Equidistribution counts abelian varieties

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VaNTAGe
Curves and abelian varieties over finite fields

Prologue

Motivating question

How big is an isogeny class of elliptic curves over a finite field?

*

[*] Joint work with S. Ali Altuğ (Radix), Luis Garcia (UCL), and Julia Gordon (UBC)

Isogenous elliptic curves

If $E_1, E_2/\mathbb{F}_q$, the following are equivalent:

- E_1 and E_2 are isogenous;
- $\#E_1(\mathbb{F}_q) = \#E_2(\mathbb{F}_q)$;
- $a(E_1) = a(E_2)$, where characteristic polynomial of Frobenius is

$$f_{E_i/\mathbb{F}_q}(T) = T^2 - a(E_i)T + q.$$

Let

$$I(a, \mathbb{F}_q) = \left\{ E/\mathbb{F}_q : a(E) = a \right\}.$$

Motivating question

What is $\#I(a, \mathbb{F}_q)$?

Or $\widetilde{\#}I(a, \mathbb{F}_q)$, *where* E *has weight* $1/\#\operatorname{Aut}(E)$.

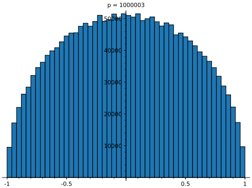
First guess: uniform

- $a \in [-2\sqrt{q}, 2\sqrt{q}]$ *(Hasse)*
- $\asymp q$ elliptic curves over $\mathbb{F}_q$. *(Exact, if we weight by automorphism.)*
- Suppose $a(E)$ uniformly distributed on $[-2\sqrt{q}, 2\sqrt{q}]$.

Heuristic

$$\widetilde{\#} I(a, \mathbb{F}_q) \asymp q/\sqrt{q} = \sqrt{q}.$$

This can't be exactly right. The distribution is *not* uniform.



Sato–Tate distribution

- Frobenius angles:

$$f_E(T) = T^2 - a_E T + q = (T - \sqrt{q}\exp(i\theta_E))(T - \sqrt{q}\exp(-i\theta_E))$$

$$a_E = 2\sqrt{q}\cos(\theta_E)$$

$$\widetilde{a}_E := \frac{a_E}{2\sqrt{q}}$$

- Sato–Tate distribution:

$$\begin{tikzcd} \mathrm{SU}(2) \arrow[d, "\frac{1}{2}\operatorname{tr}"] & \mu^{\mathrm{Haar}} \arrow[d, squiggly] \\ {[-1,1]} & \mu^{\mathrm{ST}} = \frac{1}{2}\operatorname{tr}_* \mu^{\mathrm{Haar}} \end{tikzcd}$$

Second guess: Sato–Tate

Heuristic

$$\widetilde{\#}I(a,\mathbb{F}_q) \approx \#\left\{E/\mathbb{F}_q\right\} \cdot \mathbb{P}(a_E = a)$$
$$\asymp 2q \cdot \frac{1}{2\sqrt{q}} \frac{1}{2\pi} \sqrt{1 - \left(\frac{a}{2\sqrt{q}}\right)^2}$$
$$= \sqrt{q} \frac{1}{2\pi} \sqrt{1 - \widetilde{a}^2}.$$

Seems closer, unlikely to be literally correct.

Frobenius elements

- $E/\mathbb{F}_q$
- For $\ell \neq p$, Frobenius gives

$$\mathrm{Fr}_{E/\mathbb{F}_q,E_\ell} \in \mathrm{Aut}(E_\ell) \cong \mathrm{GL}_2(\mathbb{Z}/\ell)$$
$$\mathrm{Fr}_{E/\mathbb{F}_q,\ell} \in \mathrm{Aut}(T_\ell E) \cong \mathrm{GL}_2(\mathbb{Z}_\ell)$$

and thus conjugacy classes

$$\gamma_{E/\mathbb{F}_q,\mathbb{Z}/\ell} \in \mathrm{GL}_2(\mathbb{Z}/\ell)^{\#}$$
$$\gamma_{E/\mathbb{F}_q,\ell} \in \mathrm{GL}_2(\mathbb{Z}_\ell)^{\#}.$$

- These Frobenius elements are:
  - Equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell)$ and $\mathrm{GL}_2(\mathbb{Z}_\ell)$; and
  - Independent: equidistributed in $\mathrm{GL}_2(\mathbb{Z}/\ell_1) \times \mathrm{GL}_2(\mathbb{Z}/\ell_2)$.

