Dynamical Belyi Maps

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Definition

Let X be a smooth projective curve over $\mathbb{C}$. A **Belyi map** is a nonconstant morphism

$$f : X \to \mathbb{P}^1$$

that is branched exactly over $0, 1, \infty$. A Belyi map has genus zero if X has genus $g(X) = 0$.

A **dynamical Belyi map** is a Belyi map

$$f : \mathbb{P}^1 \to \mathbb{P}^1$$

such that $f(\{0, 1, \infty\}) \subset \{0, 1, \infty\}$.

Examples:

1. $f(x) = 2x^3 + 3x^2$ is not a Dynamical Belyi map but $f(x) = -2x^3 + 3x^2$ is a Dynamical Belyi map!
2. $f(x) = \frac{(x-1)^4}{(1-(x-1)^2)^2}$ is also a Dynamical Belyi map, $f(0) = \infty, f(\infty) = 1, f(1) = 0$.

Dynamical Belyi Maps

- Dynamical Belyi maps allow for iteration and hence the study of the dynamical behaviour. By **dynamical behavior**, we mean classification of points according to their orbits.

- Dynamical Belyi maps are **postcritically finite** (PCF) maps, i.e., the orbit of each critical point is finite. The PCF maps have been an object of interest in arithmetic dynamics. Informally, the behavior of the orbits of the critical points largely determines the function's dynamics.

- We can study the **iterated monodromy groups** of Dynamical Belyi maps and the **Arboreal representations**.

Motivation

Let $f(x) \in \mathbb{Z}[x]$ and fix $a_0 \in \mathbb{Z}$. Consider the sequence $\{a_n\}$ defined as $a_n = f(a_{n-1})$ for all $n \geq 1$.
Example: Let $f(x) = x^2 - 2x + 2$ and $a_0 = 3$. We obtain the sequence of Fermat numbers.

1. Let $\mathcal{P} = \{p \text{ prime in } \mathbb{Z} : p \text{ divides } a_n \text{ at least for one } n\}$. What is the natural density of the set $\mathcal{P}$?

$$\lim_{x\to\infty} \frac{\#\{p \in \mathcal{P} : p \leq x\}}{\#\{p \text{ prime in } \mathbb{Z} : p \leq x\}} =?$$

2. (Odoni) Let $f(x) = x^2 - x + 1$. Then

$$\mathcal{P} \cap [1, x] = O(x(\log x)^{-1}(\log\log\log x)^{-1}) \text{ as } x \to \infty$$

Generating Systems

A **generating system** of degree d is a triple $\rho = (\sigma_1, \sigma_2, \sigma_3)$ of permutations $\sigma_i \in S_d$ that satisfy

- $\sigma_1\sigma_2\sigma_3 = \mathrm{id}$,
- $G := \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subset S_d$ acts transitively on $\{1, 2, \ldots, d\}$.

We call the group G the (geometric) **monodromy group** of f.

The **combinatorial type** of ρ is a tuple $\underline{C} := (d; C(\sigma_1), C(\sigma_2), C(\sigma_3))$, where d is the degree and $C(\sigma_i)$ is the conjugacy class of σ_i in S_d.

The conjugacy class $C(i)$ can be represented by a sequence of integers which denote the length of each cycle in σ_i.

Normalized, single cycle dynamical Belyi Maps

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a Dynamical Belyi map.

- Assume $f(0) = 0, f(1) = 1, f(\infty) = \infty$.
- There is a unique ramification point above each of the points $0, 1,$ and ∞ and these unique points are $0, 1, \infty$.

Combinatorial type of such a Belyi map is described by $\underline{C} := (d; e_1, e_2, e_3)$ such that $e_1 + e_2 + e_3 = 2d + 1$.

An **abstract combinatorial type** of genus 0 is $(d; e_1, e_2, e_3)$ such that $2 \leq e_i \leq d$ and $e_1 + e_2 + e_3 = 2d + 1$.

Proposition(Anderson-Bouw-Girgin-E.-,Karemaker-Manes)
For each abstract combinatorial type $\underline{C} := (d; e_1, e_2, e_3)$ of genus 0, there exists a unique normalized Belyi map $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ of combinatorial type $\underline{C}$. Moreover, the rational map f may be red defined over $\mathbb{Q}$.

