Geometric Serre weight conjectures and Θ-operators

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Introduction

I'll discuss work largely inspired by Edixhoven's work on the weight part of Serre's Conjecture, and in particular his "geometric" variant (which extends the formulation to include weight one modular forms).

I'll describe an analogue in the context of Hilbert modular forms.

Motivation: the geometric version reveals interesting aspects missing from (previously studied) "algebraic" Serre weight conjectures.

Plan for the talk

1. The classical setting:
   - Serre's Conjecture.
   - Edixhoven's variant
2. Algebraic Serre weight conjectures
3. The Hilbert modular setting:
   - Mod p HMF's and Galois representations
   - Weight-shifting
   - A geometric Serre weight conjecture

Theorem (Serre's Conjecture)

If $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous, odd and irreducible, then ρ is modular of weight k_ρ and level N_ρ, where

- *N_ρ is the (prime-to-p) Artin conductor of ρ,*
- *k_ρ (≥ 2) is given by an explicit recipe in terms of $\rho|_{G_{\mathbb{Q}_p}}$.*

Proved by Khare–Wintenberger, building on work of many (Serre, Mazur, Ribet, Carayol, Coleman–Voloch, Gross, Edixhoven, Wiles, Taylor–Wiles, Kisin, ...)

Modular forms

Recall $Y_1(N) = \Gamma_1(N) \backslash \mathfrak{H}$ parametrizes pairs (E, P), where:

- E is an elliptic curve;
- $P \in E$ is a point of order N.

Extends to $X_1(N)$ using generalized elliptic curves (for $N > 4$).

Let $E \to X_1(N)$ be the universal generalized elliptic curve, and $\omega = 0^* \Omega^1_{E/X_1(N)}$.

The space of modular forms of weight k and level N is

$$M_k(\Gamma_1(N)) = H^0(X_1(N), \omega^k),$$

equipped with Hecke operators T_v for all primes v, S_v for $v \nmid N$.

Fix a prime p, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Associated Galois representations

Theorem (Eichler–Shimura, Deligne, Deligne–Serre)

If $f \in M_k(\Gamma_1(N))$ is a Hecke eigenform, then there is a continuous representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that if $v \nmid pN$, then ρ_f is unramified at v and $\rho_f(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - a_v X + d_v v$, where $T_v = a_v f$ and $S_v f = d_v f$.

Choose a basis so that $\rho_f(G_{\mathbb{Q}}) \subset \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ and define

$$\overline{\rho}_f : G_{\mathbb{Q}} \xrightarrow{\rho_f} \mathrm{GL}_2(\overline{\mathbb{Z}}_p) \twoheadrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

(well-defined up to semi-simplification).

We say an irreducible $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular of weight k and level N if $\rho \sim \overline{\rho}_f$ for some f as above.

Geometric modularity

If $N > 4$, then the moduli problem defining $X_1(N)$ is representable by a smooth scheme X over $\mathbb{Z}[1/N]$.
For $p \nmid N$, let $\overline{X} = X_{\overline{\mathbb{F}}_p}$, $\omega = 0^*\Omega^1_{\overline{E}/\overline{X}}$ and define:

$$M_k(\Gamma_1(N), \overline{\mathbb{F}}_p) := H^0(\overline{X}, \overline{\omega}^k),$$

We again have Hecke operators, and Galois representations ρ_f associated to Hecke eigenforms $f \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)$.
We say ρ is geometrically modular of weight k and level N if $\rho \sim \rho_f$ for such an f.
If $k \geq 2$, this is equivalent to being modular of weight k and level N (since $M_k(\Gamma_1(N), \overline{\mathbb{Z}}_p) \twoheadrightarrow M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)$).

Weight-shifting

The Hasse invariant $\mathrm{Ha} \in M_{p-1}(\Gamma_1(N), \overline{\mathbb{F}}_p) = \mathrm{Hom}_{\mathcal{O}_{\overline{X}}}(\omega, \omega^p)$ is the section induced by $\mathrm{Ver} : \overline{E}^{(p)} \to \overline{E}$.
Multiplication by Ha defines a Hecke-equivariant

$$M_k(\Gamma_1(N), \overline{\mathbb{F}}_p) \hookrightarrow M_{k+p-1}(\Gamma_1(N), \overline{\mathbb{F}}_p).$$

So if ρ geometrically modular of weight k,
then ρ is geometrically modular of weight $k + p - 1$.