Local factors

Set

$$\nu_\ell(a,q) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,q) \bmod \ell^n\}}{\#\mathrm{GL}_2(\mathbb{Z}/\ell^n)/(\ell^n \cdot \ell^{n-1}(\ell-1))}.$$

Rationale (sic)

- Denominator is average number of elements with given charpoly.
- Equivalently, ν_ℓ comes from pushforward of Haar:

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{Z}_\ell) & & \gamma \\ \downarrow \mathfrak{c} & & \downarrow \\ (\mathbb{A}^1 \times \mathbb{G}_m)(\mathbb{Z}_\ell) & & (\mathrm{tr}(\gamma), \det(\gamma)), \end{array}$$

Third guess: local corrections

Heuristic

$$\widetilde{\#}I(a,\mathbb{F}_q) \asymp \sqrt{q}\nu_\infty(a,q)\prod_{\ell<\infty}\nu_\ell(a,q)$$

$$\nu_\infty(a,q) = \frac{2}{\pi}\sqrt{1-\frac{a^2}{4q}}$$

$$\nu_\ell(a,q) = \lim_{n\to\infty}\frac{\#\{\gamma\in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma),\det(\gamma)) \equiv (a,q) \bmod \ell^n\}}{\#\mathrm{GL}_2(\mathbb{Z}/\ell^n)/(\ell^n\cdot\ell^{n-1}(\ell-1))}.$$

This can't be right.

Equidistribution only holds for $\ell \ll q$.

Gekeler's Theorem

Set

$$\nu_\ell(a,p) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,p) \bmod \ell^n\}}{\#\mathrm{GL}_2(\mathbb{Z}/\ell^n)/(\ell^n \cdot \ell^{n-1}(\ell-1))}$$

$$\nu_p(a,p) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{Mat}_2(\mathbb{Z}/p^n) : (\mathrm{tr}(\gamma), \det(\gamma)) \equiv (a,p) \bmod p^n\}}{\#\mathrm{GL}_2(\mathbb{Z}/p^n)/(p^n \cdot p^{n-1}(p-1))}$$

$$\nu_\infty(a,p) = \frac{2}{\pi}\sqrt{1 - \frac{a^2}{4p}}$$

Theorem (Gekeler)

If $|a| < 2\sqrt{p}$ *and* $a \neq 0$*, then*

$$\widetilde{\#} I(a, \mathbb{F}_p) = \sqrt{p}\,\nu_\infty(a,p) \prod_\ell \nu_\ell(a,p).$$

Counterfactual equidistribution predicts the right answer!

Why does it work?

~~Why~~ How does it work?

- $\Delta := \Delta_{a,q} = a^2 - 4q$, $\mathcal{O}_{a,q} = \mathbb{Z}[\sqrt{\Delta}] = \mathbb{Z}[T]/(T^2 - aT + q)$.
- $K = \mathbb{Q}(\sqrt{\Delta})$, $\Delta_0 = \Delta_{K/\mathbb{Q}}$, $\chi = \chi_K = \left(\frac{\Delta_0}{\cdot}\right)$.
- $\Delta = \mathfrak{f}^2 \Delta_0$.

Theorem (Deuring)

$$\widetilde{\#} I(a,q) = \sum_{b \mid \mathfrak{f}} \frac{h(b^2\Delta_0)}{w(b^2\Delta_0)} = \sum_{\mathcal{O}' \supseteq \mathcal{O}_{a,q}} \widetilde{h}(\mathcal{O}').$$

A class number counts the isogeny class.

Special case

Suppose $\mathcal{O}_{a,q} = \mathcal{O}_K$.

$$
\begin{aligned}
\widetilde{\#} I(a,q) &= h(K) \quad \textit{Deuring} \\
&= \frac{\#\mathcal{O}_K^\times \sqrt{|\Delta_K|}}{2\pi} L(1,\chi) \quad \textit{analytic class number formula} \\
&= \frac{\#\mathcal{O}_K^\times \sqrt{|\Delta_K|}}{2\pi} \prod_\ell \frac{1}{1-\chi(\ell)/\ell}.
\end{aligned}
$$

Key matrix calculation

Suppose $\ell \nmid \Delta = a^2 - 4q$.