Examples of Dynamical Belyi Maps

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2. The unique normalized Belyi map of type $(d; 2, d-1, d)$ is

$$f(x) = -(d-1)x^d + dx^{d-1}, \text{ where}$$

3. The unique normalized Belyi map of type $(d; d-k, k+1, d)$ is

$$f(x) = cx^{d-k}(a_0x^k + \cdots + a_{k-1}x + a_k), \text{ where}$$

$$a_i := \frac{(-1)^{k-i}}{(d-i)}\binom{k}{i} \text{ and } c = \frac{1}{k!}\prod_{j=0}^{k}(d-j).$$

More Examples

4. The unique normalized Belyi map of type $(d; d-k, 2k+1, d-k)$ is

$$f(x) = x^{d-k} \left(\frac{a_0 x^k - a_1 x^{k-1} + \cdots + (-1)^k a_k}{(-1)^k a_k x^k + \cdots - a_1 x + a_0} \right),$$

where

$$a_i := \binom{k}{i} \prod_{k+i+1 \leq j \leq 2k} (d-j) \prod_{0 \leq j \leq i-1} (d-j) = k! \binom{d}{i} \binom{d-k-i-1}{k-i}.$$

Iterated Monodromy Groups

Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a Belyi map defined over K.
Let $F_0 := K(t)$ and let F_n be the extension of F_0 corresponding to the map $f^n : \mathbb{P}^1_K \to \mathbb{P}^1_K$. For $n \geq 1$, define

$$G_{n,\mathbb{Q}} := \mathrm{Gal}(\tilde{F}_n / F_0)$$

and

$$G_{n,\bar{\mathbb{Q}}} := \mathrm{Gal}(\tilde{F}_n \otimes_K \bar{\mathbb{Q}} / \bar{\mathbb{Q}}(t))$$

We can describe $G_{n,\bar{\mathbb{Q}}}$ as a quotient of the fundamental group:

$$\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{0, 1, \infty\}, t) \to \mathrm{Aut}(f^{-n}(t)) \subset S_{d^n}$$

Taking inverse limit of $G_{n,\bar{\mathbb{Q}}}$ and $G_{n,\mathbb{Q}}$, we obtain the groups $G_{\infty,\bar{\mathbb{Q}}}$ and $G_{\infty,\mathbb{Q}}$ respectively. They fit into the following exact sequence:

$$1 \to G_{\infty,\bar{\mathbb{Q}}} \to G_{\infty,\mathbb{Q}} \to \mathrm{Gal}(L/k) \to 1$$

where L is an algebraic extension of k (possibly infinite).
For $a \in \mathbb{Q} - \{0,1\}$ and for any $n \geq 1$, we consider the following groups:

$$G_{n,a} := \mathrm{Gal}(\mathbb{Q}(f^n(x) - a)/\mathbb{Q})$$

Taking inverse limit over n, we define

$$G_{\infty,a} := \varprojlim_n \mathrm{Gal}(\mathbb{Q}(f^n(x) - a)/\mathbb{Q})$$

By Hilbert's irreducibility theorem, for a dense subset of $\mathbb{P}^1_{\mathbb{Q}}$,

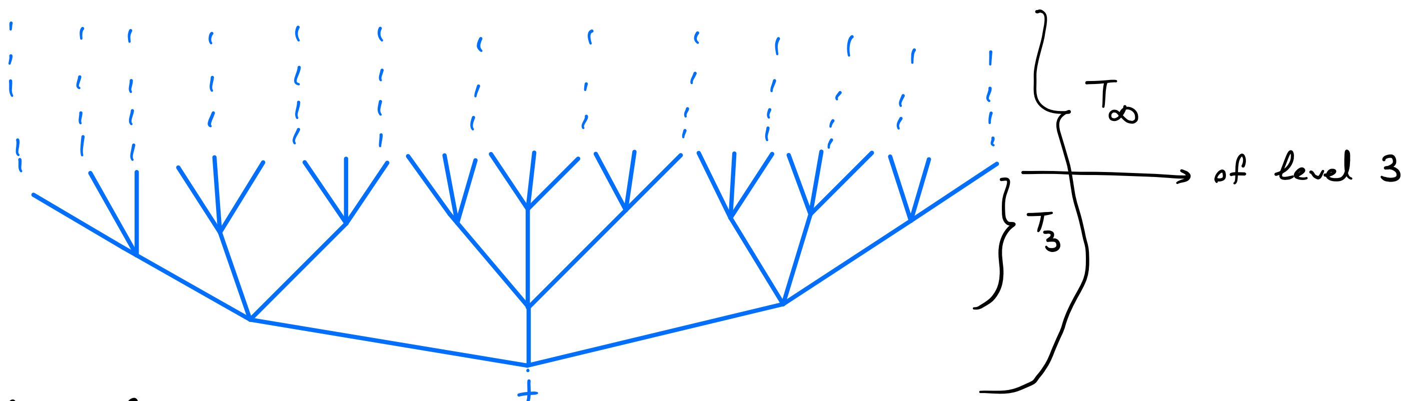
$$G_{n,a} = G_{n,\mathbb{Q}}.$$

Automorphism Group of Trees

Construct a regular d-ary rooted tree with root t where

- vertices of the tree are the roots of $f^n(x) = t$,
- two vertices p, q are connected by an edge if $f(p) = q$.

We can embed $G_{n,\bar{\mathbb{Q}}}$ and $G_{n,\mathbb{Q}}$ into $\mathrm{Aut}(T_n)$.



for $n \geq m$,

- $\mathrm{Aut}(T_n) \longrightarrow \mathrm{Aut}(T_m)$

for $n \geq 2$,

- $\mathrm{Aut}(T_n) \simeq \mathrm{Aut}(T_{n-1}) \text{ wr } S_d$

What is known already?

