# Kim's Conjecture and Effective Faltings 

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## Talk Overview

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## Introduction: Effective Faltings and Kim's Conjecture

Part I: Introduction: Effective Faltings and Kim's Conjecture

## Background: Faltings' Theorem

## Question

Given a polynomial $f(x, y) \in \mathbb{Z}[x, y]$, can we find $(x, y) \in \mathbb{Q}^{2}$ such that $f(x, y)=0$ ? And if so, how many?

Such a polynomial defines a (smooth projective) algebraic curve $X$ - which has a geometric genus $g$

## Theorem (Faltings '83)

If $g \geq 2$, the set $X(\mathbb{Q})^{a}$ of rational solutions is finite.
${ }^{a}$ Technically this might differ from the set of solutions to $f(x, y)=0$ by a
computable finite set.

In addition:

- If $g=0$, there are 0 or infinitely many solutions
- If $g=1$, there is an elliptic curve associated with $X$


## Effective Faltings

## Problem (Effective Faltings)

When $X(\mathbb{Q})$ is finite, find all solutions.

- If we have a list of solutions, how do we know if our list is complete?
- For some $X$ (more details later), Chabauty-Coleman produces $p$-adic analytic functions that vanish on the set of all solutions
- Given a function, Newton's method determines its finite number of zeroes
- Kim's "non-abelian Chabauty's method" is expected to do this for all $X$


## Refined Problem (Chabauty-Kim Theory)

Find $p$-adic analytic (Coleman) functions on $X\left(\mathbb{Q}_{p}\right)$ that vanish on $X(\mathbb{Q})$ using non-abelian Chabauty.

## Example 1: S-Integral Points on $\mathbb{P}^{1} \backslash\{0,1, \infty\}$

- For a positive integer $N$, we may ask about $\mathbb{P}^{1} \backslash\{0,1, \infty\}(\mathbb{Z}[1 / N])$
- This is equivalent to asking for $x, y \in \mathbb{Z}[1 / N]^{\times}$such that $x+y=1$.
- Simplest case and testing ground for more general methods
- For $N=2$, we have the solutions $x=2,-1,1 / 2$


## Theorem (Dan-Cohen, Wewers, 2013)

- For $p \neq 2$, the following function vanishes on $\mathbb{P}^{1} \backslash\{0,1, \infty\}(\mathbb{Z}[1 / 2])$ :

$$
2 \mathrm{Li}_{2}^{p}(z)-\log ^{p}(z) \mathrm{Li}_{1}^{p}(z)
$$

- For $p=5,7$, this function has only 3 zeroes in $\mathbb{P}^{1} \backslash\{0,1, \infty\}\left(\mathbb{Z}_{p}\right)$


## Example 2: Split Cartan Modular Curve of Level 13

- The modular curve $X_{s}(13)$ parametrizes elliptic curves $E$ whose Galois action on $E[13]$ factors through a split Cartan subgroup
- An homogeneous equation is given by

$$
\begin{aligned}
Y^{4}+ & 5 X^{4}-6 X^{2} Y^{2}+6 X^{3} Z+26 X^{2} Y Z+10 X Y^{2} Z-10 Y^{3} Z \\
& -32 X^{2} Z^{2}-40 X Y Z^{2}+24 Y^{2} Z^{2}+32 X Z^{3}-16 Y Z^{3}=0
\end{aligned}
$$

- There were seven known points, $(1: 1: 1),(1: 1: 2),(0: 0: 1),(-3:$ $3: 2),(1: 1: 0),(0: 2: 1),(-1: 1: 0)$.
- Non-abelian Chabauty (specifically Quadratic Chabauty) showed that these were the only rational points:

> Theorem (Balakrishnan, Dogra, Müller, Tuitman, Vonk)
> $\# X_{s}(13)(\mathbb{Q})=7$

## Effectivity and Kim's Conjecture

- For each positive integer $n$, Kim's method produces a a set $\mathcal{I}_{C K, n}$ of p-adic analytic Coleman functions
- Conjectures on Galois cohomology imply that $\mathcal{I}_{C K, n}$ has a nonzero element for sufficiently large $n$
- Such functions have finitely many zeroes, so a single nonzero function gives finiteness!


## Conjecture (Kim et al., 2014)

For sufficiently large $n$, the set of common zeroes of functions in $\mathcal{I}_{C K, n}$ is precisely $X(\mathbb{Q})$.

- $\mathcal{I}_{C K, 1}$ corresponds to classical Chabauty's method
- The Quadratic Chabauty method of Balakrishnan et al computes part of $\mathcal{I}_{C K, 2}$ in many cases
- Work of C-Dan-Cohen-Wewers hopes to apply to $\mathcal{I}_{C K, n}$ for all $n$


## Classical Chabauty and Chabauty-Kim

## Part II: Classical Chabauty and Chabauty-Kim

## Classical Chabauty's Method

- Mordell conjectured in 1922 that $X(\mathbb{Q})$ is finite if $X$ has genus $g \geq 2$
- First proof in some cases by Chabauty in 1940's using the following method:
- Embed $X$ into a abelian variety $J$ and considers the diagram:

- $J$ has an abelian group structure and dimension $g=\operatorname{dim} \operatorname{Lie}\left(J_{\mathbb{Q}_{p}}\right)$
- The theorem of Mordell-Weil states that $J(\mathbb{Q})$ is finitely generated
- Let $r$ denote the rank of $J(\mathbb{Q})$ as a f.g. abelian group
- When $r<g$, Chabauty shows that $X\left(\mathbb{Q}_{p}\right) \cap J(\mathbb{Q}) \subseteq J\left(\mathbb{Q}_{p}\right)$ is finite