$f(T) \bmod \ell$	split	irreducible
centralizer T	$\begin{pmatrix} * & \\ & *' \end{pmatrix}$	$\begin{pmatrix} * & \epsilon *' \\ *' & * \end{pmatrix}$
$T(\mathbb{F}_\ell)$	$\mathbb{F}_\ell^\times \times \mathbb{F}_\ell^\times$	$\mathbb{F}_{\ell^2}^\times$
$\#T(\mathbb{F}_\ell)$	$(\ell-1)^2$	ℓ^2-1

$$\frac{\#\{\text{conjugacy class of} \gamma_0 \bmod \ell\}}{\#\mathrm{GL}_2(\mathbb{F}_\ell)/(\ell(\ell-1))}$$

$$= \frac{\#\mathrm{GL}_2(\mathbb{F}_\ell)/\#T(\mathbb{F}_\ell)}{\#\mathrm{GL}_2(\mathbb{F}_\ell)/(\ell(\ell-1))} = \frac{\ell(\ell-1)}{\#T(\mathbb{F}_\ell)}$$

$$= \begin{cases} \frac{1}{1-1/\ell} & \text{split} \\ \frac{1}{1+1/\ell} & \text{irreducible} \end{cases} = \frac{1}{1-\chi(\ell)/\ell}.$$

Extensions

Similar strategy used for certain:

- Abelian surfaces [Williams, Rauch]
- Abelian varieties of prime dimension [Gerhard–Williams]

Count isogeny class using something like $h(K)/h(K^+)$.

Two questions

Can we find...

- a pure-thought proof of Gekeler's theorem?
- an analogue for isogeny classes of principally polarized abelian varieties?

Remark 8.7. Perhaps with a more conceptual approach, one could also verify the conjecture for the more general moduli problems we considered, where we allow also a $\Gamma_0(L)$-structure and a $\Gamma_1(M)$-structure. What is the relation of the conjectured formula to the formula, in terms of orbital integrals, given by Kottwitz in [Ko1, §16, pp. 432-433] and [Ko2, p. 205], when that general formula is specialized to the case of elliptic curves; cf. also [Cl, §3,§4]?

N. Katz, Lang–Trotter Revisited, 2008.

One answer

Theorem (AAGG)

$[X, \lambda] \in \mathcal{A}_g(\mathbb{F}_q)$ a principally polarized abelian variety over $\mathbb{F}_q$ with commutative endomorphism ring. Suppose X is ordinary or that $\mathbb{F}_q = \mathbb{F}_p$. Then

$$\widetilde{\#}I([X,\lambda],\mathbb{F}_q) = q^{\frac{\dim(\mathcal{A}_g)}{2}} \tau_T \nu_\infty([X,\lambda],\mathbb{F}_q) \prod_\ell \nu_\ell([X,\lambda],\mathbb{F}_q).$$

Proof is close to pure thought:

- No actual calculation of local terms.
- No appeal to analytic class number formula.

Theorem (Remix)

$$\widetilde{\#}I([X,\lambda],\mathbb{F}_q) = \tau_T \frac{q^{-\frac{g(3g-1)}{4}}}{(2\pi)^g}\sqrt{\left|\frac{\operatorname{disc}(f)}{\operatorname{disc}(f^+)}\right|}B([X,\lambda]) \lim_{s\to 1^+}\frac{\zeta_K(s)}{\zeta_{K^+}(s)}$$

where

$$B([X,\lambda]) = \prod_{\ell \mid 2p\,\operatorname{disc}(f)} \frac{\zeta_{K^+,\ell}(1)}{\zeta_{K,\ell}(1)}\nu_\ell([X,\lambda]).$$

Class numbers emerge, but we never used the class number formula.

