Grothendieck–Teichümller avatars and the absolute Galois actions

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Route

This talk addresses various aspects of Grothendieck–Teichmüller theory, with an emphasis on the recent developments that I have done in collaboration with Yuri I. Manin and by myself. In particular, this evolves around Hurwitz spaces, Galois extensions, strongly inspired by Emmy Noether's program and Grothendieck–Teichmüller groups.

Plan of talk

– Part 1: 30-40' survey on classical results on GT theory and related topics;
– Part 2: highlight deep interactions between GT and QC, Hurwitz spaces, Emmy Noether's program (concerning hypercomplex numbers) + absolute Galois group/ $\mathbb{Q}$.

The mysterious Gal_Q

Interest in solving arithmetical problems by geometric means is not exactly new: it goes back already to Diophantus of Alexandria (about 250 C.E)

Nowadays, one big long standing problem that we are trying to understand is the mysterious

$$Gal_Q = Gal(\overline{\mathbb{Q}}/\mathbb{Q}),$$

being the group of automorphisms of the algebraic closure of the field of rational numbers $\overline{\mathbb{Q}}$ that fix $\mathbb{Q}$).[1].

[1] Formally:
$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) := \{\phi \,|\, \phi$ field automorphisms of $\overline{\mathbb{Q}}$ $s.t: \phi(x) = x \, \forall x \in \mathbb{Q}\}$

Brief History.

The mystery of Gal_Q has lead mathematicians to explore different paths, flavoured by geometry. Some took the path of Grothendieck–Teichmüller theory; others chose p-adic numbers and Hodge theory of p-adic varieties.

80's : Grothendieck–Teichmüller theory started to take shape.

- ★ Grothendieck proposed to study the mysterious absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ via its actions on *geometrical/ topological objects* rather than on algebraic numbers.

Grothendieck's idea

- Grothendieck noticed that $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts upon the algebraic fundamental groups of all the stacks $\mathcal{M}_{g,n}$ (and the natural maps relating these for various g and n forming a tower).
- He conjectured that $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ coincides with the *symmetry group* of this pro-finite completion.

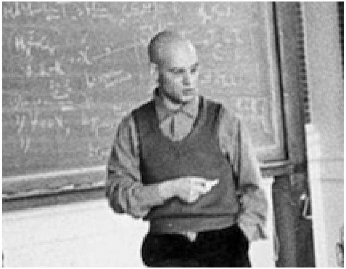


Figure: A. Grothendieck, IHES

Action on algebraic fundamental group

The **algebraic fundamental group**[2] $\pi_1^{alg}(\mathcal{M}_{g,n} \times_{\mathbb{Q}} \overline{\mathbb{Q}})$ has an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Theorem[SGA 1, Exp. IX, Theoreme 6.1]
Let V be a quasi-compact and quasi-separated scheme over $\mathbb{Q}$. Then, there is a short exact sequence

$$1 \to \pi_1^{alg}(V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \to \pi_1^{alg}(V) \to Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1,$$

of pro-finite topological groups[3].

This induces a homomorphism of Gal_Q into outer automorphism group $Out(\pi_1^{alg}(V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}))$

[2] Algebraic fundamental group = fundamental group for a scheme. It is the fundamental group seen through étale homotopy.

[3] Topological group G is pro-finite if it is homeomorphic to an inverse limit of finite groups $G \cong \lim_{\leftarrow} G_i$, where G_i are endowed with discrete topology

About $\overline{\mathcal{M}}_{g,n}$: Geometry & Arithmetics aspects

Divisorial stratification of $\overline{\mathcal{M}}_{g,n}$ supports at the same time *arithmetic* and *geometric* properties.

- Geometry: There is a question of defining, on smooth objects, an *operadic structure*, given by singular degeneracies;
- Arithmetic: Determine how $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is encoded within the geometric symmetries of $\overline{\mathcal{M}}_{g,n}$. This precisely is the Grothendieck–Teichmüller theory.

The symmetry group of this pro-finite completion

★ Drinfeld (90's): introduced the object that we call nowadays the Grothendieck–Teichmüller group $\widehat{GT}$, encapsulating the symmetries of the "profinite completion", where the following relation is given:

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \quad \text{injects into} \quad \widehat{GT}$$

The Grothendieck–Teichmüller group:

- acts on a wide variety of objects in many different fields of maths;
- remains mysterious: structure + relation to many of the objects it acts on is unclear and forms ongoing research.

1. The garden of divergent paths: a bird's eye view

Different shades of GT

Note that there exist three different versions of GT:

1. pro-finite $\widehat{GT}$ (Galois theory)– **invented first**;
2. pro-ℓ version GT_ℓ (Galois theory);
3. pro-unipotent GT^{un} (homological algebra).

★★★ conjectures ★★★

The conjecture outlined by Grothendieck is still open:

- the absolute Galois group Gal_Q and the pro-finite version of the Grothendieck–Teichmuller group are in bijection,
- Somehow, the conjecture also extends to the motivic Galois group and the pro-unipotent Grothendieck–Teichmüller group, conjectured also to be in bijection correspondence.

At the ♡ of other deep conjectures

Many developments have stimulated relations to other objects, showing thus the richness of the problem and of Grothendieck–Teichmüller groups. This is how, Grothendieck–Teichmüller groups find themselves at the heart of numerous deep and beautiful conjectures relating:

— MZVs, deformation quantization, motivic Galois group
— Double shuffle group, Kashiwara–Vergne's group, etc.

