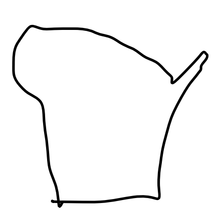
$\boxed{\text{VaNTAGe}}$

6 Apr 2021

(Jordan Ellenberg, )

"Counting points on (some) stacks: progress and problems"

A stack is a generalization of a scheme - first developed to study moduli problems

$X(1)$ (moduli stack of elliptic curves)

$X(1)(K) = \{$elliptic curves over $K\}$

Not the same as the j-line: when K is not $\mathbb{C}$, many non-$\cong$ elliptic curves with same j-invariant!

BG (moduli stack of G-covers)

$BG(K) = \{$Galois G-extensions of $K\}$

Some stacks (like these two) have infinitely many points.

Count in what order?

Custom in ye olde scheme theory: count by <u>height</u>.

(Non-negative real # attached to a point, e.g. $H(a:b:c) = \max(|a|, |b|, |c|)$)

What is the height of a point on a stack?

Q: How many isomorphism classes of elliptic curves $E/\mathbb{Q}$ are there such that:

- E has a rational 5-isogeny
- $\max(|A|^3, |B|^2) < X$

($E: y^2 = x^3 + Ax + B$ minimal Weierstrass form)

A: $\Theta(X^{1/6} \log^2 X)$

(Boggess-Sankar, 2020)

(Remark: for rational 5-torsion instead of 5-isogeny, $\Theta(X^{1/6})$ by Harron-Snowden (2017))

Also in Boggess-Sankar: N-isogenies for $N = 2, 3, 4, 6, 8, 9, 12, 16, 18$.

Q: What about 7-isogenies?

These are questions about rational points on the moduli stacks $X_0(N)$ (Boggess-Sankar)

schemes $X_1(N)$ (Harron-Snowden)

Define $\mathrm{sqf}(a)$ = squarefree m such that ma is a square.

($\mathrm{sqf}(12)=3$, $\mathrm{sqf}(15)=15$, etc.)

Q: How many pairs of coprime a,b such that

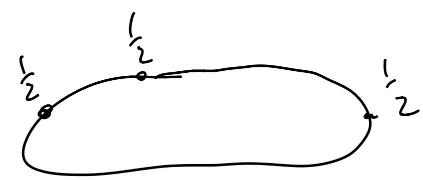
$$\mathrm{sqf}(a)\,\mathrm{sqf}(b)\,\mathrm{sqf}(a-b)\max(a,b) < X\,?$$

Le Boudec (2020)

Nasserden-Xiao (2020)

$c_1 X \log^3 X < \text{count} < c_2 X \log^3 X$

"Stacky curve" $\mathcal{X}$ of genus $3/4$



Q: Are there only finitely many 5-term coprime arithmetic progressions a_1, a_2, a_3, a_4, a_5 with

$$\mathrm{sqf}(a_1 a_2 a_3 a_4 a_5) < \max(|a_i|)^{1/2-\delta}\,?$$

Q: How many points $P \in \mathbb{P}^2(\overline{\mathbb{Q}})$ such that

- $Ht_{abs}(P) < X$
- $[\mathbb{Q}(P) : \mathbb{Q}] = 3$?

Guignard (2017): $\sim X^{12}$, but

\# P not on a $\mathbb{Q}$-rational line

$\sim X^{9+\varepsilon}$

Stack: $\mathrm{Sym}^3 \mathbb{P}^2 = (\mathbb{P}^2)^3 / S_3$

Q: How many degree-3 number fields $K/\mathbb{Q}$ of discriminant $< X$ ...

such that $\mathcal{O}_K \setminus \mathbb{Z}$ contains an element with all archimedean absolute values $< .001 X^{1/2}$?

$< X^{0.499}$?

(For the first question, see Bhargava-H 2016)

The cubic fields are points on $B(S_3)$ - but what is <u>height</u> here?

Key idea: for schemes, height of a point on $X(K)$ is defined in terms of a <u>line bundle</u> on X. (When $X = \mathbb{P}^n$, use $\mathcal{O}(1)$.)

For stacks, we need to use <u>vector bundles</u>.

Consider $B(A_5)$!

Malle predicts:

$\#$ G-extns, disc $< X$
of K

$$\sim C_{K,G} X^{a(G)} (\log X)^{b(G,K)}$$

Which vector bundle on BG?

(i.e. which rep'n of G)

Regular rep of G $\longrightarrow$ disc. of Galois G-extension

Perm rep of S_n (or $G < S_n$) $\longrightarrow$ disc of degree-n extension with Galois closure having group S_n.

2-dim'l rep of D_4 $\longrightarrow$ new count of D_4-extns by Varma, Altug, Shankar, Wilson 2017

But one can do more! Consider an $\mathcal{X}$ with a metrized vector bundle V. In fact, let $\mathcal{X}$ be a proper scheme X!

$$\begin{tikzcd}
f^*V \arrow[r] \arrow[d] & V \quad \text{rank } n \arrow[d, dash] \\
\operatorname{Spec} \mathbb{Z} \arrow[r, "x"'] & X
\end{tikzcd}$$

x^*V is a metrized rank-n vector bundle on $\operatorname{Spec} \mathbb{Z}$ - i.e.

a lattice in $\mathbb{R}^n$

The covolume of this lattice is the height in the usual sense. But why not remember the whole lattice?

e.g.

- $X = \mathbb{P}^2$, V = tangent bundle:

  $H_V(a:b:c) = (a,b,c)^\perp \subset \mathbb{Z}$

  covolume $\sqrt{a^2+b^2+c^2}$

  but shape depends on whether x is on a low-height line!

- $X/_{k(t)}$: $x : \mathbb{P}^1 \to X$

  $H_V(x)$ keeps track of stability of vector bundle $x^*V = \mathcal{O}(a_1) + \ldots + \mathcal{O}(a_k)$

- $\mathcal{X} = B(S_n)$, V: permutation rep

  x: a degree-n extension $K/\mathbb{Q}$

  $H_V(x)$ = integer lattice $\mathcal{O}_K$

  (covol^2 = discriminant!)

  Shape of this lattice studied by

  H, Bhargava, Harron, Mantilla-Soler, ...

WILL'S TALK 5/4

Peyre, "Beyond heights" (2018) – keeping track of these lattices may be the key to understanding accumulation phenomena in Batyrev-Manin: "free" rational points are those with "round" lattices....

YURI'S TALK 5/11

How to make good zeta functions when height is a lattice instead of a number?

(see e.g. Aka-Einsiedler-Shapira-Zhang 2016)