}}_p$ is ordinary if:

- $E[p](\overline{\mathbb{F}}_p) = \mathbb{Z}/p\mathbb{Z}$
- $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ is an order in a quad. im. field
- Slopes 0 and 1

An e.c. E over $\overline{\mathbb{F}}_p$ is supersingular if:

- $E[p](\overline{\mathbb{F}}_p) = \mathrm{id}$
- $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ is an order in a quat. alg.
- Slope $\frac{1}{2}$

The p-divisible group of E is:

$$E[p^\infty] = \varinjlim E[p^k].$$

E_1 and E_2 are in the same Newton stratum if $E_1[p^\infty]$ is isogenous to $E_2[p^\infty]$.

Newton Stratification of Modular Curve

- Two points E_1 and E_2 of $\mathcal{Y}_p$ are in the same Newton stratum if $E_1[p^\infty]$ is isogenous to $E_2[p^\infty]$.
- $\mathcal{Y}$ has two Newton strata - ordinary and supersingular

Cor. to Eichler-Deuring Mass Formula:

There are approx. $\frac{p}{12}$ supersingular elliptic curves over $\overline{\mathbb{F}}_p$.

Newton Stratification of $\mathcal{M}(a, b)_p$

- Two points $(A_1, \iota_1, \lambda_1, \eta_1)$ and $(A_2, \iota_2, \lambda_2, \eta_2)$ of $\mathcal{M}(a, b)_p$ (or of the modular curve $\mathcal{Y}$) are in the same Newton stratum if $A_1[p^\infty]$ is isogenous to $A_2[p^\infty]$

- $\mathcal{M}(a, b)_p$ has a unique closed Newton stratum - the supersingular locus $\mathcal{M}(a, b)_p^{ss}$

- How to describe the geometry of $\mathcal{M}(a, b)_p^{ss}$?

Results on Geometry of $\mathcal{M}(a,b)_p^{ss}$

- The geometry of $\mathcal{M}(a,b)_p^{ss}$ depends on the signature (a,b). $\mathcal{M}(a,b) \cong \mathcal{M}(b,a)$, so take $b \geq a \geq 0$.
- The supersingular loci $\mathcal{M}(a,b)_p^{ss}$ have been described by...

$(0,m)$	$(1,1)$	$(1,2)$	$(1,m-1)$	$(2,2)$	$(2,m-2)$	$(a,m-a)$ $a \geq 3$
0-dim'l	0-dim'l	Vollaard 2008	Vollaard-Wedhorn 2010	Howard-Pappas 2014	Imai-F. 2021 after perf.; Howard-Imai-F.	Incomplete

- We'll consider the $(1,2) = (2,1)$, $(2,2)$, and $(2,3)$ cases.

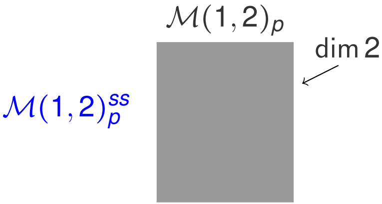
Ex: The Supersingular Locus of $\mathcal{M}(1,2)$

Theorem (Vollaard '08)

Assume η is suff. small. Each irreducible component of $\mathcal{M}(1,2)_p^{ss}$ is isomorphic to the Fermat curve

$$C : x_0^{p+1} + x_1^{p+1} + x_2^{p+1} \subset \mathbb{P}^2_{\overline{\mathbb{F}}_p}.$$

There are $p^3 + 1$ int. pts on each irr. comp., and each int. point is the intersection of $p + 1$ irr. comps.



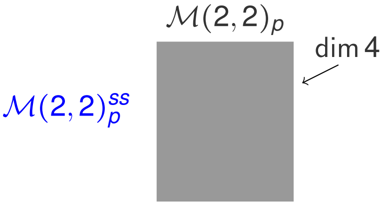
Ex: The Supersingular Locus of $\mathcal{M}(2,2)$

Theorem (Howard-Pappas '14)

Assume η is suff. small. Each irreducible component of $\mathcal{M}(2,2)_p^{ss}$ is isomorphic to the Fermat surface

$$S : x_0^{p+1} + x_1^{p+1} + x_2^{p+1} + x_3^{p+1} \subset \mathbb{P}^3_{\overline{\mathbb{F}}_p}.$$

Any two irr. components intersect trivially, in a projective line, or in a point.