Weil polynomials and isogeny classes

- $(X, \lambda)/\mathbb{F}_q$ a g-dimensional principally polarized abelian variety.
- Frobenius $\mathrm{Fr}_{X/\mathbb{F}_q,\ell}$ acts on $V_\ell X = T_\ell X \otimes \mathbb{Q}_\ell$.
- Tate: (unpolarized) isogeny class determined by charpoly of $\mathrm{Fr}_{X/\mathbb{F}_q,\ell}$:

$$f_{X/\mathbb{F}_q}(T) \in \mathbb{Z}[T].$$

- λ induces $\langle\cdot,\cdot\rangle_\lambda : V_\ell X \times V_\ell X \to \mathbb{Q}_\ell^\times$.
- (X, λ) determines
  - $\gamma_0 = \gamma_{X/\mathbb{F}_q,\ell} \in \mathrm{GSp}(V_\ell, \langle\cdot,\cdot\rangle_\lambda) \cong \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$, up to conjugacy.
  - $\delta_0 = \delta_{X/\mathbb{F}_q} \in \mathrm{GSp}(H^1_{\mathrm{cris}}(X), \langle\cdot,\cdot\rangle) \cong \mathrm{GSp}_{2g}(\mathbb{Q}_q)$ up to σ-conjugacy.

If $g = 1$, λ unique, $\langle\cdot,\cdot\rangle$ unique, $\mathrm{GSp}_2 = \mathrm{GL}_2$, conjugacy determined by charpoly.

Groups attached to (X, λ)

- $G = \mathrm{GSp}_{2g}$, $g = \dim X$;
- $\gamma_0 \in G(\mathbb{A}^p_{\mathrm{fin}})$, $\delta_0 \in G(\mathbb{Q}_q)$;
-
  - Polarization induces involution $(\dagger)$ on $\mathrm{End}(X)$.
  - Define group scheme $T = T_{(X,\lambda)}$:

$$T_{(X,\lambda)}(R) = \left\{ \alpha \in (\mathrm{End}(X) \otimes R)^\times : \alpha\alpha^{(\dagger)} \in R^\times \right\}.$$

  -

$$T(\mathbb{Q}_\ell) = \mathrm{GSp}(V_\ell X) \cap (\mathrm{End}(X) \otimes \mathbb{Q}_\ell)$$
$$T_{\mathbb{Q}_\ell} = (\mathrm{GSp}_{2g})_{\gamma_{X/\mathbb{F}_q,\ell}} \quad \textit{(Tate)}$$

Orbital integrals

Theorem (Kottwitz)

We have

$$\widetilde{\#}I([X,\lambda],\mathbb{F}_q) = \mathrm{vol}(T_{(X,\lambda)}(\mathbb{Q})\backslash T_{(X,\lambda)}(\mathbb{A}_{\mathrm{fin}})) \times \int_{G_{\gamma_0}(\mathbb{A}^p_{\mathrm{fin}})\backslash G(\mathbb{A}^p_{\mathrm{fin}})} \phi_0(g^{-1}\gamma_0 g)\, d\mu^{\mathrm{can}}(g) \times \int_{G_{\delta_0\sigma}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_q)} \psi_{q,p}(h^{-1}\delta_0 h^{\sigma})\, d\mu^{\mathrm{can}}(h)$$

where the measure on the orbit is the canonical measure.

Idea

Proof is elementary; count sub-lattices of $H^1(X)$ stable under Frobenius.

Towards local terms

Want natural local factors $\nu_\ell(X,\lambda)$ which compute

$$\int_{G_{\gamma_0}(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \mathbf{1}_{G(\mathbb{Z}_\ell)}(g^{-1}\gamma_0 g)d\mu^{\mathrm{can}}(g).$$

$G(\mathbb{Q}_\ell)$ vs. $G(\overline{\mathbb{Q}}_\ell)$ Can't use $f_{X/\mathbb{F}_q}(T)$; conjugacy and stable conjugacy are different.