A bird's eye view of incarnations of GT^{un}

- Drinfeld: pioneered applications of GT in Nb th.+ th. of quasi-Hopf algebras;
- Etingof–Kazhdan: via GT, solved the Drinfeld quantization conj. for Lie bialgebras;
- Kontsevich–Tamarkin: the grp GRT is central in defo. quantizations of Poisson struct.;
- Alekseev–Torossian: GT gives sol. of Kashiwara–Vergne pb;
- Merkulov–Willwacher: interpret. as a sym. group of the invol. Lie bialgebra properad (important in string topology);
- Alekseev–Kawazumi–Kuno–Naef–Massuyeau: GT enters the Goldman–Turaev th.;
- Chan–Galatius–Payne: *grt* central in th. of cohomology grps of moduli spaces $\mathcal{M}_g$
- Willwacher: link grt and cohomologies of Kontsevich's graph.

The geometric (operadic) side of the story

★ Drinfeld has explicitly written down systems of algebraic equations defining elements of both pro-unipotent groups GT^{un} and GRT.

★ Due to Bar-Natan's works: this system of algebro–geometric meanings of these two systems of equations are well understood using theory of operads.

★ In short: GT is the automorphism group of the topological operad $\widehat{PaB}$ of parenthesized braids; GRT is the automorphism group of a much simpler graded analogue $\widehat{PaCD}$ of the operad $\widehat{PaB}$.

The set of Drinfeld associators can be identified with the set of isomorphisms $\widehat{PaB} \to \widehat{PaCD}$.

About $\widehat{PaB}$

An endomorphism of $\widehat{PaB}$ fixing the objects is uniquely specified by a pair (β, α), where:

- β is a morphism in $\widehat{PaB}(2)$ from (12) to (21)
- α is a morphism in $\widehat{PaB}(3)$ from (12)3 to 1(23) i.e. $\beta \in \hat{\mathbb{Z}}$ and $\alpha = (n, f) \in \hat{\mathbb{Z}} \times \hat{F}_2$.

The pair (β, α) is subject to the hexagon and pentagon relations.

Some known results

- Geometry: An element of the $\widehat{GT}$ group represents an automorphism (of the Malcev completion) of the operad of **parenthesized braids** PaB.

  Theorem *The monoid $\widehat{GT}$ is the monoid of endomorphisms of $\widehat{PaB}$ fixing objects (base points)*[4].
- Arithmetics: GT theory provides a finitely presented group $\widehat{GT}$ that contains $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

♣ This induces a $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $\widehat{PaB}$ that is group-theoretically defined and from topological origin.

[4] In the sense of Grothendieck

1.2. *Avant-propos*

Yu. Manin once said to me:

"[$\cdots$]During several last milleniums of development of modern geometry and geometric models of theoretical physics, it was gradually understood that geometry appears on the scene in two different guises:
– as space (or space–time) domains,
– and as their symmetries.

The idea of symmetry has mathematically crystallised relatively late: arguably, simultaneously with Galois's discovery of the role of symmetry in the theory of algebraic equations.

Revolutions of Relativity Theory and Quantum Physics brought with them comparatively fast understanding that space–time models and their symmetries are related by a kind of **duality***".*

This talk is an attempt to introduce listeners to the respective models and their dualities in their most abstruse contemporary incarnations ... as is usual for algebraists.

It is based upon our papers:

N. C. Combe, Y. I. Manin, *Hidden symmetry of genus 0 modular operad and absolute Galois group*, North-Western European Journal of Mathematics, vol. 8. (2022)

N. C. Combe, Y. I. Manin, *Hidden symmetry of genus 0 modular operad and its stacky versions*, Integrability, Quantization, and Geometry, Proceedings Symposia in Pure Mathematics, American Mathematical Society. DOI: 10.1090/pspum/103.2/01854

N. C. Combe, Y. I. Manin, M. Marcolli, *Dessins for Modular Operad and Grothendieck–Teichmuller Group*, July 2021,Volume 33, 537–560, Topology Geometry, European Maths Society.

and earlier Yu. I. Manin's Montreal lecture notes from 1988.

The topic I have chosen to expose today

- will not be about the motivic Galois group nor the pro-unipotent Grothendieck–Teichmüller group side of the story.
- But rather the pro-finite Grothendieck–Teichmüller group, allowing a slightly more number theory flavoured side unravelling some hidden symmetries. This is partially motivated also by the research of a notion of "duality" mentioned above.

Plan of the discussion

1. Recollections on the original pro-finite Grothendieck–Teichmüller group
2. Modular operads and Quantum cohomology
3. The *hidden symmetries* of $\overline{\mathcal{M}}_{0,n}$: a fruitful interaction between quadratic extensions investigated by Ihara–Deligne and symmetries coming from Quantum Field Theory and Gravity.
4. A modified GT group: the avatar of Grothendieck–Teichmüller groups

Origins

For today: consider the pro-finite $\widehat{GT}$. But $\widehat{GT}$ is difficult to handle

One of our joint works with Yu. I. Manin concern one remarkable fruit of the following interaction: creation of the theory of **quantum cohomology** (Kontsevich–Manin, 94') and subsequent discovery of its connections with one central objects of number theory: **(absolute) Galois group of the field of algebraic numbers**.

