# The arithmetic of zero-cycles on products of K3 surfaces and Kummer varieties

**VaNTAGe Seminar**

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# Our objects of interest

# K3 surfaces

**Definition**

A(n algebraic) K3 surface $X$ over a number field $k$ is a smooth projective 2-dimensional variety over $k$ such that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

e.g. the Fermat quartic surface $x^4 + y^4 + z^4 + w^4 = 0 \subset \mathbb{P}^3$.

# Kummer surfaces and Kummer varieties

Let $A$ be an abelian variety of dimension $d$ over $k$.

Consider the involution $\iota : A \to A$ given by $x \mapsto -x$.

**Definition**

The minimal desingularisation

$$X := \widetilde{A/\iota}$$

is a Kummer variety of dimension $d$. When $d = 2$, $X$ is called a Kummer surface.

**Kummer surfaces are K3 surfaces!**

**But...** notice that $X(k) \neq \emptyset$ always! For what we want to do next, this makes things a bit too easy! So...

## Twisted Kummer surfaces and twisted Kummer varieties

Let $A$ be an abelian variety of dimension $d$ over $k$.

Let $[T] \in H^1_{\text{ét}}(k, A[2])$. This induces a 2-covering $V \to A$.

Consider the involution $\iota : A \to A$ given by $x \mapsto -x$.

Then $\iota$ induces an involution $\iota_V : V \to V$.

**Definition**

The minimal desingularisation

$$X := \widetilde{V/\iota_V}$$

is the twisted Kummer variety of dimension $d$ associated to $V$.

**From now on, when we talk about about "Kummer varieties" we really mean "twisted Kummer varieties".**

# A classical goal: understanding the arithmetic of rational points

# The Hasse principle...

**Basic question**: when is $X(k)$ empty? (**HARD QUESTION!**)

**Definition (HP)**

Let $X$ be a smooth, projective, geometrically integral variety over a number field $k$. We say that $X$ satisfies the Hasse principle if

$$X(k) = \emptyset \iff X(\mathbf{A}_k) = \emptyset.$$

E.g. smooth quadrics satisfy the HP.

**But there are numerous examples of K3 surfaces failing the Hasse principle!**

**Problem:** the set of adelic points is a bit too coarse to capture the emptiness of the set of rational points. Hence, we need to refine it!

# ...the Brauer group...

Let us consider $\mathrm{Br}\, X := H^2_{\text{ét}}(X, \mathbf{G}_m)$, the Brauer group of $X$.

- $\mathrm{Br}(-) : \mathbf{Sch}_k \to \mathbf{Ab}$ is a contravariant functor
- Fix $\alpha \in \mathrm{Br}\, X$. Let $x \in X(k)$, say $x : \mathrm{Spec}\, k \to X$. Then we can evaluate $\alpha$ at $x$ as follows:
  - Apply $\mathrm{Br}(-)$ to $x : \mathrm{Spec}\, k \to X$ to get $\mathrm{Br}(x) : \mathrm{Br}\, X \to \mathrm{Br}\, k$.
  - The evaluation $\alpha(x)$ is $\mathrm{Br}(x)(\alpha) \in \mathrm{Br}\, k$.

  (Similarly, we get evaluation maps for $k_v$-points, etc.)
- Using these evaluation maps and class field theory, we get the very useful commutative diagram

$$\begin{tikzcd} & X(k) \arrow[r] \arrow[d, "\alpha"] & X(\mathbf{A}_k) \arrow[d, "\alpha"] \arrow[dr, "\langle\ ,\ \alpha\rangle_{BM}"] & & \\ 0 \arrow[r] & \mathrm{Br}\, k \arrow[r] & \oplus_v \mathrm{Br}\, k_v \arrow[r, "\oplus_v \mathrm{inv}_v"] & \mathbf{Q}/\mathbf{Z} \arrow[r] & 0 \end{tikzcd}$$

# ...and a refinement of the Hasse principle

We can easily check that

$$X(k) \subset X(\mathbf{A}_k)^{\operatorname{Br} X} := \bigcap_{\alpha \in \operatorname{Br} X} \{(x_v) \in X(\mathbf{A}_k) : \langle (x_v), \alpha \rangle_{BM} = 0\} \subset X(\mathbf{A}_k).$$

The set $X(\mathbf{A}_k)^{\operatorname{Br} X}$ is called the Brauer-Manin set for $X$.