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3. The cubic polynomial $f(x) = -2x^3 + 3x^2$ is studied by Benedetto-Faber-Hutz-Juul-Yasufuku. This is the first example of a dynamical Belyi map we consider in this talk.

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4. (Bouw-E.-Karemaker) We describe the iterated monodromy groups of Belyi maps of combinatorial type $(d; e_1, e_2, e_3)$.

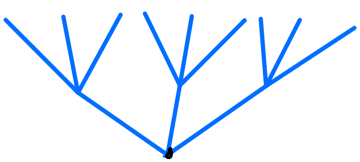
Geometric Monodromy Groups

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a normalized Belyi map of combinatorial type $(d; e_1, e_2, e_3)$.

(Liu-Osserman) For $n = 1$, it is known that $G_{1,\bar{\mathbb{Q}}}$ is either S_d or A_d. If all e_i are odd, then it is A_d. Otherwise it is S_d.

Theorem 1 (Bouw-E.-Karemaker)

1. If $G_{1,\bar{\mathbb{Q}}} = A_d$, then $G_{n,\bar{\mathbb{Q}}}$ is the iterated wreath product of A_d with itself.
2. If $G_{1,\bar{\mathbb{Q}}} = S_d$, then $G_{2,\bar{\mathbb{Q}}}$ is an index two subgroup of $\mathrm{Aut}(T_2)$. For $n \geq 3$, $G_{n,\bar{\mathbb{Q}}}$ is as large as possible allowed by $n = 2$.



$d=3$, $n=2$.

Define $\mathrm{sgn}_2((\sigma_1, \ldots, \sigma_d)\tau) = \mathrm{sgn}(\tau) \prod_{i=1}^{d} \mathrm{sgn}(\sigma_i)$

$G_{2,\bar{\mathbb{Q}}} = \mathrm{Aut}(T_2) \cap \ker(\mathrm{sgn}_2)$.

Inductively, $G_{n,\bar{\mathbb{Q}}} = (G_{n-1,\bar{\mathbb{Q}}} \wr S_d) \cap \ker(\mathrm{sgn}_2)$

Arithmetic Monodromy

Theorem 2 (Bouw-E.-Karemaker)

1. For $n \geq 2$, $G_{n,\bar{\mathbb{Q}}} = G_{n,\mathbb{Q}}$ if and only if $G_{2,\mathbb{Q}} = G_{2,\bar{\mathbb{Q}}}$.
2. The field L in the exact sequence is at most of degree two over $\mathbb{Q}$.

$$1 \to G_{n,\bar{\mathbb{Q}}} \to G_{n,\mathbb{Q}} \to \mathrm{Gal}(L/\mathbb{Q}) \to 1$$

How do we check that $G_{2,\mathbb{Q}} = G_{2,\bar{\mathbb{Q}}}$?
Idea: Define a product discriminant.

Let $t_1, \ldots, t_d$ in $\tilde{F}_1$ such that $f(t_i) = t$.

Then we define

$$\Delta := \Delta(f(x) - t) \prod_{i=1}^{d} \Delta(f(x) - t_i)$$

$G_{2,\mathbb{Q}} \subseteq \ker(\mathrm{sgn}_2) \iff \Delta$ is a square in $\mathbb{Q}(t)$.

Arboreal Representations

Theorem 3 (Bouw-E.-Karemaker)
Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3)$. Let $a \in \mathbb{Q} - \{0, 1\}$ be such that there exist distinct primes p, q_1, q_2, q_3 satisfying the following conditions:

- $f(x) \equiv x^d \mod p$
- f has good separable reduction at q_1, q_2, q_3 and we have

$$\nu_p(a) = 1, \quad \nu_{q_1}(a) > 0, \quad \nu_{q_2}(1 - a) > 0, \quad \nu_{q_3}(a) < 0$$

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Sketch of the proof:

- The condition of good monomial reduction is necessary for the Eisenstein condition and hence the irreducibility.
- The good separable reduction condition guarantees that the reduction of f at p has the same combinatorial type as f. We use this to show that there is a tower of primes above p, with exactly one ramified prime at each step and the ramification index is e_1.

Dynamical Applications

Theorem (Bouw-E.-Karemaker)

Let f be a normalized Belyi map such that $G_{n,\bar{\mathbb{Q}}} = G_{n,\mathbb{Q}}$ for all $n \geq 1$. Let K be the splitting field of f and consider the non-zero preimages of zero under f, denoted $c_j \in K$. Suppose that for each c_j, $G_{n,K} = G_{n,c_j}$.
Define a dynamical sequence $\{b_i\}_{i\geq 0}$ by $b_0 \in \mathbb{Q} \setminus \{0,1\}$ and $b_i = f(b_{i-1})$ for $i \geq 1$. Then the set of prime divisors of the entries of this sequence has natural density zero.

Thank you!