- Consider the situation from a new and innovative point of view, allowing to work with a more **combinatorial** version of $\widehat{GT}$: that I call an AVATAR of $\widehat{GT}$.
- **Features of GT's avatar**: more "usable" version, encapsulating the arithmetic data of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.
- **Characteristics**: relies on hidden symmetries of $\widehat{GT}$ and $Aut(\mathcal{M}_{0,n})$.

Pro-finite Grothendieck–Teichmüller group

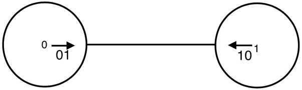
• Consider the case: $V = \mathbb{P}^1 \setminus \{\infty, 0, 1\}$

• Consider the morphism from the absolute Galois group $Gal_Q = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ to the outer automorphism group of the étale fundamental group of this scheme $\mathbb{P}^1 \setminus \{\infty, 0, 1\}$.

♣ (Topological) fundamental group: identified with the free group $Free(x, y)$, where x (respectively, y) is a loop turning around 0 (respectively, 1).

♣ (Algebraic) fundamental group:
$\pi_1^{et}(\mathbb{P}^1 \setminus \{\infty, 0, 1\}) = \widehat{Free}(x, y)$

We consider the variety $V = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ defined over $\mathbb{Q}$.
Naturally: $\pi_1(V) = Free(x, y)$, $Free(x, y)$ is the free group



generated by 2 elements x and y
where x is the loop around 0 and y is the loop around 1.

Hence, we have the following group-morphism:

$$\varphi : Gal_Q \to Out(\widehat{Free}(x, y)).$$

$\diamond$ By Belyi's theorem it is known that φ is *injective* $\diamond$.

The Deligne tangential base point approach

In the tangential base point approach (Deligne), one considers a loop x based at the tangent vector $\overrightarrow{01}$, the image of this loop under the map $\rho(z) = 1 - z$ (which forms a loop $\rho(x)$ based at $\overrightarrow{10}$ in the fundamental groupoid).

The loops x and $y = \gamma^{-1}\rho(x)\gamma$ (where γ is a path from $\overrightarrow{01}$ to $\overrightarrow{10}$) correspond to the previously considered generators of the fundamental group $\pi_1^{alg}(\mathbb{P}^1 \setminus \{\infty, 0, 1\})$, based at a point near 0.

The morphism $\varphi : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Out(\widehat{Free}(x,y))$ admits a lifting to the group of *automorphisms* of the free group $\widehat{Free}(x,y)$.

By **Drinfeld**, the $\widehat{GT} \subset Aut(\widehat{Free}(x,y))$ consisting of automorphisms ψ satisfying the following. For each $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, we have an automorphism $\psi(\sigma) : \widehat{Free}(x,y) \to \widehat{Free}(x,y)$ such that: $\lambda = \xi(\sigma)$ denotes the image of σ under the cyclotomic character $\xi : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^{\times}$.

♣ $f(x,y)$ satisfies these (pro-finite analogues of the) unit, involution, pentagon and hexagon relations:

$$\begin{cases} f(y,x) = f(x,y)^{-1} \\ f(z,x)z^m f(y,z)y^m f(x,y)x^m = 1, (xyz = 1) \\ f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23}) \end{cases}$$

where the last equation takes place in $\widehat{PB}_4$, the pro-finite completion of the pure braid group with 4 strands, with generators x_{ij}. Furthermore, $m = (1-\lambda)/2$ and $\lambda \in 1 + 2\widehat{\mathbb{Z}}$.

Drinfel'd:

$\widehat{GT}$ acts on the genus zero profinite groups via the formula

$$(\lambda, f)(\sigma_i) = f(\sigma_i^2, y_i)^{-1}\sigma_i^{\lambda} f(y_i, \sigma_i^2),$$

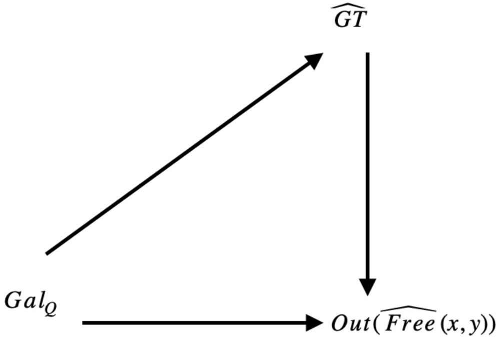
where $y_i = \sigma_{i-1} \dots \sigma_1^2 \dots \sigma_{i-1}$ (here notation has nothing to do with σ defined above).

Ihara–Matsumoto:

This action extends the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-action on these groups which occurs naturally by considering them as fundamental groups of moduli spaces and using a tangential base point.

Diagram

The outer action of the absolute Galois group Gal_Q factors uniquely through $\widehat{GT}$ fitting into the following commutative diagram:



1.3. One step higher with operads

One step higher with operads

The collection of objects $\overline{\mathcal{M}}_{0,n+1}$ belongs to the symmetric monoidal category of topological spaces.

More globally: Moduli spaces/stacks of stable curves of all genera with a finite number of marked points endowed with natural correspondences between them form a (modular) operad.

An **operad** is simply a type of (higher structure) algebra that can be defined over any symmetric monoidal category.