The Θ-operator (defined by Katz) is a differential operator

$$\Theta : M_k(\Gamma_1(N), \overline{\mathbb{F}}_p) \to M_{k+p+1}(\Gamma_1(N), \overline{\mathbb{F}}_p)$$

- effect on q-expansions is $q\frac{d}{dq}$;
- implies that if ρ is geometrically modular of weight k, then $\rho \otimes \chi_{\mathrm{cyc}}$ is geometrically modular of weight $k + p + 1$.

Edixhoven's variant

Define $k(\rho) = \begin{cases} 1, & \text{if } \rho \text{ is unramified at } p, \\ k_\rho, & \text{otherwise.} \end{cases}$

Theorem (Edixhoven)

If $p \neq 2$ and ρ is geometrically modular of any weight k, level N, then ρ is geometrically modular of weight $k(\rho)$, level N. Furthermore ρ is geometrically modular of weight k' and level N if and only if $k' = k(\rho) + t(p-1)$ for some $t \geq 0$.

- $k(\rho)$ is the smallest weight for which ρ can be geometrically modular (of some level prime to p);
- it's determined by $\rho|_{G_{\mathbb{Q}_p}}$, and may be 1;
- determines all weights for which ρ is geometrically modular;
- Serre's Conjecture holds for geometric modularity with $k(\rho)$ instead of k_ρ (at least if $p > 2$).

Generalizations?

The weight part of Serre's Conjecture can be viewed as a manifestation of local-global compatibility at p in a hypothetical mod p Langlands Programme. This is one motivation for considering it in the context of more general automorphic forms. Analogues have been formulated by:

- Ash–Sinnott, Ash–Doud–Pollack for $\mathrm{GL}_n/\mathbb{Q}$;
- Buzzard-D-Jarvis, Schein, Gee for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, F totally real (really for certain inner forms);
- Herzig for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$, Herzig–Tilouine for $\mathrm{GSp}_4/\mathbb{Q}$;
- Gee–Herzig–Savitt for connected reductive $G/\mathbb{Q}$ unramified at p.

However these are all based on an "algebraic" reinterpretation of the weight in Serre's Conjecture.

Algebraic Serre weights

For $k \geq 2$, consider the locally constant sheaf

$$\mathcal{F}_k := \Gamma_1(N)\backslash(\mathfrak{H} \times \mathrm{Sym}^{k-2}(\overline{\mathbb{F}}_p)^2)$$

where $\Gamma_1(N)$ acts on $\mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2$ via projection to $\mathrm{SL}_2(\mathbb{F}_p)$.
Hecke operators act on

$$H^1(Y_1(N), \mathcal{F}_k) \cong H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}(\overline{\mathbb{F}}_p^2)),$$

and ρ is modular of weight k and level N if and only if the corresponding Hecke eigensystem arises.
Consider instead the irreducible representations

$$V_{t,s} := \det{}^s \otimes \mathrm{Sym}^{t-1}\overline{\mathbb{F}}_p^2,$$

and say ρ is algebraically modular of weight V if the eigensystem arises in $H^1(\Gamma_1(N), V)$.The set of such V determines the set of $k \geq 2$ for which ρ is modular.

Algebraic Serre weight conjectures

The algebraic version of the weight part of Serre's Conjecture (in the classical setting) is a recipe in terms of $\rho|_{G_{\mathbb{Q}_p}}$ for

$$W(\rho) = \{ V \mid \rho \text{ is algebraically modular of weight } V \}.$$

- This completely ignores $k = 1$,
- and obscures the existence of a minimal such k (but see Hanneke's talk).

For Hilbert modular forms (i.e., $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$, F totally real):

- even more weights are ignored (e.g., partial weight 1);
- minimality is even more obscure (but see Hanneke's talk).

Aim: Recover these, i.e., generalize "Edixhoven's variant."

Notation

Let F a totally real number field, $d = [F : \mathbb{Q}] > 1$,
p unramified in F (inert for notational simplicity),and
$U \subset \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ be an open compact subgroup

- of level prime to p (i.e., $\supset \mathrm{GL}_2(\mathcal{O}_{F,p})$),
- sufficiently small (e.g. $\subset U_1(\mathfrak{n})$ for sufficiently small $\mathfrak{n}$).