**Definition (HP-Br)**

Let $X$ be a smooth, projective, geometrically integral variety over a number field $k$. We say that $X$ satisfies the Hasse principle with Brauer-Manin obstruction if

$$X(k) = \emptyset \iff X(\mathbf{A}_k)^{\operatorname{Br} X} = \emptyset.$$

# Skorobogatov's conjecture for K3 surfaces

**Question**: Is the HP-Br enough for K3 surfaces?

**Conjecture (Skorobogatov)**

Let $X$ be a K3 surface over a number field $k$. Then $X$ satisfies the Hasse principle with Brauer-Manin obstruction, i.e.

$$X(k) = \emptyset \iff X(\mathbf{A}_k)^{\mathrm{Br}\, X} = \emptyset.$$

Hence, conjecturally, for K3 surfaces the Brauer-Manin set $X(\mathbf{A}_k)^{\mathrm{Br}\, X}$ is a perfect approximation to the set $X(k)$ with respect to the question of whether $X(k)$ is empty or not.

**Evidence towards Skorobogatov's conjecture:**

- For some specific Kummer surfaces over some specific number fields, e.g. when the underlying abelian surface $A$ is the product of two elliptic curves or the Jacobian of a genus 2 curve with a rational Weierstrass point (Harpaz and Skorobogatov)
- For some specific elliptic fibrations over some specific number fields (Colliot-Thélène, Swinnerton-Dyer, Skorobogatov)

# A more general goal: understanding the arithmetic of 0-cycles

# What are 0-cycles?

0-cycles are generalisations (and "abelianisations") of rat'l points.

**Definition (0-cycles of degree $d$)**

Let $X$ be a smooth, projective, geometrically integral variety over a number field $k$. Fix $d \in \mathbf{Z}_{>0}$. A 0-cycle $z$ of degree $d$ is a formal $\mathbf{Z}$-sum

$$z = \sum_{\substack{x \in X \\ \text{closed pt}}} n_x x$$

with $n_x \in \mathbf{Z}$ and with $n_x = 0$ for all but finitely many $x \in X$ such that

$$\deg(z) := \sum_{\substack{x \in X \\ \text{closed pt}}} n_x [\kappa(x) : k] = d,$$

where $\kappa(x)$ is the residue field of $x$.

# The Hasse principle for 0-cycles and its refinements

| $k$-rational points | | 0-cycles of degree $d$ |
|---|---|---|
| $X(k)$ <br> set of $k$-rational points | $\rightsquigarrow$ | $Z_0^d(X_k)$ <br> set of 0-cycles of degree $d$ |
| $X(\mathbf{A}_k) = \prod_{v\in\Omega_k} X(k_v)$ <br> set of $k$-adelic points | $\rightsquigarrow$ | $Z_0^d(X_{\mathbf{A}_k}) = \prod_{v\in\Omega_k} Z_0^d(X_{k_v})$ <br> set of adelic 0-cycles of degree $d$ |
| $X(\mathbf{A}_k)^{\mathrm{Br}\,X}$ <br> Brauer-Manin set for $k$-rational points | $\rightsquigarrow$ | $Z_0^d(X_{\mathbf{A}_k})^{\mathrm{Br}}$ <br> Brauer-Manin set for 0-cycles of degree $d$ |
| $X(k) = \emptyset \iff X(\mathbf{A}_k) = \emptyset$ <br> HP for $k$-rational points | $\rightsquigarrow$ | $Z_0^d(X_k) = \emptyset \iff Z_0^d(X_{\mathbf{A}_k}) = \emptyset$ <br> HP for 0-cycles of degree $d$ |
| $X(k) = \emptyset \iff X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$ <br> HP-Br for $k$-rational points | $\rightsquigarrow$ | $Z_0^d(X_k) = \emptyset \iff Z_0^d(X_{\mathbf{A}_k})^{\mathrm{Br}} = \emptyset$ <br> HP-Br for 0-cycles of degree $d$ |

# (The Brauer-Manin set for 0-cycles)

The Brauer-Manin set $Z_0^d(X_{\mathbf{A}_k})^{\mathrm{Br}}$ is defined as the set of adelic 0-cycles $(z_v = \sum_{x_v \in X_{k_v}} n_{x_v} x_v)_v$ of degree $d$ such that, for any $\alpha \in \operatorname{Br} X$, we have

$$\sum_v \sum_{x_v \in X_{k_v}} n_{x_v} \operatorname{inv}_v \left(\operatorname{cores}_{\kappa(x_v)/k_v} (\alpha(x_v))\right) = 0.$$

**Question**: Is the HP-Br for 0-cycles enough for K3 surfaces?

**Conjecture (Colliot-Thélène)**

Let $X$ be a smooth, projective, geometrically integral variety over a number field $k$. (**So not just K3 surfaces!**) Then $X$ satisfies the Hasse principle with Brauer-Manin obstruction for 0-cycles of degree 1, i.e. $Z_0^1(X_k) = \emptyset \iff Z_0^1(X_{\mathbf{A}_k})^{\mathrm{Br}} = \emptyset$