Definition (Borisov–Manin)

An operad $\mathcal{P}$ over a symmetric monoidal category $(A, \otimes)$ (called the ground category) is a **tensor functor** $(\Gamma, \sqcup) \to (A, \otimes)$, *where Γ is a category of finite (eventually labeled) graphs with disjoint union $\sqcup$ and morphisms including graftings.*

— In our context, graphs are forests having one labelled root at each connected component and a numbering $\{1, \cdots, n\}$ of all leaves on each connected component.
— Grafting means that one can connect roots to leaves.
— The notation $\mathcal{P}(n)$, $n \geq 1$ stands for the image of a tree with one root and n leaves totally ordered by the labels $\{1, \cdots, n\}$.
— The data completely determining such an operad are the set of morphisms in the ground category:

$$\mathcal{P}(k) \otimes \mathcal{P}(m_1) \otimes \mathcal{P}(m_2) \otimes \cdots \otimes \mathcal{P}(m_k) \to \mathcal{P}(n),$$

where $n = m_1 + \cdots + m_k$ called *operadic* multiplications.

The genus zero modular operad

♣ **genus zero modular operad**, (differently called the tree level part) of the big modular operad.
Structure morphisms (operadic multiplications) of genus zero modular operad is given by maps of moduli spaces defined point-wise by a glueing of the respective stable curves:

$$\overline{\mathcal{M}}_{0,k+1} \times \overline{\mathcal{M}}_{0,m_1+1} \times \cdots \times \overline{\mathcal{M}}_{0,m_k+1} \to \overline{\mathcal{M}}_{0,m_1+\cdots+m_k}$$

♣ Interest: comes from the fact that **mathematical formalism of quantum cohomology** involves the construction of an operad whose components are (co)homology groups of moduli spaces $\mathcal{M}_{g,n}$ of stable pointed curves.

Relation to GT

Quantum cohomology is:

$$QC = H^*(V, \mathbb{Q}) + DATA$$

One can take V as being $\overline{\mathcal{M}}_{0,n+1}$, the moduli space (a projective manifold) parametrising stable curves of genus zero with $n+1$ labelled points.[5]

We can decorate this object $H^*(\overline{\mathcal{M}}_{0,n+1})$ with the additional algebraic structure: called the *modular operad.*

[5] Taking genus 0 the coefficients of Φ of QC are genus zero Gromov–Witten invariants.

OPERADIC QUADRATIC DATA

Here the components of the genus zero modular operad are $\mathcal{P}(n) := H^*(\overline{\mathcal{M}}_{0,n+1}, \mathbb{Q})$. Structure morphisms (co-operadic comultiplications) :

$$\mathcal{P}(m_1+m_2+\cdots+m_k) \to \mathcal{P}(k)\otimes\mathcal{P}(m_1)\otimes\mathcal{P}(m_2)\otimes\cdots\otimes\mathcal{P}(m_k),$$

are maps induced by the maps of moduli spaces defined point-wise by a gluing of the respective stable curves:

$$\overline{\mathcal{M}}_{0,k+1} \times \overline{\mathcal{M}}_{0,m_1+1} \times \cdots \times \overline{\mathcal{M}}_{0,m_k+1} \to \overline{\mathcal{M}}_{0,m_1+\cdots m_k+1}.$$

Note that $\mathcal{P}(n) = H^*(\overline{\mathcal{M}}_{0,n+1}, \mathbb{Q})$ is a **quadratic algebra** and we are working in the **monoidal category** $(QA, \bullet)$ of quadratic algebras.

1.3 Short digression on Quantum cohomology and relations to GT

Considerable attention was attracted by the studies of interaction of modular operad (playing the central role in quantum cohomology constructions) with the celebrated Grothendieck–Teichmüller group.

Quantum cohomology can be defined as

$$QC = H^*(V, k) + \text{DATA}$$

where V is a projective algebraic variety with coefficients in k a field of char. 0.
The **"DATA"** can be described in at least three seemingly different ways.

- The simplest (to me) is to define it as: a formal series Φ (which will be called a "*potential*") in coordinates on H and such that its third derivative Φ_{abc} can be used to define the structure of *a commutative $\mathbb{Z}_2$-graded algebra on $k[[H^t]] \otimes H$*, where H^t is the (dual of H^*, by the Poincaré map).

Quantum cohomology: a (type of) formal Frobenius manifold

This potential function will sound familiar to those who studied the notion of **Frobenius manifolds**, given that quantum cohomology forms a class of *formal* Frobenius manifolds, if Φ is potential.

$\rightarrow$ It equips $k[[H^t]] \otimes H$, with the structure of a **Frobenius algebra** (commutative + associative and unital algebra with a symmetric bilinear map $\langle a \circ b, c\rangle = \langle a, b \circ c\rangle$, where a, b, c are elements of the algebra).

In other words: the ring $QC^*(V)$ is a formal deformation of the ring $H^*(V)$.

Quantum cohomology and number theory

Quantum cohomology operad has as its components cohomology spaces of $\overline{\mathcal{M}}_{g,n}$; its structure morphism corresponds to well defined morphisms of these moduli spaces. Taking for instance étale cohomology, the Galois group of algebraic numbers acts upon all operadic components, creating thus the connection between the theory of **quantum cohomology** and **number theory.**

Back to our initial Motivation

The considerable attention given to the interaction of the modular operad with the Grothendieck–Teichmüller group stimulated some of our research. We started looking more closely upon symmetries of the genus zero components of this operad and the respective quotient operad.

