Belyi maps in positive characteristic

VaNTAGe Seminar

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Belyi maps

Let $k = \overline{k}$ be a field of char. ≥ 0.
A Belyi map over k is a cover $\varphi : X \to \mathbb{P}^1_k$ unramified outside $\{0, 1, \infty\}$.

$\mathrm{char}(k) = 0$: Combinatorial description of Belyi maps

$$\{\text{Belyi maps}\}/_{\simeq} \quad \leftrightarrow \quad \{(\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3 \mid \prod_i \sigma_i = 1, \langle \sigma_i \rangle \text{ transitive}\}/\sim$$

Set $h_0(d; \lambda) = \#$ covers of type $(d, \lambda = (C(\sigma_0), C(\sigma_1), C(\sigma_\infty))$.

([talk Voight]) The result uses the description of the topological fundamental group

$$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, *)$$

$\mathrm{char}(k) = p > 0$: Can one give a similar description for Belyi in positive characteristic?

Belyi maps in characteristic $p > 0$

Two very different cases!

The wild case: $p \mid$ ramification index for some point.
Example: $X : x^p - x = t^h$ with $p \nmid h$. Then $X \to \mathbb{P}^1, (x, t) \mapsto t$ is only branched at $t = \infty$ and $g(X) = (h-1)(p-1)/2$.

$$\rightsquigarrow \qquad \#\{\text{covers in characteristic } p\} = \infty.$$

The tame case: $p \nmid \text{order}(\sigma_i)$ for all i. Then

$$h_p(d, \lambda) \leq h_0(d, \lambda)$$

Reason: Every tame cover can be lifted to a cover of the same type in characteristic 0.

Existence of tame Belyi maps via reduction: $d < p$

If $d < p$ then $h_p(d, \lambda) = h_0(d, \lambda)$.

Reason: Every Galois cover in char. 0 with $p \nmid |G|$ for $G = \langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ has good reduction to char. p. ([SGA1])

Good reduction of Galois covers: Let $\psi : Y \to \mathbb{P}^1$ be the Galois closure of the Belyi map $\varphi : X \to \mathbb{P}^1$ over a local field K of mixed char. p.

For simplicity assume $g(Y) \geq 2$.

After extending K there is a smooth model $\mathcal{Y}$ of Y over $\mathcal{O}_K$. The curve $\mathcal{Y}/G$ is a smooth model of $\mathbb{P}^1$.

The special fiber $\overline{\psi} : \overline{Y} \to \mathbb{P}^1_k$ is a Galois cover of the same type.

Existence of Belyi maps via reduction: single-cycle case

([Talk Ejder]): Fix $(d; C_0, C_1, C_\infty)$, where $C_j = e_j$ is the conjugacy class in S_d of a single cycle of length e_j and $e_0 + e_1 + e_\infty = 2d + 1$ (i.e. $g(X) = 0$.) Normalization condition: $\varphi(0) = 0, \varphi(1) = 1, \varphi(\infty) = \infty$ are the ramification points.

Then $h_0(d; e_0, e_1, e_\infty) = 1$. (Rigid triple)

Osserman: combinatorial description for $h_p(d; e_0, e_1, e_\infty)$ in the tame genus-0 case.

Idea of proof: There exists a Belyi map of type $(d; e_0, e_1, e_\infty)$ iff $\nexists$ inseparable linear system "with this ramification".

More precisely: Instead of ramification points, we ask that $x = 0, 1, \infty$ are base points of the linear system with at least the required multiplicity. Also works if $\varphi = f/g$ is a rational function.

Example

The unique normalized Belyi map of type $(d; d-1, 2, d)$ is

$$\varphi(x) = -(d-1)x^d + dx^{d-1}.$$

Reduction mod p of φ is:

- inseparable of degree d if $p \mid d$: $\overline{\varphi}(x) = x^d$,
- inseparable of degree $d-1$ if $p \mid (d-1)$: $\overline{\varphi}(x) = x^{d-1}$,
- separable of type $(d; d-1, 2, d)$ otherwise.