Let $\mathcal{O} = \mathcal{O}_K$ for sufficiently large $K \supset \mathbb{Q}_p$, and

$$\begin{aligned} \Sigma &= \{\, F \hookrightarrow \overline{\mathbb{Q}} \,\} = \{\, F \hookrightarrow K \,\} \\ &\leftrightarrow \{\, \mathcal{O}_F/p \hookrightarrow \overline{\mathbb{F}}_p \,\} = \{\tau_0, \tau_1, \ldots, \tau_{d-1}\,\} \end{aligned}$$

where $\tau_i = \phi^i \circ \tau_0$ (and view $i \in \mathbb{Z}/d\mathbb{Z}$).

Hilbert modular varieties

Let $Y = Y_U$ denote the (coarse) moduli space parametrizing:

- abelian varieties $A \to S$ of dimension d;
- $\mathcal{O}_F \to \mathrm{End}_S(A)$ such that $0^*\Omega^1_{A/S}$ is locally free over $\mathcal{O}_S \otimes \mathcal{O}_F$;
- level U-structure (e.g., $\mathcal{O}_F/\mathfrak{n} \hookrightarrow A$ if $U = U_1(\mathfrak{n})$).

Then Y is a smooth scheme of dimension d over $\mathcal{O}$, with

$$\begin{aligned} Y(\mathbb{C}) &= \mathrm{GL}_2^+(F)\backslash((\mathfrak{H}^\Sigma \times \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})/U) \\ &\cong \textstyle\coprod_t \Gamma_t\backslash\mathfrak{H}^\Sigma \end{aligned}$$

for congruence subgroups $\Gamma_t \subset \mathrm{GL}_2^+(\mathcal{O}_F)$ depending on U.

Hilbert modular forms

For $\vec{k}, \vec{m} \in \mathbb{Z}^{\Sigma}$, define a line bundle on the fine moduli scheme S (covering Y_U) by

$$\mathcal{A}_{\vec{k},\vec{m}} := \bigotimes_{i=0}^{d-1} (\omega_i^{k_i} \otimes \delta_i^{m_i}),$$

where $0^*\Omega^1_{A/S} = \oplus \omega_i$ and $\wedge^2_{\mathcal{O}_S \otimes \mathcal{O}_F} \mathcal{H}^1_{\mathrm{dr}}(A/S) = \oplus \delta_i$.
This descends to Y if $k_i + 2m_i$ is independent of i, more generally to Y_R for any $\mathcal{O}$-algebra R in which $\prod_i \tau_i(\mu)^{k_i+2m_i} = 1$ for all $\mu \in U \cap \mathcal{O}_F^{\times}$.
In particular for $R = \overline{\mathbb{F}}_p$ and any $\vec{k}, \vec{m}$ and sufficiently small U, have $\mathcal{A}_{\vec{k},\vec{m}}$ over $\overline{Y} := Y_{\overline{\mathbb{F}}_p}$, and let

$$M_{\vec{k},\vec{m}}(U; \overline{\mathbb{F}}_p) := H^0(\overline{Y}, \mathcal{A}_{\vec{k},\vec{m}}).$$

Acted on by Hecke operators T_v, S_v for $v \nmid \mathfrak{n}p$.

Associated Galois representations

Theorem (D–Sasaki)

If $f \in M_{\vec{k},\vec{m}}(U;\overline{\mathbb{F}}_p)$ is a Hecke eigenform, then there exists

$$\rho_f : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

such that if $v \nmid \mathfrak{n}p$, then ρ_f is unramified at v and $\rho_f(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - a_v X + d_v \mathrm{Nm}_{F/\mathbb{Q}} v$ (where $T_v = a_v f$, $S_v = d_v f$).

- Approach: multiply by partial Hasse invariants and lift to characteristic zero (for which Galois representations were constructed by Taylor).
- Proved by Goldring–Koskivirta, Emerton–Reduzzi–Xiao under parity hypotheses, which we removed using lifts with level $\mathfrak{n}p$.
- Say such a ρ is geometrically modular of weight $(\vec{k}, \vec{m})$.