$G(\mathbb{Q}_\ell)$ vs. $G(\mathbb{Z}_\ell)$ Can't use $G(\mathbb{Z}_\ell)$-conjugacy;

$$\left\{g^{-1}\gamma_0 g : g \in G(\mathbb{Z}_\ell)\right\} \subsetneq \left\{g^{-1}\gamma_0 g : g \in G(\mathbb{Q}_\ell)\right\} \cap G(\mathbb{Z}_\ell).$$

Local terms

The key definition is:

Local factor

$$\nu_\ell([X,\lambda]) = \lim_{d\to\infty} \lim_{n\to\infty} \frac{\# C_{(d,n,\ell)}(\gamma_0)}{\# G(\mathbb{Z}_\ell/\ell^n)/\# A(\mathbb{Z}_\ell/\ell^n)}.$$

Lemma

If $\ell \nmid \operatorname{disc}(f(T))$, *then*

$$\nu_\ell([X,\lambda]) = \frac{\#\{\gamma \in G(\mathbb{Z}_\ell/\ell) : \gamma \sim \gamma_0\}}{\# G(\mathbb{Z}_\ell/\ell)/\# A(\mathbb{Z}_\ell/\ell)}.$$

Strategy

Rewrite Kottwitz formula as Gekeler-type product.

But for the unquiet heart and brain
A use in measured language lies;
The sad mechanic exercise
Like dull narcotics numbing pain.

Alfred, Lord Tennyson

Serre–Oesterlé measure (on analytic sets)

μ^{SO} is essentially a point-counting measure on ℓ-adic analytic sets.
If $\mathcal{Z} \subset \mathbb{A}^m_{\mathbb{Z}_\ell}$,

$$\mu^{\mathrm{SO}}(\mathcal{Z}(\mathbb{Z}_\ell)) = \int_{\mathcal{Z}(\mathbb{Z}_\ell)} d\mu^{\mathrm{SO}} = \lim_{n\to\infty} \frac{\#\mathcal{Z}(\mathbb{Z}_\ell/\ell^n)}{\ell^{n \dim \mathcal{Z}}}.$$

Example

If $\mathcal{Z}$ smooth, then

$$\mu^{\mathrm{SO}}(\mathcal{Z}(\mathbb{Z}_\ell)) = \frac{\#\mathcal{Z}(\mathbb{Z}/\ell)}{\ell^{\dim \mathcal{Z}}}.$$

Gauge measures (on groups)

$G/\mathbb{Q}_\ell$ an algebraic group.

- ω_G an invariant top-degree form; then

$$\mu^\omega(S) = \int_S |\omega_G|.$$

Example

G split, ω_G Gross's canonical form; then

$$\mu^\omega(G(\mathbb{Z}_\ell)) = \int_{G(\mathbb{Z}_\ell)} |\omega_G|_\ell = \mu^{\mathrm{SO}}(G(\mathbb{Z}_\ell)) = \frac{\#G(\mathbb{Z}_\ell/\ell)}{\ell^{\dim G}}.$$

Canonical measure (on groups)

- Normalize Haar measure so a certain maximal compact subgroup has volume 1.

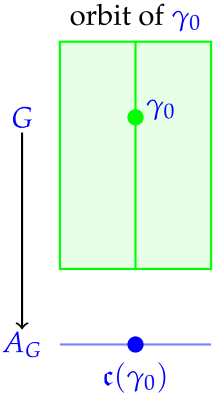
Example

$\mathcal{G}/\mathbb{Z}_\ell$ smooth; choose Haar measure with

$$\mu_G^{\mathrm{can}}(\mathcal{G}(\mathbb{Z}_\ell)) = 1.$$

Geometric measure (on orbit)

Use Steinberg fibration ("characteristic polynomial map") to define a measure on the orbit of γ_0:



$$\omega_G = \omega_{A_G} \wedge \omega^{\text{geom}}_{\mathfrak{c}(\gamma_0)}$$

$$\mu^{\text{geom}}(S) = \int_S \left| \omega^{\text{geom}}_{\mathfrak{c}(\gamma_0)} \right|$$

Tamagawa measure (on orbit)

- $S/\mathbb{Q}$ a torus, ω a gauge form on S.
- $$\omega^{\mathrm{Tama}} = \omega_\infty \prod_\ell L_\ell(1, \sigma_S)\omega_\ell.$$
- Define Tamagawa measure on orbit of γ_0:

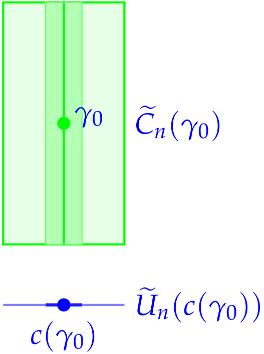
$$\mu^{\mathrm{Tama}}_{\gamma_0} = \frac{\left|\omega_G^{\mathrm{can}}\right|}{\left|\omega_T^{\mathrm{Tama}}\right|}.$$