This lead us to the full automorphism group of genus 0 operad and opened questions about the possibility of transfer, in this combinatorial context, also an action of the absolute Galois group Gal_{Q}. This is how we introduced an avatar of GT i.e. the modified GT group.

Hidden symmetries of the Automorphism group $Aut(\overline{\mathcal{M}_{0,n+1}})$

Emmy Noether's program–ICM–Zurich 1932 ★

This project comes naturally as a continuation and development of problems that Emmy Noether outlined in 1932, at the International Congress of mathematics (Zurich):

$[\cdots]$ *First of all, it should be remarked that the main difficulty in obtaining the formulation for general Galois fields lies in the fact that no starting point at all will exist without the hypercomplex method.*[6]

[6] Hypercomplex numbers are a generalisation of complex numbers. Tessarines (bi-complex) are hypercomplex numbers, for instance.

Emmy Noether at ICM, Zurich 1932 ★

"I would, at the same time, like to explain the application of the non- commutative ideas to commutative ones: One seeks to arrive at invariant and simple formulations of the known facts regarding quadratic forms or cyclic fields by means of the theory of algebras — i.e., those formulations that depend upon only the structural properties of algebras. Once one has verified those invariant formulations (and that will be the case in the examples that were given above), one will have then obtained an adaptation of those facts to arbitrary Galois fields in doing so."

In the spirit of the words marked by Emmy Noether, we equip the moduli space $\overline{\mathcal{M}}_{0,n+1}$ with an algebraic structure. This algebraic structure is at the same time:

1. inspired from a construction by Deligne–Ihara (in particular, there is a symmetry which starts to dawn in their papers, given by $z \mapsto 1 - z$).
2. motivated by mathematical physics:

- used in quantum theory (Baez 2012, Varadarajan 1985)
- in general relativity and gravity (Gogberashvili 2014; Kulyabov Korolkova Geovorkyan 2020; Ulrych 2006.)

From a more algebraic perspective:
those physical requirements mean that to describe the geometry of the space we need a *composition algebra*.

For physical reasons it is not necessary to use field extensions, quadratic extensions over real numbers are enough.

So, let us equip $\overline{\mathcal{M}}_{0,n+1}$ with a given holomorphic involution θ.

The underlying algebra, over which are defined the corresponding modules, is a **composite normed algebra** and in particular a **split algebra**.

Here, the hidden symmetry is associated to the **split quaternion algebra**.

Table: Split-quaternion multiplication table

×	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	1	-i
k	k	j	i	1

We think that working on spaces being realisations of modules over these split algebras offers an interesting model to consider, encoding also physical data. We introduce this construction in order to define what we call a NY-gravity operad for this model and investigate the geometry around it.

About split algebras and composition algebras

About split algebras and composition algebras

Given A a finite dimensional; unital non-necessarily associative algebra over the field k (of characteristic not equal to 2), one can equip it with $N(x)$ a quadratic form defined on A.

Definition

A composition algebra A over a field k is an algebra equipped with a (non-degenerate) quadratic form N defined over A and such that for all x, y in A, we have

$$N(xy) = N(x)N(y).$$

A composition algebra can be classified either as a **division algebra** or as a **split algebra**.

Split algebras

Split algebras arise whenever there exist zero divisors which are non-null, i.e there exists a non-zero $x \in A$ such that $N(x) = 0$.

Example: split-complex algebra (also belongs to hypercomplex numbers).
It is a rank 2 algebra, generated by

$$\langle 1, \epsilon \,|\, \epsilon^2 = 1 \rangle.$$

We can see that it contains zero-divisors which are non null:

$$(1 - \epsilon)(1 + \epsilon) = 1 - \epsilon^2 = 1 - 1 = 0.$$

Here we see that $(1 - \epsilon)$ and $(1 + \epsilon)$ both divide 0 (and are different from 0).

Division algebras

A division algebra is an algebra over a field where every non-zero element has an inverse.
In particular: *a finite-dimensional, unital, associative algebra (over any field) is a division algebra if and only if it has no zero divisors.*
In fact, by Milnor + Kervaire we have that: *any finite-dimensional real division algebra must be of dimension 1, 2, 4, or 8*.

About DM stack with special θ symmetry.

DM Stacks

Consider the DM–stack $\mathcal{M}_{0,n+1}$. Define a stable S–labeled curve $(C/T, \pi, (x_i) \mid i \in S, \, |S| = n+1)$, where:

- T is a base scheme in the category $Sch_{\mathbb{Q}}$,
- C is a scheme (in the category $\mathcal{F}$),
- $\pi : C \to T$ is a proper flat morphism.

For any geometric point $t \in T$, the sections $x_i(t)$ are smooth on C_t and for a given $i \neq j$, the section $x_i(t)$ is different from $x_j(t)$. The stack $\mathcal{M}_{0,S}$ has as S-fiber an $n+1$-pointed curve $C \to T$ with $n+1$ sections $T \to C$ (having disjoint images). A section of C is a morphism of T-schemes defined from T to C such that composed with π one obtains the identity Id_T.

