Integral points on elliptic curves

Elliptic curve $/\mathbb{Q}$ $\quad E: y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{Z}$

Assume $p^4 \mid A \Rightarrow p^6 \nmid B$ (quasi minimal)

$\Delta(E) = -16(4A^3 + 27B^2) \neq 0$

Integral points $E(\mathbb{Z}) = \{(x,y) \in \mathbb{Z}^2 : y^2 = x^3 + Ax + B\}$

Theorem (Mordell) $\#E(\mathbb{Z})$ is finite

Conjecture (Lang) $\#E(\mathbb{Z})$ can be bounded above only in terms of $r = \operatorname{rank} E(\mathbb{Q})$

Theorem (Hindry-Silverman) $\#E(\mathbb{Z}) \ll C^{(r+1)\sigma}$, $\sigma = \dfrac{\log \Delta}{\log \text{conductor}}$
(Szpiro ratio)

Average / density results

Order by height $H(E) = \max(4|A|^3, 27B^2)$

Theorem (Alpoge) ① 100% of curves satisfy $\#E(\mathbb{Z}) \ll 3^r$

② $\overline{\mathrm{Av}}(\#E(\mathbb{Z})) = \limsup_{N \to \infty} \dfrac{1}{\#\{H(E) \leq N\}} \sum_{H(E) \leq N} \#E(\mathbb{Z}) < 66$

Idea Average pointwise bound ① then apply

$\overline{\mathrm{Av}}(5^r) \leq \mathrm{Av}(5\text{-Selmer}) = 6$ (Bhargava-Shankar)

Conjecture ① 100% of curves $\#E(\mathbb{Z}) = 0$

② $\mathrm{Av}\, \#E(\mathbb{Z}) = 0$

Note ② $\Rightarrow$ ①

Expect the conj. to hold for any "reasonable" family after removing any "trivial" points

Related open question: Estimate $\#\{E/\mathbb{Q} : |\Delta(E)| \leq N\} = o(N)$

Equivalent to counting integral points $y^2 = x^3 - 1728\Delta$, $|\Delta| \leq N$

<u>Heuristics</u> (Brumer–McGuiness) $\sim c N^{5/6}$

Restricting to quadratic twists families

Fix $E: y^2 = x^3 + Ax + B$, consider $E_d: y^2 = x^3 + Ad^2x + Bd^3$

Order by size of $d \in \mathbb{Z}$, $d > 0$ sqfree

Define the non-trivial integral points $E^*(\mathbb{Z}) := \{(x,y) \in E(\mathbb{Z}) : y \neq 0\}$

<u>Example</u> $E_d: y^2 = x^3 - d^2x$, $(0,0), (\pm d, 0) \in E_d(\mathbb{Z})$

<u>Conjecture</u> Fix E, ① 100% of E_d satisfy $\#E^*(\mathbb{Z}) = 0$

(Granville)

② $\mathrm{Av}(\#E_d^*(\mathbb{Z})) = \lim_{N\to\infty} \frac{1}{\#\{d \leq N \text{ sqfree}\}} \sum_{\substack{d \leq N \\ \text{sqfree}}} \#E_d^*(\mathbb{Z}) = 0$

<u>Theorem</u> (Matscke–Mudigonda) Suppose $x^3 + Ax + B$ is reducible$/\mathbb{Q}$ then abc conjecture implies a version of ①

<u>Theorem</u> (C.) Let $E_d: y^2 = x^3 - d^2x$. $\sum_{\substack{d \leq N \\ \text{sqfree}}} \#E_d^*(\mathbb{Z}) \ll N(\log N)^{-\frac{1}{8}+\varepsilon}$

so $\mathrm{Av}(\#E_d^*(\mathbb{Z})) = 0$

Proof for $E_d: y^2 = x^3 - d^2x$

<u>Theorem</u> (Heath-Brown) $\mathrm{Av}(\mathrm{Sel}_2(E_d)^k) \sim c_k$

Combine with known cases of Lang's conjecture

$\#E_d(\mathbb{Z}) \ll c^r$

Since rank $\leq$ 2-Selmer rank, $\quad C^r \leq (\mathrm{Sel}_2(E_d))^k$

as long as k is large enough.

Then $\overline{\mathrm{Av}}(\# E_d(\mathbb{Z})^k)$ are all bounded in terms of k

<u>Mordell correspondence</u>

$$\{(E,(x,y)) : (x,y) \in E(\mathbb{Z})\} \longleftrightarrow \left\{ \begin{array}{l} f(X,Y) = X^4 + 6cXY^2 + 8dXY^3 + eY^4 \\ c,d,e \in \mathbb{Z}, \quad e \equiv c^2 \bmod 4 \\ \Delta(f) = 2^8 \Delta(E) \end{array} \right\}$$

($SL_2(\mathbb{Z})$-equivalence classes of integral binary quartic forms that represents 1)

<u>Δ-reducing Lemma</u> Suppose $(x,y) \in E_d(\mathbb{Z})$, take f to be the quartic form in the image of (x,y).

Let p be a prime $\boxed{p \mid d, \; p \nmid 6x.}$

Then $\exists\, k \in \mathbb{Z}$ such that

$$F(X,Y) = \frac{1}{p^3} f(pX + kY, Y) \text{ is a int. binary quartic}$$

with
- $\Delta(F) = \dfrac{\Delta(f)}{p^6} \in \mathbb{Z}$, and $(\Delta(E_d) = (2d)^6)$
- represents p

<u>Main ideas</u>

$\approx \exp((\log d)^{\varepsilon})$

$\star$ Reduce Δ by some large p, then we are left with $o(N)$ quartic forms.

For 100% of E_d, we can find p of good size for every $(x,y) \in E_d^*(\mathbb{Z})$ by Heath-Brown.

* Each Δ-reduced form can only be the image of very few integral points.
Thue's inequality $0 < |F(X,Y)| < h$ have very few solutions as long as h is small relative to Δ.

Cubic twists

Theorem (C.) Fix $k \neq 0, 1$ squarefree.
Let $E_d : y^2 = x^3 + kd^2$
then $\#\{1 \leq d \leq N : E_d(\mathbb{Z}) = \emptyset\} \ll_k N(\log N)^{-\frac{1}{2}+\varepsilon}$