Partial Hasse invariants

Now $\mathrm{Ver} : \overline{A}^{(p)} \to \overline{A}$ induces

$$0^*\Omega^1_{\overline{A}/\overline{S}} \to 0^*\Omega^1_{\overline{A}^{(p)}/\overline{S}} = (0^*\Omega^1_{\overline{A}/\overline{S}})^{(p)},$$

restricting to $\omega_i \to \omega_{i-1}^p$ for each i, hence a section of $\omega_i^{-1}\omega_{i-1}^p$ descending to the partial Hasse invariant

$$\mathrm{Ha}_i \in M_{\vec{h}_i,\vec{0}}(U, \overline{\mathbb{F}}_p), \quad \text{where } \vec{h}_i = (0, \ldots, 0, p, -1, 0, \ldots, 0).$$

Multiplication by Ha_i defines a Hecke-equivariant:

$$M_{\vec{k},\vec{m}}(U; \overline{\mathbb{F}}_p) \hookrightarrow M_{\vec{k}+\vec{h}_i,\vec{m}}(U; \overline{\mathbb{F}}_p).$$

So if ρ is geometrically modular of weight $(\vec{k}, \vec{m})$,
then ρ is geometrically modular of weight $(\vec{k} + \vec{h}_i, \vec{m})$.

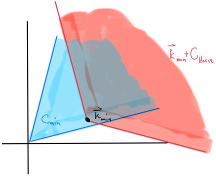
Minimal weights

For $0 \neq f \in M_{\vec{k},\vec{m}}(U; \overline{\mathbb{F}}_p)$, define its minimal weight so that $\vec{k}_{\min}(f) = \vec{k}$ if no $\mathrm{Ha}_i \,|\, f$, and $\vec{k}_{\min}(\mathrm{Ha}_i\, f) = \vec{k}_{\min}(f)$ in general. (The vanishing loci of the Ha_i meet every component of $\overline{Y}$, and have no common components. By contrast similarly defined $\delta_i \to \delta_{i-1}^p$ are isomorphisms, so $\delta_i^{p^d - 1} = \mathcal{O}_{\overline{Y}}$.)

Theorem (D–Kassaei)

$\vec{k}_{\min}(f)$ lies in the (positive) minimal cone (if f is non-Eienstein):

$$C_{\min}^{(+)} = \{\, \vec{k} \mid (0 <)\, k_{i-1} \leq p k_i \text{ for all } i \,\}.$$



Partial Θ-operators

Theorem (Andreatta–Goren)

There's a Hecke-equivariant $\Theta_i : M_{\vec{k},\vec{m}}(U;\overline{\mathbb{F}}_p) \longrightarrow M_{\vec{k}',\vec{m}'}(U;\overline{\mathbb{F}}_p)$, *where* $\vec{k}' = \vec{k} + (\ldots, 0, p, 1, 0, \ldots)$, $\vec{m}' = \vec{m} + (\ldots, 0, -1, 0, \ldots)$. *Furthermore* $\mathrm{Ha}_i | \Theta_i(f)$ *if and only if* $\mathrm{Ha}_i | f$ *or* $p | k_i$.

Idea of construction ($\vec{m} = \vec{0}$ for simplicity):

- Pull back to level $U \cap U_1(p)$,
- divide by $\prod_j h_j^{k_j}$, where h_j is a fundamental Hasse invariant (level p, weight $(\ldots, 0, 1, 0, \ldots)$) to get a rational function,
- differentiate and apply Kodaira–Spencer to get a (meromorphic) form of weight $(\ldots, 0, 2, 0, \ldots)$, $(\ldots, 0, -1, 0, \ldots)$,
- multiply by $\mathrm{Ha}_i \prod_j h_j^{k_j}$ to remove poles, descend to level U.