**Evidence towards Colliot-Thélène's conjecture:**

- Not much yet!
- Curves such that the Tate-Shafarevich group of their Jacobian is finite (Saito)
- Conic bundle surfaces over $\mathbf{P}^1$ (Salberger)
- Smooth compactifications of homogeneous spaces of connected linear algebraic groups with connected geometric stabilisers (Liang)
- Varieties with a morphism to a curve such that the geometric generic fibre is rationally connected and for which the BM obstruction is the only one for weak approximation for the fibres above "enough" (Harpaz and Wittenberg)

# Relating the arithmetic of 0-cycles to the arithmetic of rational points

# The general idea

The general question behind Liang's strategy is:

If we know that $X_K(\mathbf{A}_K)^{\mathrm{Br}(X_K)} = \emptyset \iff X_K(K) = \emptyset$ for all (or "enough") finite extensions $K/k$, can we conclude that $Z_0^1(X_{\mathbf{A}_k})^{\mathrm{Br}} = \emptyset \iff Z_0^1(X_k) = \emptyset$?

In other words, can we use the knowledge of the arithmetic of rational points over enough field extensions $K/k$ to get information about the arithmetic of 0-cycles?

**Liang showed that, in some cases, this transfer of knowledge is possible!**

# Liang's strategy

Step 1. It suffices to prove the result for $X \times \mathbf{P}^1$. We have now available the trivial fibration $X \times \mathbf{P}^1 \to \mathbf{P}^1$.

Step 2. To start the process, we need that $\operatorname{Br} X / \operatorname{Br} k$ is finite. Fix a closed point $\tilde{x} \in X$. Fix $(z_v) \in Z_0^1(X_{\mathbf{A}_k})^{\operatorname{Br}}$. We can manipulate $(z_v)$ so to get a new effective adelic zero-cycles $(z'_v)$ still compatible with the Brauer-Manin set and with the property that, for all $v$, $\deg(z'_v) = \Delta$ and $\Delta \equiv 1 \bmod [\kappa(\tilde{x}) : k] \cdot \# \operatorname{Br} X / \operatorname{Br} k$.

Step 3. By cleverly using the trivial fibration $X \times \mathbf{P}^1 \to \mathbf{P}^1$, one obtains a field extension $K/k$ of degree $[K : k] = \Delta$ and an adelic point $(x_w) \in X_K(\mathbf{A}_K)^{\operatorname{Br} X}$.

Step 4. We somehow show that for such a $K/k$, the natural restriction map

$$\mathrm{res}_{K/k} : \operatorname{Br} X / \operatorname{Br} k \to \operatorname{Br}(X_K)/ \operatorname{Br} K$$

is surjective. Hence, $(x_w) \in X_K(\mathbf{A}_K)^{\operatorname{Br}(X_K)}$.

Step 5. We now use the assumption that the BM obstruction to the HP is the only one for rational points (over **any*** number field) to deduce the existence of a $K$-rational point $x \in X_K(K)$.

Step 6. By exploiting the coprimality conditions and by taking a suitable combination of the points $x$ and $\tilde{x}$, we obtain a 0-cycles of degree 1, i.e. $Z_0^1(X_k) \neq \emptyset$.

# Can we adapt Liang's strategy to K3 surfaces and Kummer varieties?

| | K3's | Kummer vars |
|---|---|---|
| $\# \operatorname{Br} X / \operatorname{Br} k < \infty$? | ✓ Skorobogatov-Zarhin | ✓ Skorobogatov-Zarhin |
| $\mathrm{res}_{K/k}$ surj? | ✓ Ieronymou (based on Orr-Skorobogatov) | ✓* B.-Newton (based on Creutz-Viray, Skorobogatov-Zarhin) |
| HP-Br for $K$-rat pts? | ??? (✓ by Skorobogatov's conj) | ??? (Probably true) |

# Results for K3 surfaces

**Theorem (Ieronymou)**

*Conditionally on Skorobogatov's conjecture, if $X$ is a K3 surface over a number field $k$, then*

$$Z_0^d(X_k) = \emptyset \iff Z_0^d(X_{\mathbf{A}_k})^{\mathrm{Br}} = \emptyset$$

*for any $d \in \mathbf{Z}_{>0}$.*

This result makes fundamental use of the following result:

**Theorem (Orr-Skorobogatov)**

*Let $X$ be a K3 surface over $k$. Then there exists a constant $C_{B,X}$ such that $\# \mathrm{Br}(\overline{X})^{\mathrm{Gal}(\overline{k}/K)} \leq C_{B,X}$ for any field extension $K/k$ with $[K:k] \leq B$.*

## How about Kummer varieties?

- We don't know, in general, whether $\mathrm{res}_{K/k} : \mathrm{Br}\, X / \mathrm{Br}\, k \to \mathrm{Br}(X_K) / \mathrm{Br}\, K$ is surjective for the *whole* Brauer group. But luckily, we have the following result by Creutz-Viray, so Liang's strategy still works if we restrict our attention to the 2-primary part of the Brauer group.