Key observation

Recall:

$$\nu_\ell([X,\lambda]) = \lim_{d\to\infty}\lim_{n\to\infty} \frac{\#C_{(d,n,\ell)}(\gamma_0)}{\#G(\mathbb{Z}_\ell/\ell^n)/\#A(\mathbb{Z}_\ell/\ell^n)}.$$

Let $\widetilde{C}_{(d,n,\ell)}(\gamma_0) = \left\{\gamma \in \gamma_0 : (\gamma \bmod \ell^n) \in C_{(d,n,\ell)}(\gamma_0)\right\}$.



Lemma

$$\begin{aligned}\#C_{(d,n,\ell)}(\gamma_0) &= \ell^{n\dim G}\mu^{\mathrm{SO}}(\widetilde{C}_{(d,n,\ell)}(\gamma_0))\\ &= \ell^{n\dim G}\mu^{|\omega_G|}(\widetilde{C}_{(d,n,\ell)}(\gamma_0))\end{aligned}$$

Use this to relate ν_ℓ to $\mathcal{O}^{\mathrm{geom}}$.

Main result

Theorem (AAGG)

$[X, \lambda] \in \mathcal{A}_g(\mathbb{F}_q)$ a principally polarized abelian variety, $\mathrm{End}(X)$ commutative. Suppose X is ordinary or that $\mathbb{F}_q = \mathbb{F}_p$. Then

$$\widetilde{\#}I([X,\lambda],\mathbb{F}_q) = q^{\frac{\dim(\mathcal{A}_g)}{2}}\tau_T \nu_\infty([X,\lambda],\mathbb{F}_q)\prod_\ell \nu_\ell([X,\lambda],\mathbb{F}_q).$$

Proof.

$$\prod_\ell \nu_\ell \rightsquigarrow \mathcal{O}^{\mathrm{geom}} \rightsquigarrow \mathcal{O}^{\mathrm{Tama}} \rightsquigarrow \mathrm{LHS}$$

Kottwitz formula

□

A numerical example

- Consider the 3-Weil polynomial

$$f(T) = T^8 - 6T^7 + 13T^6 - 10T^5 + T^4 - 30T^3 + 117T^2 - 162T + 81.$$

  Isogeny class is ordinary, contains principally polarized $(X, \lambda)/\mathbb{F}_3$.

- $K := \mathbb{Q}[T]/f(T)$; $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$.
- $\tau_T = 2$ [Rüd]
- Since

$$\mathrm{disc}(f(T))/\,\mathrm{disc}(K) = 3^{4(4-1)},$$

  for all finite ℓ, including $\ell = p$,

$$\nu_\ell([X, \lambda], \mathbb{F}_3) = \frac{\zeta_{K,\ell}(1)}{\zeta_{K^+,\ell}(1)}.$$

A numerical example

- Evaluate numerically:

$$\widetilde{\#}I([X,\lambda],\mathbb{F}_3) = 3^{3(3+1)/4}\tau_T\nu_\infty([X,\lambda],\mathbb{F}_3)\lim_{s\to 1}\frac{\zeta_K(s)}{\zeta_{K^+}(s)}$$
$$\approx 3^3\cdot 2\cdot 0.000111808\cdot 0.871253$$

which of course is

$$\approx 0.0500000.$$

- Check LMFDB: (X,λ) is unique in $I([X,\lambda],\mathbb{F}_3)$, and

$$\#\operatorname{Aut}([X,\lambda]) \cong (\mathcal{O}_K^\times)_{\mathrm{tors}} \cong \mathbb{Z}/20.$$

Questions?

Putting the p back in Prym[†]

Credit: BBC Two

† Joint work with S. Casalaina-Martin (Boulder)

Inclusions of abelian schemes

- S reduced, connected, locally Noetherian
- $Y \overset{\iota_Y}{\hookrightarrow} X$ an inclusion of abelian schemes over S
- $\lambda : X \to \widehat{X}$ a polarization of X.