Deligne-Lusztig Varieties

Moduli spaces of flags in char. p vector spaces, with "fixed relative position" to Frobenius-twist:

Given G over $\mathbb{F}_p$, B, and $w \in N_G(T)/T$:

$$X(w) = \{gB \in G/B \mid g^{-1}Fr(g) \in BwB\}.$$

Example: $G = \mathrm{SL}_2$, B upper-tri, $N_G(T)/T = \{1, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$.

- $G/B \cong \{$ lines $\ell \subseteq \overline{\mathbb{F}}_p^2\}$
- $\mathrm{rel}(\ell_1, \ell_2) = 1$ if and only if $\ell_1 = \ell_2$
- If $\ell = \langle c_o e_1 + c_1 e_2 \rangle$, $\mathrm{Fr}(\ell) = \langle c_o^p e_1 + c_1^p e_2 \rangle$
- So,

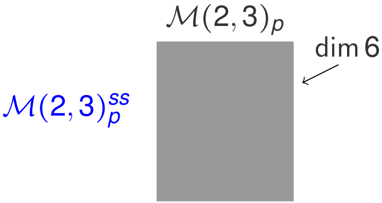
$$X(1) = \{\ell \subseteq \overline{\mathbb{F}}_p^2 \mid \mathrm{rel}(\ell, \mathrm{Fr}(\ell)) = 1\} = \mathbb{P}^1(\mathbb{F}_p)$$

$$X\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \{\ell \subseteq \overline{\mathbb{F}}_p^2 \mid \mathrm{rel}(\ell, \mathrm{Fr}(\ell)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\} = \mathbb{P}^1(\overline{\mathbb{F}}_p) \setminus \mathbb{P}^1(\mathbb{F}_p).$$

Ex: The Supersingular Locus of $\mathcal{M}(2,3)$

Theorem (Howard-Imai-F.)

Assume η is suff. small. There are two isomorphism classes of irreducible components in $\mathcal{M}(2,3)_p^{ss}$: X^1 and X^2.
X^1 is isomorphic to a Deligne-Lusztig variety, and X^2 is not.
We describe X^2 explicitly via a map to a Deligne-Lusztig variety.



Next

- Introduce main technique for study of supersingular loci
- Explain why one might expect Deligne-Lusztig varieties
- Explain what “goes wrong” for signatures beyond $(2,2)$

Unitary Rapoport-Zink Spaces

Unitary Rapoport-Zink Space: $\mathcal{N}(a,b)(S) = \{(G,\iota,\lambda,\rho)\}/\cong$,

- G a supersingular p-div. gp over S of dim $a+b$
- $\iota : \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}(G)$ of sign. (a,b)
- $\rho : G_{S_0} \to \mathbb{G}_{S_0}$, quasi-isog

Rapoport-Zink Uniformization

$$\mathcal{M}(a,b)_p^{ss} \cong \bigsqcup_{j=1}^{m} \Gamma_j \backslash \mathcal{N}(a,b)$$

The Γ_j are discrete groups (depending on level structure) acting on $\mathcal{N}(a,b)$.

Can study the (more "linear-algebraic") Rapoport-Zink spaces $\mathcal{N}(a,b)$ to understand the supersingular loci $\mathcal{M}(a,b)_p^{ss}$

Unitary Rapoport-Zink Spaces

(p-adic) Dieudonné Theory:

There is an equivalence of categories between:

(G, ι, λ) in $\mathcal{N}(a,b)(\overline{\mathbb{F}}_p)$

p-adic Dieudonné modules (M, F, V)

- M free rk $2(a+b)$ over $\breve{\mathbb{Z}}_p$
- F has "slopes $\frac{1}{2}$"
- M has action of sign. $(1,2)$
- $M = p^i M^\vee$