Idea of proof: Write $d = p^n d'$ with $p \nmid d'$. The "ramification indices" $\overline{e}_i$ of $\overline{\varphi} = x^d$ satisfy: $\overline{e}_0 = d \geq d-1$, $\overline{e}_1 = p^n \geq 2$, $\overline{e}_\infty = d \geq d$. It follows: $h_p(d; d-1, 2, d) = 1$ iff $p \nmid d(d-1)$.

Let φ be a genus-0 single-cycle Belyi map, normalized as before.

- The rational function φ has coefficients in $\mathbb{Z}$.
- May show: reducing the coefficients of φ mod p yields a nonconstant rational function $\overline{\varphi}$.
- If $\overline{\varphi}$ is separable, then it has the same type as φ.
- The possible inseparable maps $\overline{\varphi}$ occurring may be counted without knowing an equation for φ. This yields $h_p(d; e_0, e_1, e_\infty)$.

A sample result

Osserman, Anderson-B-Ejder-Girgin-Karemaker-Manes

Let $\varphi : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ be the unique genus-0 Belyi map of type $(d; e_0, e_1, e_\infty)$ such that $\varphi(0) = 0, \varphi(1) = 1, \varphi(\infty) = \infty$. Write $d = p^n d'$ with $p \nmid d'$. Then $\overline{\varphi}(x) = x^d$ if and only if $e_1 \leq p^n$.

May be used to determine the preperiodic points of the dynamical system defined by iterating φ.

What changes for arbitrary conjugacy classes?

Example: $h_0(5; 2\text{-}3, 3, 4) = 2 > h_5(5; 2\text{-}3, 3, 4) = 1$. Maps in char. 0:

$$f(x) = c\frac{x^3(x-\alpha)^2}{(x-\beta)}, \quad \text{with}$$

$$15\beta^2 - 24\beta + 8 = 0,\ \alpha = 4 - 5\beta/2,\ c = (1-\beta)/(1-\alpha)^2$$

Maps in char. $p = 5$: unique separable map

$$\overline{f} = \frac{x^3(x-1)^2}{(x-2)} = 1 + \frac{(x-1)^3(x^2-2)}{x-2}$$

and a unique inseparable map $\overline{f} = x^5$ (corresponds to $\beta \rightsquigarrow \infty$)

An existence result

B-Osserman

Assume $1 < e_1 < e_2, e_1 + e_2 \leq p, 1 < e_3 \leq e_4 < p, e_3 + e_4 = p + 2$, and either $e_1 + e_2$ or e_3 odd. Then

$$h_0(p; e_1\text{-}e_2, e_3, e_4) = (p + 1 - e_1 - e_2) \min(e_1, p + 1 - e_4),$$
$$h_p(p; e_1\text{-}e_2, e_3, e_4) = p + 1 - e_1 - e_2.$$

Osserman's method does not apply here. For example;

$$h_0(7; 2\text{-}3, 5, 6) = 4, \qquad h_7(7; 2\text{-}3, 5, 6) = 2.$$

However, there is only one possible inseparable map $\overline{\varphi} = x^7$. It should be counted with multiplicity 2.

Method of proof We determine all possibilities for the stable reduction of the Galois closure of φ in the case of bad reduction.

Results of Wewers "explains" the multiplicity; no obvious interpretation in terms of $\overline{\varphi}$.

Counting tame Belyi maps of given type in char. p

Direct method: works as explained in [Talk Schiavone].

Osserman's method: using linear series.
Works in the genus-0 single-cycle case:

$$h_p(d,\lambda) = \underbrace{h_0(d,\lambda)}_{=1} - h_p^{\mathrm{insep}}(d,\lambda).$$

Wewers' method: using stable reduction of Galois covers.
Works if $p^2 \nmid |G|$ (in particular if $p \leq d < p^2$):

$$h_p(d,\lambda) = h_0(d,\lambda) - \sum \mu(\overline{\varphi}).$$

The sum runs over the possibilities $\overline{\varphi}$ for the stable reduction and $\mu(\overline{\varphi}) = \#\{\text{covers in char 0 with this reduction}\}$.
Can be computed from the type by characteristic p information.

Counting covers via reduction: To count tame covers in char. p one counts inseparable covers instead.