Example

$\varpi : C \to C'$ a finite morphism of curves over S.
Then $\varpi^* : \mathrm{Pic}^0_{C'/S} \to \mathrm{Pic}^0_{C/S}$ has finite kernel, and factors as

$$\begin{tikzcd} \mathrm{Pic}^0_{C'/S} \arrow[r, "\varpi^*"] \arrow[d, two heads] & \mathrm{Pic}^0_{C/S} \\ Y \arrow[ur, hook, "\iota_Y"'] & \end{tikzcd}$$

Complements

- The *exponent* $e = e_{Y \subset X, \lambda}$ is the exponent of $\iota_Y^* \lambda$:

$$\begin{tikzcd} Y \arrow[r, "\iota^* \lambda"] \arrow[d, "\iota"] & \widehat{Y} \\ X \arrow[r, "\lambda"] & \widehat{X} \arrow[u, "\widehat{\iota}"'] \end{tikzcd}$$

- Define a *norm endomorphism* $N_Y : X \to X$ with image Y.
- Let $Z = \mathrm{im}([e]_X - N_Y)$.

Question

When does λ induce a principal polarization on Y?

Definition (Prym–Tyurin)

A Prym–Tyurin scheme of exponent e is:

- $(Z, \xi)/S$ a principally polarized abelian scheme;
- C/S a smooth projective curve; and

$$Z \xhookrightarrow{\iota_Z} \mathrm{Pic}^0_{C/S}$$

- such that

$$\iota_Z^* \lambda_C = e\xi.$$

Definition (Prym)

(Z, ξ) is further a Prym scheme if there is a finite separable $\varpi : C \to C'$, and Z is the complement of $\operatorname{im} \mathrm{Pic}^0_{C'/S}$ in $\mathrm{Pic}^0_{C/S}$.

Welters' Criterion

Welters' Criterion (proved for varieties over an algebraically closed field of characteristic zero) holds over an arbitrary field:

Theorem

- $(Z, \xi)/K$ *PPAV,* $\Xi \subset Z$ *a divisor,* $\phi_\Xi = \xi$*;*
- C/K *a smooth projective curve,* $\Theta \subset \mathrm{Pic}^0_{C/K}$ *a divisor,* $\phi_\Theta = \theta$*.*

Suppose there is a morphism $\beta : C \to Z$ *such that:*

- $\beta^* : \widehat{Z} \hookrightarrow \mathrm{Pic}^0_{C/K}$ *is an inclusion; and*
- $\beta_*[C] \equiv \frac{e}{(g_Z - 1)!}[\Xi]^{g_Z - 1}$.

Then there is a morphism $Z \hookrightarrow \mathrm{Pic}^0_{C/K}$ *making* (Z, ξ) *a Prym–Tyurin variety of exponent* e*.*
If $C(K) \neq \emptyset$*, then the converse holds.*

Everything is a Prym–Tyurin

Corollary

Let $(Z,\xi)/K$ be a principally polarized abelian variety.
Then (Z,ξ) is a Prym–Tyurin variety of exponent $n^{g-1}(g-1)!$ for infinitely many n.
If $\mathrm{char}(K)=0$, this holds for all $n\geq 3$.

Classification of Prym varieties

Theorem

- *$\varpi : C \to C'/K$ finite separable morphism of degree d*
- *Z the complement of $\varpi^* \operatorname{Pic}^0_{C'/K}$ in $\operatorname{Pic}^0_{C/K}$.*

Then Z is a Prym variety of exponent e and dimension f if and only if one of:

a. *$d = 2$ and ϖ is étale; then $(e, f) = (2, g' - 1)$;*
b. *$d = 2$, $\deg \operatorname{ram}(\varpi) = 2$; then $(e, f) = (2, g')$;*
c. *$d = 3$, ϖ étale noncyclic, and $g' = 2$; then $(e, f) = (3, 2)$;*
d. *$g = 2$ and $g' = 1$; then $(e, f) = (d, 1)$.*

In case (b), either

- $\operatorname{char}(K) \neq 2$, ϖ is tamely ramified at exactly two points; or
- $\operatorname{char}(K) = 2$, and f is weakly wildly ramified at exactly one point. $K[\![x]\!] \hookrightarrow K[\![x]\!][y]/(y^2 - y - x)$

Thanks!