Unitary Rapoport-Zink Space: $\mathcal{N}(a,b)(\overline{\mathbb{F}}_p) = \{(G, \iota, \lambda, \rho)\}/\cong$,

$$\mathcal{N}(a,b)(\overline{\mathbb{F}}_p) = \{M \subseteq \mathbb{N} \mid pM \subseteq FM \subseteq M,\ \mathcal{O}-\text{stable},\ M = p^i M^\vee\}$$

The $\mathrm{GU}(1,2)$ Rapoport-Zink Space

Thm (Vollaard): Geometry of $\mathcal{N}(1,2)$

- $\mathcal{N}(1,2)$ decomposes as:

  $$\mathcal{N}(1,2) = \bigcup_{\Lambda \in \mathcal{L}} \mathcal{N}_\Lambda$$

  indexed by "vertex lattices" Λ.
- The irr. comp of $\mathcal{N}(1,2)$ are precisely the $\mathcal{N}_\Lambda$. Each is isomorphic to

  $$C : x_0^{p+1} + x_1^{p+1} + x_2^{p+1} = 0 \subset \mathbb{P}^2_{\overline{\mathbb{F}}_p}.$$

  (a Deligne-Lusztig variety).
- (Further results)

The $\mathrm{GU}(2,2)$ Rapoport-Zink Space

Thm (Howard-Pappas):

- $\mathcal{N}(2,2)$ decomposes as:

$$\mathcal{N}(2,2) = \bigcup_{\Lambda \in \mathcal{L}} \mathcal{N}_\Lambda$$

  indexed by "vertex lattices" Λ.
- The irr. comp of $\mathcal{N}(2,2)$ are precisely the $\mathcal{N}_\Lambda$. Each is isomorphic to

$$S \ : \ x_0^{p+1} + x_1^{p+1} + x_2^{p+1} + x_3^{p+1} = 0 \subset \mathbb{P}^3_{\overline{\mathbb{F}}_p},$$

  (a Deligne-Lusztig variety).
- (Further results)

Why Expect Deligne-Lusztig Varieties?

- Replace $(G, \iota, \lambda, \rho)$ with p-adic Dieudonné module M to identify:

$$\mathcal{N}(1,2)(\overline{\mathbb{F}}_p) = \{M \subseteq \mathbb{N} \mid \text{ conditions wrt } F\}.$$

- (Convert from $\mathbb{N}$ to an alternative Hermitian space W:)

$$\mathcal{N}(1,2)(\overline{\mathbb{F}}_p) = \{L \subseteq W \mid \text{ conditions wrt } F\}.$$

- Irreducible components $\mathcal{N}_\Lambda \subset \mathcal{N}(1,2)$ defined as:

$$\mathcal{N}_\Lambda(\overline{\mathbb{F}}_p) = \{L \subseteq W \mid p\Lambda \subseteq L \subseteq \Lambda \text{ \& conditions wrt } F\}.$$

- Replace L with $\ell = L/p\Lambda$

$$\mathcal{N}_\Lambda(\overline{\mathbb{F}}_p) = \{\ell \subseteq (\Lambda/p\Lambda)_{\overline{\mathbb{F}}_p} \mid \text{conditions wrt } F\},$$

a Deligne-Lusztig variety.

The $\mathrm{GU}(2,3)$ Rapoport-Zink Space

Thm (Howard-Imai-F.):

- $\mathcal{N}(2,3)$ decomposes as:

$$\mathcal{N}(2,3) = \bigcup_{\Lambda \in \mathcal{L}} \mathcal{N}_\Lambda$$

  indexed by "vertex lattices" Λ.

- Each $\mathcal{N}_\Lambda$ further decomposes as:

$$\mathcal{N}_\Lambda = \mathcal{N}_\Lambda^1 \bigsqcup \mathcal{N}_\Lambda^2.$$

- $\mathcal{N}_\Lambda^1$ is a Deligne-Lusztig variety. We describe $\mathcal{N}_\Lambda^2$ via a map to a Deligne-Lusztig variety.