The special θ symmetry

Consider an affine orientable symmetry group

$$G = \langle \theta | \theta^2 = Id \rangle,$$

where for any $x \in \mathbb{C}$,

$$\theta : x \mapsto 1 - x;$$

and consider the representation of this group as follows. Let C/T be an n-pointed stable curve:

$$G \to \mathrm{Aut}(C/T, (x_i)).$$

Proposition

Let $\overline{\mathcal{M}}_{0,S}$ be the category of S–labeled stable curves of genus 0 and $\overline{\mathcal{M}}^{\theta}_{0,S}$ the category of S–labeled stable curves of genus 0, obtained by the action of G on objects of $\overline{\mathcal{M}}_{0,S}$. Then, $b : \overline{\mathcal{M}}_{0,S} \times \overline{\mathcal{M}}^{\theta}_{0,S} \to Sch_{\mathbb{Q}}$ is a groupoid.

Theorem

The category fibered in groupoid $\overline{\mathcal{M}}_{0,S}^{\theta}$ over $Sch_{\mathbb{Q}}$ of S-pointed stable curves with θ-symmetry is a stack.

Theorem (Comparison theorem)

Consider the NY Gravity operad $Grav_{NY}$. Then, for $n \geq 3$ we have:

$$Grav_{NY}(n) = sH_*(\mathcal{M}_{0,n} \setminus Fix_\theta) \otimes H_*(\mathcal{M}_{0,n}^{\theta} \setminus Fix_\theta)$$

where Fix_θ is the set of fixed points of the automorphism θ.

Beyond the actual versions of GT

Aim: introduce an avatar of GT keeping track of arithmetical data but allowing also some combinatorial structure.

Previously, considerations of **symmetries of the genus zero components** with focus upon a very particular symmetry of order two have been done.

★ Take into account the **whole family** of **automorphisms of $\overline{\mathcal{M}}_{0,n}$** + combine them into two "locally finite" combinatorial objects (posets in groupoids) reflecting their compatibilities with operadic compositions.

This leads us to two versions of the notion of "full automorphism group" of the genus zero operad and opens a question about possibility to transfer in this combinatorial context also an action of the absolute Galois group.

Two more steps towards the "higher level" structures

[7]

[7](but without using ∞ categories)

• Denote by $\mathbb{N}$ the set of natural numbers.

• Subset $S \subset \mathbb{N}$ is called *cofinite* (cf for brevity), if its complement is finite.

• A map $f : \mathbb{N} \to \mathbb{N}$ is a *cf–map*, if it is identical on an appropriate cf–set (that may depend on f).

Claim.

(a) Composition of two cf–maps is a cf–map.

(b) Bijective cf–maps form a group wrt composition.

Denote the latter group $\mathbb{S}_\infty$.

Part 1

$\mathbb{S}_\infty$ coincides with the union $\cup_{n=1}^{\infty}\mathbb{S}_n$, in which every symmetric group $\mathbb{S}_n$ is the full group of bijective cf–maps identical outside $\{1,\dots,n\}$.

Proposition

1. $\mathbb{S}_\infty$ *acts upon the family of operadic components* $\overline{\mathcal{M}}_{0,n}$, $n \geq 3$.
2. *This family of automorphisms* $\{\sigma_{\mathcal{M}}\}$ *can be naturally and uniquely extended to the family of all operadic compositions between the components.*

Description of the action on $\overline{\mathcal{M}}_{0,n}$

- Indeed, let $C_{0,n} \to \overline{\mathcal{M}}_{0,n}$ be the **universal family** of stable curves of genus zero endowed with n structure sections $s_i : \overline{\mathcal{M}}_{0,n} \to C_{0,n}$.

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- This map $\mathbb{S}_n \to Aut\,\overline{\mathcal{M}}_{0,n}$ is surjective for all $n \geq 3$ and bijective for $n \geq 5$.

Thin categories (defined for expressing stable trees)

Definition

A category $\mathcal{C}$ is called thin if

- *for any two objects X, Y, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ consists of ≤ 1 element, and*
- *if both $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ are non–empty, then $X = Y$.*

Remarks

It follows that all automorphisms of X, Y act upon $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ as identity, so $\mathcal{C}$ is equivalent to a category for which $\mathrm{Hom}_{\mathcal{C}}(X, X) = \{id_X\}$ for any object X, which we will temporarily assume.

Remark: *For such a category,* $\mathrm{Ob}\,\mathcal{C}$ *has a canonical structure of a poset*[8]*:* $X \leq Y$ *iff* $\mathrm{Hom}(X, Y)$ *is nonempty.*

[8] A *poset* (partially ordered set) is a set $\mathcal{S}$ endowed with a binary relation $\leq$ which is reflexive, transitive and anti–symmetric.

Poset in groupoids

A poset in groupoids[9] is a category $\mathcal{PG}$, satisfying the following:

1. For any object X, the **full subcategory** of $\mathcal{PG}$ consisting of all objects isomorphic to X is a **groupoid**.
2. If X and Y are **not isomorphic** and $\mathrm{Hom}(X, Y)$ is **non–empty,** then $\mathrm{Hom}(X, Y)$ has a single orbit with respect to the precomposition by the automorphism group of X and postcomposition by the automorphism group of Y.

