Iwasawa Theory for Class Group Schemes in characteristic p

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k - finite field of char $p>0$

X/k curve (smooth, proj. geom conn'd)

$$Cl_X = \frac{\{\text{degree 0 divisors on } X\}}{\{\text{principal divisors}\}} = \underbrace{J_X}_{\text{Jacobian}}(k)$$

= finite, abelian group.

Iwasawa theory: How do class groups grow in $\mathbb{Z}_\ell$ - towers of curves?

Def: ℓ prime. A $\mathbb{Z}_\ell$-tower over X is: $\{X_n\}_{n\geq 0}$, $X_n \to X_0 = X$ branched Galois cover of curves with group $\mathbb{Z}/p^n\mathbb{Z}$, unram'd outside $S \subseteq X(k)$, totally ram'd over S.

CFT: No such towers if $\ell \neq p$ or $S=\emptyset$!

$$\left\{ \begin{matrix} \mathbb{Z}_p\text{-towers}/X \\ \text{ram'd over } S\neq\emptyset \end{matrix} \right\} \leftrightarrow \left\{ \rho : \pi_1^{ét}(\underbrace{X_{\bar{k}} - S}_{\text{Affine!}}) \twoheadrightarrow \mathbb{Z}_p \right\}$$

$$\pi_1^{ét}(\text{Affine})^{(p)} = \text{free pro-}p \text{ on countably infinite gens!}$$

max'l pro-p qt.

Thm (Gold–Kiselevsky '84; Mazur–Wiles '86)
Let $\{X_n\}$ be a $\mathbb{Z}_p$-tower. Then

$$|Cl_{X_n}[p]| = p^{\mu p^n + r} \quad \text{for some } \mu, r \in \mathbb{Z} \text{ and all } n \gg 0$$

$Cl_{X_n}[p] = J_{X_n}[p](k) = k$-points of

$$\underbrace{J_{X_n}[p]}_{\text{finite group scheme}} := \operatorname{Ker}(p : J_{X_n} \to J_{X_n})$$

Algebraic group!

Any finite gp. scheme G/k decomposes

$$G = G^{\text{ét}} \times G^{\circ}$$

$G(k) = G^{\text{ét}}(k)$ <u>totally</u> <u>misses</u> G°!

<u>Ex</u>: Any $\mathbb{Z}_p$-tower $\{X_n\}$ with $X_0 = \mathbb{P}^1$ and $S = \{\infty\}$ has $Cl_{X_n}[p] = 0$, all n, yet $J_{X_n}[p]$ has dimension $g_n = \text{genus } X_n$ and $g_n \geq c p^{2n}$, some $c > 0$.

<u>Refined</u> <u>Iwasawa</u> <u>Theory</u>: How do the group schemes $J_{X_n}[p]$ grow in $\mathbb{Z}_p$-towers of curves?

<u>No</u> analogue for number fields!

Basic tool: Dieudonné Theory

$$\left\{\begin{matrix}\text{finite gp. sch}/k\\ \text{killed by } p\end{matrix}\right\} \xrightarrow[G \rightsquigarrow \mathbb{D}(G)]{\cong} \left\{\begin{matrix}\text{finite dim'l } k\text{-v.s. } M + \\ F, V : M \to M \\ F\lambda = \lambda^p F,\ \lambda V = V\lambda^p,\ FV = VF = 0\end{matrix}\right\}$$

<u>Thm</u> (Oda) X/k curve.

$$0 \leftarrow J_X[F] \xleftarrow{V} J_X^{(p)}[p] \leftarrow J_X^{(p)}[V] \leftarrow 0$$

$$\Big\downarrow \mathbb{D}(-)$$

$$0 \to H^0(\Omega^1_X) \to H^1_{dR}(X) \otimes_{k \circlearrowleft \sigma} k \to H^1(\mathcal{O}_X) \otimes_{k \circlearrowleft \sigma} k \to 0$$

$F \equiv 0$ $\quad$ F = Frobenius, V = Cartier $= F^\vee$ $\quad$ $V \equiv 0$

$k[V]$-module str. of $H^0(\Omega^1_X)$ <u>determines</u> $J_X[F]$, $J_X[V]$

$$\boxed{\text{How does the } k[V]\text{-module } M_n := H^0(\Omega^1_{X_n}) \text{ grow in a } \mathbb{Z}_p\text{-tower } \{X_n\} \text{ of curves?}}$$

$\dim_k M_n = g_n$. By Riemann–Hurwitz:

$$2g_n - 2 = p^n(2g_0 - 2) + \sum_{Q \in S} \sum_{i=1}^{n} \varphi(p^i)\, s_i(Q)$$

i^{th} upper ram. break

Rem: All ramification is wild! $\Rightarrow$

$$\frac{s_{n+1}(Q)}{s_n(Q)} \geq p \text{ and can be unbounded!}$$

$\Rightarrow g_n \geq c p^{2n}$, $c > 0$ and can grow arbitrarily fast!

Def: A $\mathbb{Z}_p$-tower $\{X_n\}$ has stable monodromy if $s_n(Q) = d_Q \cdot p^{n-1} + c_Q$ for all Q, $n >> 0$.

As $k[V]$-mods: $M_n = M_n^{bij} \oplus M_n^{nil}$

and $M_n^{bij} \otimes_k \bar{k} \simeq \left(\frac{\bar{k}[V]}{(V-1)}\right)^{\oplus s_n}$ with

$$s_n = p^n(s_0 + |S| - 1) - |S| + 1$$

by the Deuring-Shafarevich formula.