Notation

- Let $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{nr}}$, with ring of integers $\breve{\mathbb{Z}}_p$.
- W is an 5-dimensional vector space over $\breve{\mathbb{Q}}_p$, with $\breve{\mathbb{Q}}_p$-valued Hermitian form.
- For any lattice $L \subseteq W$, $L^\vee$ denotes the dual lattice.
- $\Lambda \subseteq W$ a fixed self-dual $\breve{\mathbb{Z}}_p$-lattice.

Relative Position of Lattices

- Given lattices L_1 and L_2 in W, we say $\mathrm{Inv}_{L_1}(L_2) = (n_1, n_2, \ldots, n_5)$ if $L_1 = \mathrm{Span}_{\breve{\mathbb{Z}}_p}\{e_i\}_{i=1}^5$ and $L_2 = \mathrm{Span}_{\breve{\mathbb{Z}}_p}\{p^{n_i} e_i\}_{i=1}^5$.

- Example: $L_1 = \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p$, $L_2 = p^2\breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p$. Then $\mathrm{Inv}_{L_1}(L_2) = (2, 0, 0)$.

- Example: $L_1 = \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p$, $L_2 = p\breve{\mathbb{Z}}_p \oplus p\breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p$. Then $\mathrm{Inv}_{L_1}(L_2) = (1, 1, 0)$.

What goes wrong?

- Replace $(G, \iota, \lambda, \rho)$ with p-adic Dieudonné module M to identify:

$$\mathcal{N}(2,3)(\overline{\mathbb{F}}_p) = \{M \subseteq \mathbb{N} \mid \text{ conditions wrt } F\}.$$

- (Convert from $\mathbb{N}$ to an alternative Hermitian space W:)

$$\mathcal{N}(2,3)(\overline{\mathbb{F}}_p) = \{L \subseteq W \mid \text{ conditions wrt } F\}.$$

- Given a vertex lattice $\Lambda \subseteq W$, define: $\mathcal{N}_\Lambda \subset \mathcal{N}(2,3)$as:

$$\mathcal{N}_\Lambda(\overline{\mathbb{F}}_p) = \{L \subseteq W \mid p\Lambda \subseteq L \subseteq \frac{1}{p}\Lambda \text{ \& conditions wrt } F\}.$$

Two issues:

1. $\frac{1}{p}L/pL$ is not a $\overline{\mathbb{F}_p}$-vector space
2. $\mathcal{N}_\Lambda$ further decomposes

What goes wrong?

$$\mathcal{N}_\Lambda(\overline{\mathbb{F}}_p) = \left\{ \text{Lattices } L \subseteq W \;\middle|\; \begin{array}{l} p\Lambda \subseteq L \subseteq \frac{1}{p}\Lambda, \\ \mathrm{Inv}_L(L^\vee) = (1,1,0,0,0) \end{array} \right\}$$

- $\mathcal{N}_\Lambda = \mathcal{N}_\Lambda^1 \sqcup \mathcal{N}_\Lambda^2$, where:

$$\mathcal{N}_\Lambda^1(\overline{\mathbb{F}}_p) = \left\{ \text{Lattices } L \subseteq W \;\middle|\; \begin{array}{l} \mathrm{Inv}_\Lambda(L) = (0,0,0,0,-1), \\ \mathrm{Inv}_L(L^\vee) = (1,1,0,0,0) \end{array} \right\}$$

$$\mathcal{N}_\Lambda^2(\overline{\mathbb{F}}_p) = \left\{ \text{Lattices } L \subseteq W \;\middle|\; \begin{array}{l} \mathrm{Inv}_\Lambda(L) = (1,0,0,-1,-1), \\ \mathrm{Inv}_L(L^\vee) = (1,1,0,0,0) \end{array} \right\}$$

- $\mathcal{N}_\Lambda^1$ is analogous to previous cases:

$$\left\{ L \subseteq W \mid \Lambda \underset{1}{\subseteq} L \underset{3}{\subseteq} \frac{1}{p}L^\vee \underset{1}{\subseteq} \frac{1}{p}\Lambda \right\} \xrightarrow{\sim} \left\{ \ell \subseteq \left(\frac{1}{p}\Lambda/\Lambda\right)_{\overline{\mathbb{F}}_p} \mid \dim(\ell) = 1,\ \ell \subseteq \ell^\perp \right\}$$

$$L \mapsto L/\Lambda$$

What goes wrong?