[9] A category G is a groupoid, if all morphisms in it are isomorphisms

Posets in groupoids–Universal functor–Thin categories

There is a **natural universal functor** from *posets in groupoids* to *thin categories*, identical on objects and identifying all morphisms in each non–empty $\mathrm{Hom}(X, Y)$.

Statement

Consider the poset $\mathcal{N}_*$:

- elements are subsets $\mathbf{n} := \{\mathbf{1}, \dots, \mathbf{n}\}$, $n = 1, 2, \dots \in \mathbb{N}$,
- binary relation $\mathbf{m} \leq \mathbf{n}$ iff $\mathbf{m} \subseteq \mathbf{n}$.

It forms a **thin category** whose morphisms are cf–maps coinciding with usual embeddings.

Corollas and automorphism group of $\overline{\mathcal{M}}_{0,T}$

The widest categorical context in which we can imagine symmetries of the genus zero operad would involve all forests τ in a small universe and respective spaces $\overline{\mathcal{M}}_{0,\tau}$ and $\mathcal{M}_{0,\tau}$.

We restrict our-selves to *corollas*, that is **finite sets T** marking points on stable curves, and moduli spaces $\overline{\mathcal{M}}_{0,T}$.

The automorphism group of $\overline{\mathcal{M}}_{0,T}$ can be canonically identified with the group of bijections $T \to T$.

Now, in order to study the symmetries of the whole operad generated by $\overline{\mathcal{M}}_{0,T}$ we must connect these finite permutation groups $AutT$ by families of chosen morphisms with respect to which one could pass to some meaningful limits. This is what we explained previously on the structure and action of the permutation group $\mathbb{S}_{\infty}$.

This last part performs this job producing another "infinite permutation group" mGT which is a combinatorial version of the (profinite) Grothendieck–Teichmuller group. We start with showing how to include groupoids of finite sets into a different poset of groupoids.

The larger category $\mathcal{N}_*^{cf}$

Extend $\mathcal{N}_*$ to a larger category $\mathcal{N}_*^{cf}$.

- same set of objects as in $\mathcal{N}_*$,
- larger set of morphisms: $\mathrm{Hom}_{\mathcal{N}_*^{cf}}(\mathbf{m}, \mathbf{n})$ consists of **all cf–maps** obtained by precomposition of a permutation of **m**, standard embedding **m** into **n** and postcomposition with a permutation of **n**.

$\mathcal{N}_*^{cf}$: the covariant skeleton of symmetries

Proposition

1. *$\mathcal{N}_*^{cf}$ is a poset in groupoids.*
2. *This poset (restricted to $n \geq 3$) acts term-wise upon the poset of components of modular operad $\overline{\mathcal{M}}_{0,n}$ whose morphisms are generated by standard embeddings of locally closed strata.*

$\mathcal{N}_*^{cf}$ is *"the covariant skeleton of symmetries"* of our operad.

Cocovariant skeleton of symmetries

Denote by $\mathcal{N}^*$ the poset (we can call it a cocovariant skeleton of symmetries):

- same elements as $\mathcal{N}_*$
- Different ordering: $\mathbf{p} := \{\mathbf{1}, \dots, \mathbf{p}\}$ and $\mathbf{q} := \{\mathbf{1}, \dots, \mathbf{q}\}$ then in $\mathcal{N}^*$, $\mathbf{p} \leq \mathbf{q}$ means

$$p \quad \text{divides} \quad q$$

Remark: The map $\mathcal{N}^* \to \mathcal{N}_*$ identical on elements, is a **bijection** compatible with respective order relations (but the inverse map is not compatible).

Cyclotomic fields

Consider the field generated by the roots of unity.
★ It is an algebraic number field, given by the cyclotomic field $\mathbb{Q}(\zeta_n)$, where $\zeta_n = \exp(2\pi\imath/n)$ (for $n \geq 1$) is a *number field* obtained from $\mathbb{Q}$ by adjoining a primitive n-th root of unity.
★ This field contains all complex n-th roots of unity.

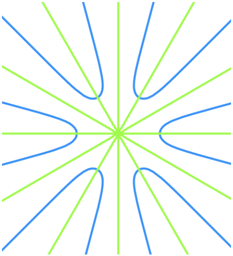


Figure: $z^6 - 1$

Highlight over some properties

The *n*th cyclotomic polynomial:

$$\prod_{1\leq k\leq n,(k,n)=1} (z - \exp(2\pi\imath k/n)) = \prod_{1\leq k\leq n(k,n)=1} (z - \zeta_n^k)$$

is the minimal polynomial of ζ_n over $\mathbb{Q}$.
— **Conjugates** of ζ_n in $\mathbb{C}$ are the other primitive *n*th roots of unity ζ_n^k for $1 \leq k \leq n$ with $(k,n) = 1$.
— The roots of $z^n - 1$ are the powers of ζ_n, so $\mathbb{Q}(\zeta_n)$ is the splitting field of $z^n - 1$ over $\mathbb{Q}$. So, $\mathbb{Q}(\zeta_n)$ is a Galois extension of $\mathbb{Q}$.

The Galois group $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is naturally isomorphic to the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$. These consist of the invertible residues modulo n, which are the residues $a \mod n$ with $1 \leq a \leq n$ and $gcd(a, n) = 1$.

