• jt work in progress w. J. Newton.

Thm A (Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor) Let $E/\mathbb{Q}$ be an elliptic curve. Then E is modular, i.e. $\exists$ $f \in S_2(\Gamma_0(N), \mathbb{C})$, Hecke eigenform s.t. $\mathcal{H}$ $\Gamma_0(N)$

(*) $a_p(f) = a_p(E)$ $\forall p \nmid N$ prime

$a_p(f)$: eigenvalue of T_p acting f

$a_p(E) = p+1-\#E(\mathbb{F}_p)$

(* can be obtained from $\rho_{f,\ell} \simeq \rho_{E,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_\ell)$)

Outline

① Prove "modularity lifting theorem":

• $\bar{\rho}_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to GL_2(\mathbb{F}_p)$

$\mathrm{Gal}(\overline{F}/F)$

assume this is modular & has

large image

· then prove that

$$\rho_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_p)$$

is modular

(works in Barsotti-Tate case, i.e. crystalline w HT wts $\{0,1\}$)

via Taylor-Wiles ~~Kisin~~ patching
Calegari-Geraghty

② $p = 3$, show that

$$\bar{\rho}_{E,3} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_3)$$

is modular

Allen-Khare-Thorne

solvable

Langlands-Tunnell thm gives modularity

(congruence from wt 1 to wt 2)

③ 3-5 modularity switch
3-7 modularity switch

$$\bar{\rho}_{E,5} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_5)$$

is modular

by finding $E'/\mathbb{Q}$ $\curvearrowright G_{\mathbb{Q}}$

s.t. $E'[5] \simeq E[5]$

& $E'[3]$ has large image $\curvearrowright \bar{\rho}_{E',3}$

to find E': produce rat'l points

$X_E(5) \simeq \mathbb{P}^1$

not able to do 3-7 switch yet

$X_E(7)$: twist of Klein quartic, harder to produce points

④ understand exceptions:

both $\bar{\rho}_{E,3}$ & $\bar{\rho}_{E,5}$ have "small image"

e.g. if they are both reducible

$\rightsquigarrow$ E gives rise ~~to~~ a point in

$X_0(15)(\mathbb{Q})$

exceptions: pts on modular

"curves of genus >1

$\hookrightarrow$ elliptic curve, MW rk 0

8 points ...

<u>Thm B</u> (Freitas - Le Hung - Siksek, 2013)

Let F be a real quadratic field & let E/F be an elliptic curve.

Then E is modular, i.e.

$\exists$ Hilbert modular form of parallel weight 2 w. right system of Hecke eigenvalues.

<u>Rk</u>: fn on $\Gamma \backslash \mathcal{H} \times \mathcal{H}$

$\Gamma \subset SL_2(\mathcal{O}_F)$ congruence subgp

contributes to

$$H^2(\Gamma\backslash\mathcal{H}\times\mathcal{H}, \mathbb{C})$$

If $F/\mathbb{Q}$ is an imaginary quadratic field, then there is a qualitative difference: want to understand

$$H^1(X_\Gamma, \mathbb{C})$$

hypabolic 3-space

where $X_\Gamma \simeq \Gamma\backslash\mathcal{H}^3$

no algebraic structure!

$\Gamma \subset SL_2(\mathcal{O}_F)$

Existence of Galois rep's more subtle: Harris-Lan-Taylor-Thorne
Scholze (applies to $H^1(X_\Gamma, \overline{\mathbb{F}}_p)$)

Thm C (C-Newton, in progress)

F im. quad, E/F ell curve,

non-CM,

Hypothesis (H) : for $p=3$ or 5, the action of $\mathrm{Gal}(\overline{F}/F(\zeta_p))$ on $E[p]$ is abs. irreducible. Then E is modular.

Remarks :

1). Zywina shows that, for a fixed F, hypothesis (H) is satisfied by 100% of elliptic curves

2). Hope to prove modularity of all E/F when F is s.t. $X_0(15)$ has MW rk 0 over F

e.g. $F = \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$

3). Previous results:

- Allen-Khare-Thorne: modularity of positive proportion of E.C. over a CM field.

- Whitmore: modularity of positive proportion of E.C. over any quad ext'n of tot. real field.

  refines: Boxer-Calegari-Gee-Pilloni

<u>Key ideas</u>

crucial for Kisin-style MLT

1). prove local-global compatibility in crystalline case for Galois representations constructed by Scholze.

New technique that combines ideas from "ten author paper" with P-ordinary parts ($GL_2/F \rightsquigarrow$ Levi inside Siegel parabolic P of $U(2,2)$)

2). refined "degree-shifting" at auxiliary primes, works in ramified case.

Lambert
A'Campo

How to understand exceptions:

$$\mathcal{C} = X((ns3)^\circ, b5)$$

E s.t. $\operatorname{Im}(\bar\rho_{E,3})$

is contained inside non-split Cartan of $GL_2(\mathbb{F}_3)$

$\mathrm{Im}(\bar\rho_{E,5})$ contained inside Borel of $GL_2(\mathbb{F}_5)$

equation:

$$y^2 = -3\,(x^4 + 2x^3 - x^2 + 10x + 25)$$

genus 1, no $\mathbb{Q}$-points

Flowe injection:

$$\begin{array}{ccc} \mathrm{Pic}^0_{\mathcal{C}}(\mathbb{Q}) & \hookrightarrow & \mathrm{Jac}_{\mathcal{C}}(\mathbb{Q}) \\ \wr\wr & & \wr\wr \\ \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \end{array}$$

(x,y)

$P + P^{\alpha}$ can be:
- trivial class
  j-invariant $\in \mathbb{Q}$, modular
- non-trivial class

$\alpha \in \mathrm{Gal}(F/\mathbb{Q})$ non-trivial

$x \cdot \bar{x} = 5$

ρ & ρ^{σ} represent isogenous ell curves

$\rightsquigarrow$ E is a $\mathbb{Q}$-curve, modular