**Theorem (Creutz-Viray)**

*Let $X$ be a Kummer variety over $k$. Then*

$$X(\mathbf{A}_k)^{\mathrm{Br}\, X\{2\}} = \emptyset \iff X(\mathbf{A}_k)^{\mathrm{Br}\, X}.$$

- In order to prove that the restriction map is surjective on the 2-primary part, we exploit the close relationship between Kummer varieties and Abelian varieties (using also results by Skorobogatov-Zarhin).

**Theorem (B.-Newton)**

*Let $d \in \mathbf{Z}_{>0}$ and let $g \in \mathbf{Z}_{>1}$. Then there exists a constant $N = N(d, g)$ such that for any $X$ a Kummer or Abelian variety over $k$ of dimension $g$, and for any $K/k$ of degree coprime to $N$, we have*

$$\mathrm{res}_{K/k} : \frac{\mathrm{Br}\, X}{\mathrm{Br}_1\, X}\{d\} \xrightarrow{\cong} \frac{\mathrm{Br}(X_K)}{\mathrm{Br}_1(X_K)}\{d\}.$$

*In particular, there exists some constant $N' = N'(d, g)$ such that for any Kummer variety $X$ over $k$ and any $K/k$ of degree coprime to $N'$, we have*

$$\mathrm{res}_{K/k} : \frac{\mathrm{Br}\, X}{\mathrm{Br}\, k}\{d\} \xrightarrow{\cong} \frac{\mathrm{Br}(X_K)}{\mathrm{Br}\, K}\{d\}.$$

- Putting everything together, we get:

**Theorem (B.-Newton)**

*Let $X$ be a Kummer variety over $k$. If*

$$X_K(\mathbf{A}_K)^{\mathrm{Br}(X_K)} = \emptyset \iff X_K(K) = \emptyset$$

*for "enough" finite extensions $K/k$, then*

$$Z_0^d(X_{\mathbf{A}_k})^{\mathrm{Br}\{2\}} = \emptyset \iff Z_0^d(X_k) = \emptyset$$

*for any odd $d \in \mathbf{Z}_{>0}$.*

## How about products of K3 surfaces and Kummer varieties?

- We have results for K3 surfaces and Kummer varieties. Can we mixed them up to get even more general results?
- Note that if $X$ and $Y$ are K3 surfaces/Kummer varieties, then $X \times Y$ is no longer a K3 surface/Kummer variety!
- But by a modification of Liang's strategy and a simple observation (using results by Skorobogatov-Zarhin) we have:

**Theorem (B.-Newton)**

*Let $W = \prod_{i=1}^{n} X_i$ where each $X_i$ is either a K3 surface or a Kummer variety over $k$. If*

$$(X_i)_K(\mathbf{A}_K)^{\mathrm{Br}((X_i)_K)} = \emptyset \iff (X_i)_K(K) = \emptyset$$

*for "enough" finite extensions $K/k$ and for all $i = 1, ..., n$, then*

$$Z_0^1(W_{\mathbf{A}_k})^{\mathrm{Br}} = \emptyset \iff Z_0^1(W_k) = \emptyset.$$

# Related questions

## Conjecture E

- So far we have only looked at the Hasse principle, but many of the conjectures and the results mentioned above hold in a greater generality!
- The Chow group of $X$ is defined as $CH_0(X) = Z_0(X_k)/\sim_{\mathrm{rat}}$.

**Conjecture (Conjecture E)**

The sequence

$$\widehat{CH_0(X)} \to \prod_v \widehat{CH'_0(X_{k_v})} \to \mathrm{Hom}(\mathrm{Br}\, X, \mathbf{Q}/\mathbf{Z})$$

is exact. Here, $CH'$ denotes the modified Chow group (at the infinite ) and $\widehat{M} = \varprojlim_n M/nM$.

(Conjecture E implies that the Brauer-Manin obstruction is the only one for the existence of 0-cycles)

## Thank you for your attention!