## Chabauty-Coleman

- When $r<g$, there is a linear function $f$ on $\operatorname{Lie}\left(J_{\mathbb{Q}_{p}}\right)$ vanishing on $J(\mathbb{Q})$
- Coleman defines a notion of $p$-adic integration; $f$ is given by

$$
P \mapsto f(P)=\int_{b}^{P} \omega
$$

for some algebraic differential form $\omega$ on $J$

- Pullback to $X\left(\mathbb{Q}_{p}\right)$ is a nonzero function on $X\left(\mathbb{Q}_{p}\right)$ that vanishes on $X(\mathbb{Q})$
- One computes $\omega$ by finding an explicit basis for $J(\mathbb{Q})$
- Newton's method finds the number of zeroes of $f$; if we find that many elements of $X(\mathbb{Q})$, then we are done!
- Two problems:
(1) Sometimes $r<g$
(2) Even if $r<g$, the function from Chabauty-Coleman might have zeroes outside $X(\mathbb{Q})$


## Towards non-abelian Chabauty

- Non-abelian Chabauty applies even with $r \geq g$.
- There is an embedding $J(\mathbb{Q}) \rightarrow \operatorname{Sel}\left(J / \mathbb{Q}, p^{\infty}\right)$ into a Selmer group ("essentially" isomorphism by BSD conjecture)
- $\operatorname{Sel}\left(J / \mathbb{Q}, p^{\infty}\right)$ may be defined as a Galois cohomology group

$$
H_{f}^{1}\left(G_{\mathbb{Q}} ; H_{1}^{\text {ett }}\left(X_{\overline{\mathbb{Q}}} ; \mathbb{Q}_{p}\right)\right)
$$

- Idea: Replace first homology $H_{1}^{\text {ét }}\left(X ; \mathbb{Q}_{p}\right)$ with a non-abelian quotient $U_{n}$ of the fundamental group of $X\left(U_{1}=H_{1}^{\text {ét }}\left(X ; \mathbb{Q}_{p}\right)\right)$
- Kim's diagram becomes:

- $\mathcal{I}_{C K, n}$ defined using this diagram the pullbacks under $j_{n}$ of functions vanishing on the image of $\mathrm{loc}_{n}$


## Quadratic Chabauty and ICERM Project

## Part III: Quadratic Chabauty and ICERM Project

## Quadratic Chabauty

- Let $X$ be a hyperbolic curve with Jacobian J
- The $p$-adic height pairing is a $\mathbb{Q}$-quadratic function:

$$
h: J(\mathbb{Q}) \rightarrow \mathbb{Q}_{p}
$$

- The $p$-adic height decomposes as a sum of local heights

$$
h_{v}: J\left(\mathbb{Q}_{v}\right) \rightarrow \mathbb{Q}_{p}
$$

for each place $v$ of $\mathbb{Q}$.

- For $v=p, h_{v}$ is given by a certain component of $j_{2}$
- For $v$ of potentially good reduction (not $p$ ), $h_{v}$ is trivial
- For $v$ not $p$ general, the image of $h_{v}$ is finite
- Recall log: $J\left(\mathbb{Q}_{p}\right) \rightarrow \operatorname{Lie}\left(J_{\mathbb{Q}_{p}}\right)$
- If $r=g$, we may write

$$
J(\mathbb{Q}) \otimes \mathbb{Q}_{p} \cong \operatorname{Lie}\left(J_{\mathbb{Q}_{p}}\right)
$$

and view $h$ as a $\mathbb{Q}_{p}$-quadratic function on $\operatorname{Lie}\left(J_{\mathbb{Q}_{p}}\right)$.

## Quadratic Chabauty (cont.)

- By computing $h$ on a basis of $J(\mathbb{Q})$, we may write a quadratic map ${ }^{1}$ $Q$ on $\operatorname{Lie}\left(J_{\mathbb{Q}_{p}}\right)$ explicitly, so that

$$
h(z)=Q(\log (z))
$$

for $z \in J(\mathbb{Q})$

- If $X$ has potentially good reduction everywhere, we may solve for $X(\mathbb{Q})$ in $X\left(\mathbb{Q}_{p}\right)$ via the equation

$$
Q(\log z)=h_{p}(z)
$$

for $z \in X\left(\mathbb{Q}_{p}\right)$

- If $X$ has permanent bad reduction, we must compute the finite images of $h_{v}: J\left(\mathbb{Q}_{v}\right) \rightarrow \mathbb{Q}_{p}$
${ }^{1}$ More precisely, $Q(\log z)=B(\log z, E \log z+c)$, for an endomorphism $E$ of $J$, $c \in \operatorname{Lie}\left(J_{\mathbb{Q}_{p}}\right)$, and a bilinear form $B$.


## Shimura Curves and Atkin-Lehner Quotients

- Shimura curves and their Atkin-Lehner quotients provide a good source of curves with permanent bad reduction
- For an integer $D$, we consider the Shimura curve $X^{D}$. This is the moduli space of abelian surfaces with action by the indefinite quaternion algebra of discriminant $D$
- If $\ell \mid D$, then $X^{D}$ and its Atkin-Lehner quotient have totally degenerate reduction at $\ell$
- This means that its semistable reduction is a union of copies of $\mathbb{P}^{1}$
- Furthermore, BSD implies that the Atkin-Lehner quotients have $r=g$


## ICERM Project

In our project, we use the $\ell$-adic geometry of curves with totally degenerate reduction to compute the local $p$-adic height at $\ell$ Joint with:

- Oana Adascalitei (BU)
- Jennifer Balakrishnan (BU)
- Netan Dogra (Kings College)
- Sachi Hashimoto (BU)
- Benjamin Matschke (BU)
- Ciaram Schembri (Dartmouth)
- Jan Vonk (IAS)
- Tian Wang (UIC)


## Thank You!