- $\mathcal{N}_\Lambda^2(\overline{\mathbb{F}}_p) = \left\{ \text{Lattices } L \subseteq W \;\middle|\; \begin{array}{l} \mathrm{Inv}_\Lambda(L) = (1,0,0,-1,-1), \\ \mathrm{Inv}_L(L^\vee) = (1,1,0,0,0) \end{array} \right\}$ is not analogous to previous cases
- However,

$$p\Lambda \subseteq (L \cap \Lambda) \subseteq \Lambda$$

$$\Lambda \subseteq (L + \Lambda) \subseteq \frac{1}{p}\Lambda.$$

- $L \mapsto (L \cap \Lambda, L + \Lambda)$ defines a map to a Deligne-Lusztig variety

$$\mathcal{N}_\Lambda^2 \to Y_\Lambda^2$$

Results on Geometry of $\mathcal{M}(a,b)_p^{ss}$

- The supersingular loci $\mathcal{M}(a,b)_p^{ss}$ (equivalently RZ spaces $\mathcal{N}(a,b)$) have been described by...

$(0,m)$	$(1,1)$	$(1,2)$	$(1,m-1)$	$(2,2)$	$(2,m-2)$	$(a,m-a)$ $a\geq 3$
0-dim'l	0-dim'l	Vollaard 2008	Vollaard-Wedhorn 2010	Howard-Pappas 2014	Imai-F. 2021 after perf.; Howard-Imai-F.	Incomplete

- Do not expect irreducible components to be Deligne-Lusztig varieties beyond $(2,2)$.

Ekedahl-Oort Stratification of Modular Curve

An e.c. E over $\overline{\mathbb{F}}_p$ is ordinary if:

- $E[p](\overline{\mathbb{F}}_p) = \mathbb{Z}/p\mathbb{Z}$
- $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ is an order in a quad. im. field
- Slopes 0 and 1

An e.c. E over $\overline{\mathbb{F}}_p$ is supersingular if:

- $E[p](\overline{\mathbb{F}}_p) = \mathrm{id}$
- $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ is an order in a quat. alg.
- Slope $\frac{1}{2}$

E_1 and E_2 are in the same EO stratum if $E_1[p] \cong E_2[p]$.

$\mathcal{Y}_p$ has two EO strata:

Ekedahl-Oort Strata

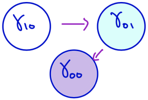
- $(A_1, \iota_1, \lambda, \eta_1)$ and $(A_2, \iota_2, \lambda, \eta_2)$ are in the same Ekedahl-Oort stratum of $\mathcal{M}(a, b)_p$ if and only if $(A_1[p], \iota_1, \lambda_1) \cong (A_2[p], \iota_2, \lambda_2)$.

- By a result of Moonen, for each $\gamma \in \mathcal{W}$ define $(H_\gamma, \iota_\gamma, \lambda_\gamma)$ and $\mathcal{M}(a, b)_\gamma$ as:

$$\{(A, \iota, \lambda, \eta) \in \mathcal{M}(a, b)_p \mid (A[p], \iota, \lambda, \eta) \cong (H_\gamma, \iota_\gamma, \lambda_\gamma)\}.$$

Ekedahl-Oort Strata of $\mathcal{M}(1,2)_p$

- Index set and closure relations (Moonen, Viehmann-Wedhorn, Wooding):



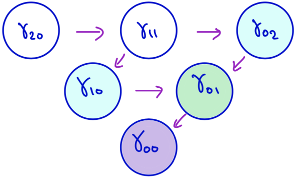
Theorem (Vollaard-Wedhorn)

$$\mathcal{M}(1,2)^{ss} = \mathcal{M}(1,2)_{00} \bigcup \mathcal{M}(1,2)_{01},$$

where $\mathcal{M}(1,2)_{00}$ consists of the intersection points and $\mathcal{M}(1,2)_{01}$ consists of their complement.