Θ-cycles

Can determine effect of Θ_i on q-expansions, describe its kernel, relate Θ_i to Θ_{i+1};in particular $\Theta_i^{p^d}(f) = \mathrm{Ha}_i^{p^d-1}\Theta_i(f)$.
For $f \notin \ker(\Theta_i)$, define its i^{th} Θ-cycle by:

$$\vec{k}_{\min}(f),\ \vec{k}_{\min}(\Theta_i f),\ \vec{k}_{\min}(\Theta_i^2 f), \ldots, \vec{k}_{\min}(\Theta_i^{p^d} f).$$

Recall that in the classical setting: if ρ is modular of weight k, then $\rho \otimes \chi_{\mathrm{cyc}}$ is modular of weight $k+p+1$.
The cyclotomic twist is a red herring — it's $\vec{m}$ that changes:

Theorem (D–Sasaki)

If ρ is geometrically modular of weight $(\vec{k}, \vec{m})$, then ρ is geometrically modular of weight $(\vec{k}', \vec{m}')$.

Geometric Serre weight conjecture

Conjecture (D–Sasaki)

If $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous, irreducible and totally odd, then for each $\vec{m} \in \mathbb{Z}^\Sigma$, there exists $\vec{k}_{\min}(\rho) \in C^+_{\min}$ such that:

1. *ρ is geometrically modular of weight $(\vec{k}, \vec{m})$ if and only if*

$$(*) \qquad \vec{k} = \vec{k}_{\min}(\rho) + \sum_i n_i \vec{h}_i \quad \text{for some } n_i \geq 0;$$

2. *if $\vec{k} \in C^+_{\min}$, then $(*)$ holds if and only if $\rho|_{G_{F_p}}$ has a crystalline lift with labelled Hodge–Tate weights $(k_i - 1, k_i + m_i - 1)_{i=0,\ldots,d-1}$.*

Remarks on the conjecture

- This generalizes Edixhoven's variant: the set of all possible weights is determined by a minimal weight, which in turn is prescribed by p-adic Hodge theory.
- The existence of $\vec{k}_{\min}(\rho)$ as in 1) should be much easier than 2), but still isn't obvious (ρ arises from forms with different q-expansions).
- Need $\vec{k} \in C^{+}_{\min}$ for 2) to be plausible.
- The existence of $\vec{k}_{\min}(\rho)$ as in 2) is related, via the Breuil–Mézard Conjecture, to conjectures in modular representation theory (as in Hanneke's talk).
- The dependence of $\vec{k}_{\min}(\rho)$ on $\vec{m}$ is described by a Θ-cycle.
- If $\vec{m} = \vec{0}$, then (conjecturally) $\vec{k}_{\min}(\rho) = \vec{1}$ if and only if ρ is unramified at p, but partial weight one is more subtle...

Evidence involving partial weight one

Theorem (D–Sasaki)

Suppose that $[F:\mathbb{Q}] = 2$, *that* $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *is geometrically modular of some weight, and* $\rho|_{G_{F_{\mathfrak{p}}}}$ *has a crystalline lift with labelled Hodge–Tate weights* $(0,0)_{i=0}, (0,k-1)_{i=1}$, *where* $k \in \{3, 5, \ldots, p\}$.
Then ρ *is geometrically modular of weight* $((1,k),(0,0))$.

Idea of proof: use integral p-adic Hodge theory and results on algebraic Serre weights to show ρ arises from an f of weight $((p+1,k-1),(0,0))$, and g of weight $((p+1,k+1),(0,-1))$. Can further ensure $\Theta_1(f)$ and g have the same q-expansions, so $\Theta_1(f) = \mathrm{Ha}_1\, g \Rightarrow \mathrm{Ha}_1 \,|\Theta_1(f) \Rightarrow \mathrm{Ha}_1\,|f$.

The ramified case

Everything above generalizes to arbitrary p, e.g.
if p is totally ramified in F (for notational simplicity):

- Define $M_{\vec{k},\vec{m}}(U;\overline{\mathbb{F}}_p)$ using smooth models constructed by Pappas–Rapoport;
- Reduzzi-Xiao define partial Hasse invariants of weights

$$(-1,0,\ldots,0,p),(1,-1,0,\ldots,0),\ldots,(0,\ldots,0,1,-1).$$

- $\vec{k}_{\min}(f)$ is in the minimal cone (D–Kassaei), defined by

$$k_0 \le k_1 \le \cdots \le k_{d-1} \le pk_0.$$

- Construction of partial Θ-operators (D), where now

$$\vec{k}' = \vec{k} + (0,\ldots,0,1,1).$$

- Construction of Galois representations, formulation of geometric Serre weight conjecture, identical result for partial weight one (D–Sasaki)