The isomorphism sends each $\sigma \in Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ to $a \mod n$, where a is an integer such that $\sigma(\zeta_n) = \zeta_n^a$.
If q is a prime not dividing n, then the Frobenius element $Frob_q \in Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ corresponds to the residue of q in $(\mathbb{Z}/n\mathbb{Z})^\times$.

$\rightarrow$ Replace each group $\mathbb{Z}/n\mathbb{Z}$ by the group of roots of unity of degree n, by applying the following map:

$$a \mod n \mapsto \exp(\frac{2\pi\imath a}{n}).$$

Then, apply the following fact that the action of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^{\times}$ then becomes the action of *the Galois group*:

$$\exp(\frac{2\pi\imath a}{n}) \mapsto (\frac{2\pi\imath da}{n}).$$

In fact, the splitting field over $\mathbb{Q}$ is $\mathbb{Q}(\zeta_n)$ and has the automorphisms

$$\zeta_n \mapsto \zeta_n^d$$

for $1 \leq d \leq n$ where $(d, n) = 1$.

Interpretation and connection to Frobenius manifolds: deformations in Saito space (aka: Hurwitz maps of genus 0)

On the operadic level, we can imagine $\mathcal{M}_{0,\mu_n}$ as *moduli spaces* of deformations of compactified G_n with roots of unity 0 and ∞as marked points.

Illustration of the deformation of z^6-1 (using my PhD).

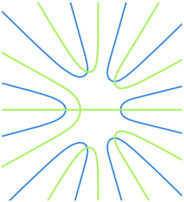
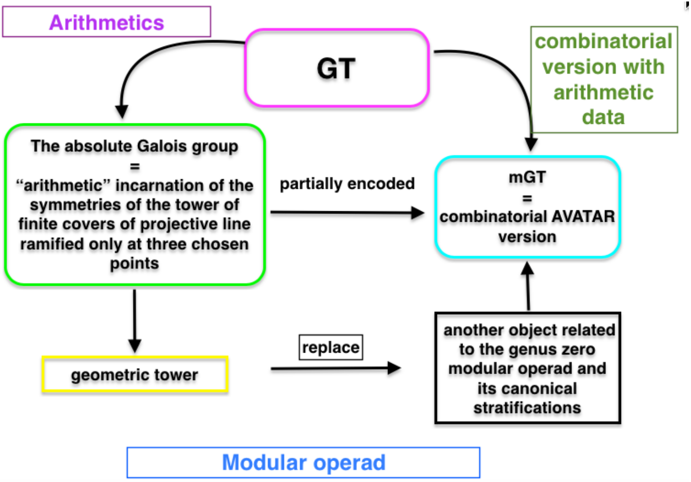


Figure: z^6-1+z^4+z

The avatar of the Grothendieck–Teichmüller group


Arithmetics
GT
combinatorial version with arithmetic data
The absolute Galois group = "arithmetic" incarnation of the symmetries of the tower of finite covers of projective line ramified only at three chosen points
partially encoded
mGT = combinatorial AVATAR version
geometric tower
replace
another object related to the genus zero modular operad and its canonical stratifications
Modular operad

• Aim: Introduce **mGT**, a combinatorial version of the (profinite) Grothendieck–Teichmüller group, pre-containing arithmetical data.

Let us introduce the family of commutative rings $\mathbb{Z}/q\mathbb{Z}$, $q \geq 3$, related by the family of (ring) homomorphisms

$$t_{q,p} : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}, \quad a \bmod q \mapsto a \bmod p$$

for each pair of natural numbers p, q such that p divides q.

Transitivity

If p divides q,r which in turn divide s, then

$$t_{q,p} \circ t_{s,q} = t_{r,p} \circ t_{s,r} = t_{s,p}.$$

Now, residue classes of all $d \bmod q$ with $\mathrm{g.c.d.}\,(d,q) = 1$, form the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$. Hence multiplications of $\mathbb{Z}/q\mathbb{Z}$ by them are permutations that also act compatibly with all $t_{q,p}$.

The avatar of the Grothendieck–Teichmüller group

Definition

The group $\mathbf{mGT_q}$ *is defined as the subgroup of permutations of* $\mathbb{Z}/q\mathbb{Z}$*, generated by the following maps:*

1. *multiplications by all elements* $d \in (\mathbb{Z}/q\mathbb{Z})^*$*;*
2. *the involution* $\theta_q : a \mapsto 1 - a$*.*

N.B: $\mathbf{mGT_q}$ is not commutative.

Proposition

For each p, q with p/q, define the homomorphism $u_{q,p} : \mathbf{mGT_q} \to \mathbf{mGT_p}$ by the following prescription: each permutation of $\mathbf{q} \in \mathcal{N}^*$ belonging to $\mathbf{mGT_q}$ is compatible with each map $t_{q,p}$ and after applying $t_{q,p}$ determines a group homomorphism

$$u_{q,p} : \mathbf{mGT_q} \to \mathbf{mGT_p}$$

These homomorphism satisfy the following relations: if p divides q,r which in turn divide s, then

$$u_{q,p} \circ u_{s,q} = u_{r,p} \circ u_{s,r} = u_{s,p}.$$

Corollary

There exists a well defined group **mGT**, *"modified profinite Grothendieck–Teichmüller group", which is the projective limit of groups* $\mathbf{mGT_q}$ *with respect to the homomorphisms* $u_{q,p}$.

Thanks