Ekedahl-Oort Strata of $\mathcal{M}(2,2)_p$

- Index set and closure relations (Moonen, Viehmann-Wedhorn, Wooding:

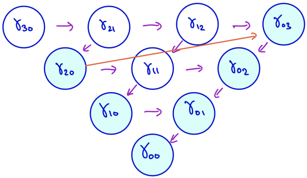


Theorem (Follows from results of Howard-Pappas):

$\mathcal{M}(2,2)^{ss} = \mathcal{M}(1,2)_{00} \cup \mathcal{M}(1,2)_{01} \cup \mathcal{M}(1,2)_{10} \cup \mathcal{M}(1,2)_{02}$, as follows:

Ekedahl-Oort Strata of $\mathcal{M}(2,3)_p$

- Index set and closure relations (Moonen, Viehmann-Wedhorn, Wooding:



Theorem (Bhamidipati, Goodson, Groen, Nair, Stacy, F.):

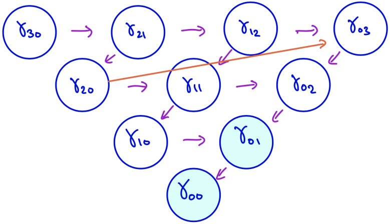
The EO strata intersecting the supersingular locus are exactly those indexed by:

$$\gamma_{00},\ \gamma_{10},\ \gamma_{01},\ \gamma_{20},\ \gamma_{02},\ \gamma_{03}$$

Further, γ_{03} intersects but is not contained in the supersingular locus.

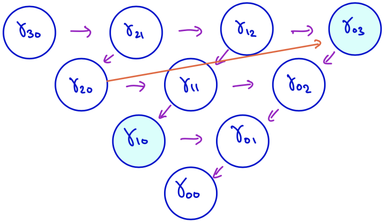
Ekedahl-Oort Strata of $\mathcal{M}(2,3)_p$

Comparison with Siegel Modular Variety:



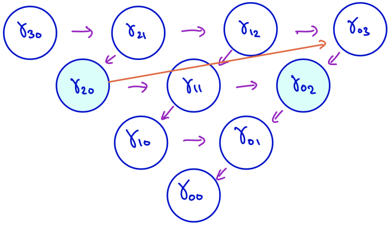
Ekedahl-Oort Strata of $\mathcal{M}(2,3)_p$

Comparison with Unitary Shimura Varieties of Smaller Signature:



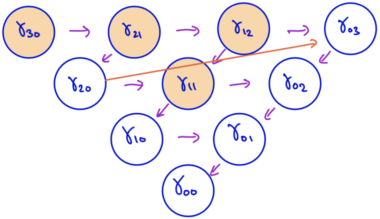
Ekedahl-Oort Strata of $\mathcal{M}(2,3)_p$

Explicit construction:



Ekedahl-Oort Strata of $\mathcal{M}(2,3)_p$

Comparison with Siegel modular variety (minimal EO strata):



General Situation

- The signature $(1, 2)$ and signature $(2, 2)$ supersingular loci have similar structure: their irr. components are Deligne-Lusztig varieties, are a union of Ekedahl-Oort Strata.
- These are examples of Coxeter Type.
- Most unitary Shimura varieties do not have this structure.

$(0, m)$	$(1, 1)$	$(1, 2)$	$(1, m-1)$	$(2, 2)$	$(2, m-2)$	$(a, m-a)$ $a \geq 3$
0-dim'l	0-dim'l	Vollaard 2008	Vollaard-Wedhorn 2010	Howard-Pappas 2014	Imai-F. 2021 after perf.; Howard-Imai-F.	Incomplete

Thank you!

Ex: The Supersingular Locus of $\mathcal{M}(2, m-2)$

Theorem (Howard-Imai-F.)

Assume η is suff. small. There are $\lfloor \frac{m}{2} \rfloor$ isomorphism classes of irreducible components in $\mathcal{M}(2, m-2)_p^{ss}$: X^i for $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$.
We describe each X^i explicitly via a morphism to a Deligne-Lusztig variety.
Only X^1 and $X^{\lfloor \frac{m}{2} \rfloor}$ (when m even) are isomorphic to Deligne-Lusztig varieties.