Since V nilpotent on M_n^{nil}, the integers

$$a_n^{(r)} = \dim_k \ker\left(V^r : M_n \to M_n\right)$$

completely determine M_n^{nil}.

The numbers $a_n^{(r)}$ are mysterious, and is no analogue of RH or DS for them!

$\underline{\text{Ex}}$ $p = 13$, $X_1 \to X_0 = \mathbb{P}^1$ given by $y^p - y = f$ with $S = \{\infty\}$ and $d_1(\infty) = 7$

f	$a_1^{(1)}$	g_1	s_1
$t^{-7} + 2t^{-6} - 6t^{-5}$	21	36	0
$t^{-7} + t^{-2} + t^{-1}$	23	$\downarrow$	$\downarrow$
$t^{-7} - 5t^{-2}$	24	RH	DS
$t^{-7} + t^{-1}$	27		
t^{-7}	36		

$\underline{\text{Rem}}$: $\mathbb{Z}_p$ - towers $\{X_n\}$ can be made "explicit": K_n = function field of X_n

$K_n = k(Y_1, Y_2, \dots, Y_n)$, $(Y_1^p, Y_2^p, \dots) - (Y_1, Y_2, \dots) = w \in W(K_0)$

in $W(K_0)$ Witt ring

If $w = (w_1, w_2, \dots)$ has $w_n = 0$ for $n \gg 0$, then $\{X_n\}$ has stable monodromy.

Based on extensive computations in MAGMA:

$\underline{\text{Conjecture}}$ (Booher - C): Assume $\{X_n\}$ has stable monodromy: $s_Q(n) = d_Q p^{n-1} + c_Q$, $n \gg 0$. Then

$$a_n^{(r)} = \frac{r}{r + \frac{p+1}{p-1}} \cdot \frac{\sum_{Q \in S} d_Q}{2(p+1)} \cdot p^{2n} + O(p^n) \text{ as } n \to \infty$$

Def: A basic tower is one given by

$$(y_1^p, y_2^p, \dots) - (y_1, y_2, \dots) = \sum_{i=1}^{d} [a_i t^{-i}]$$

← Teichmüller

$a_i \in k$, $a_i = 0$ if $i \mid p$, $a_d \neq 0$.

These are monodromy stable, with $X_0 = \mathbb{P}^1$, $S = \{\infty\}$.

$\boxed{\text{Thm (Booher, Kramer-Miller, Upton, C): Conjecture is true for basic towers.}}$

Rem For basic towers, Booher-C conjecture:

$$a_n^{(r)} = \frac{r}{r + \frac{p+1}{p-1}} \cdot \frac{d}{2(p+1)} \cdot p^{2n} + \lambda n + c(n)$$

for some $\lambda \in \mathbb{Q}$ and periodic $c : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Q}$.

Sketch of proof: Fix $\{X_n\}$, $\pi : X_n \to X_m$

$d_n = n^{th}$ upper ram. break @ ∞. For D = divisor on X_n,

$$M_n(D) := H^0(\Omega^1_{X_n}(D)), \quad M_n = M_n(0)$$

Let $M = \varprojlim_{n, \pi_*} M_n$, a $\Lambda = k[[T]]$-module

via $T = \gamma - 1$, $\mathrm{Gal}(K_\infty / K_0) = \langle \gamma \rangle$.

Magic of char. p:

$$\boxed{\begin{gathered} T^{p^m} = (\gamma-1)^{p^m} = \gamma^{p^m}-1, \quad \mathrm{Gal}(K_\infty/K_m) = \langle \gamma^{p^m} \rangle \\ \pi^* \pi_* = \sum_{i=1}^{p^{n-m}} \gamma^{p^m i} = \frac{\gamma^{p^m \cdot p^{n-m}}-1}{\gamma^{p^m}-1} = \frac{(\gamma-1)^{p^n}}{(\gamma-1)^{p^m}} = T^{p^n-p^m} \\ \pi_* \pi^* = \deg(\pi) = p^{n-m} = 0 \end{gathered}}$$

① M is a countable product of copies of Λ. Let $Q = \mathrm{Frac}(\Lambda) = k((T))$. We can make $M \hat{\otimes}_\Lambda Q$ into a Q-Banach space by declaring M = unit ball.

② $V: M \to M$ is completely continuous. Crucially uses monodromy stable hyp.

$\Rightarrow$ Fredholm determinant

$L(s) = \det(1 - sV|_M) \in \Lambda[\![s]\!]$ is defined

③ $\{X_n\} \leftrightarrow \rho: \pi_1^{\acute{e}t}(\mathbb{P}^1 - \infty) \twoheadrightarrow \mathbb{Z}_p = \Gamma \hookrightarrow \Lambda^\times$

$\gamma \mapsto 1+T$

$$L(s) = L(\rho, s) := \prod_{v \in |\mathbb{A}^1|} \frac{1}{(1 - \rho(\mathrm{Frob}_v) s^{\deg v})}$$

via crystalline interpretation of L-fns.

④ V & M has:

- Hodge polygon HP encoding SNF of V
- Newton polygon $NP = NP_T(L(s))$

Work of Upton + Kramer-Miller using ③:

* $NP \geq HP$, and touch periodically

⑤ Explicit calculation of NP, using ③, then use ④ to infer info about HP.

Further Directions

① Study $H^1_{dR}(X_n) \leftrightarrow J_{X_n}[p]$

② Study $H^1_{crys}(X_n) \longleftrightarrow J_{X_n}[p^\infty]$

③ Prove conjecture in general

④ Weaken monodromy-stable hypothesis

⑤ Γ-towers of curves, Γ = p-adic Lie gp.

Thank You!