Overview of Khare's proof.

Bas Edixhoven

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Abstract

An overview will be given of Khare's proof of Serre's conjecture in level one. Those who will not attend the rest of the seminar will have an idea of Khare's proof, and it is hoped that those who will attend the rest of the seminar will now be sufficiently motivated to digest the more technical parts that are to come.

1 Khare's result

A good reference for this lecture is Khare's survey in his preprint [9]. We should note that Taylor's results in [14] and [15] play an essential role in the papers [8], [9], [10], [6] and [5] by Khare, Khare-Wintenberger and Dieulefait. Let us start by recalling Khare's result.

1.1 Theorem. (Khare) *Let p be a prime number, and let $\overline{\rho}\colon G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous odd representation that is unramified outside p. Then $\overline{\rho}$ comes from a modular form: there is an eigenform $f = \sum a_n q^n$ of some level N, weight k and character ε, such that for all l not dividing Np the characteristic polynomial of $\overline{\rho}\mathrm{Frob}_l$ is $x^2 - a_l x + \varepsilon(l)l^{k-1}$.*

We note that in the case of a reducible $\overline{\rho}$ (that may be ramified outside p) the form f can be taken to be an Eisenstein series (easy). So, if we want, we can assume $\overline{\rho}$ to be irreducible. Moreover, if $\overline{\rho}$ is irreducible and the image $\overline{\rho}G_{\mathbb{Q}}$ is solvable, then results of Langlands and Tunnell ([11] and [16]) show that such a $\overline{\rho}$ (that may be ramified outside p) comes from a modular form, so we may even assume that $\overline{\rho}G_{\mathbb{Q}}$ is not solvable. Finally, because of the results by Tate and Serre explained in the previous two lectures by Dahmen and Beukers, we may assume that $p \geq 5$.

In what follows, we will not try to state all intermediate results in the generality in which they are known, but instead we will just focus on the strategy of the proof of Thm. 1.1. We will assume that $p \geq 5$ and that $\rho G_{\mathbb{Q}}$ is not solvable.

2 Minimal lifts

After replacing $\overline{\rho}$ with some $\overline{\rho}\otimes\overline{\chi}_p^i$ we may assume that $2\leq k(\overline{\rho})\leq p+1$; this is a well-known property of Serre's definition of $k(\overline{\rho})$.

A *lift* of $\overline{\rho}$ is then a continuous representation $\rho\colon G_{\mathbb{Q}}\to \mathrm{GL}_2(O)$, with O the ring of integers in a finite extension of $\mathbb{Q}_p$, such that ρ induces $\overline{\rho}$ via $O\to\overline{\mathbb{F}}_p$. We note that it is a priori not clear at all if lifts exist. Such a lift ρ is called *minimal* if ρ is unramified outside p and satisfies the following property at p (i.e., after restriction to a decomposition group G_p at p):

1. if $k(\overline{\rho})\neq p+1$ then $\rho|_{G_p}$ is crystalline of weights $(0,k(\overline{\rho})-1)$;
2. if $k(\overline{\rho})=p+1$ then $\rho|_{G_p}$ is semistable and non-crystalline of Hodge-Tate weights $(0,1)$, or is crystalline of Hodge-Tate weights $(0,p)$.

Some explanation is in order here, concerning the minimality property at p. If $k(\overline{\rho})=2$ then the condition means that ρ can be realised as the G_p-representation associated to a p-divisible group over $\mathbb{Z}_p$ (compare with the equivalence "$k(\overline{\rho})=2$ iff $\overline{\rho}$ finite at p"). Properties such as "crystalline" and "semi-stable" are defined with the help of certain functors of Fontaine that go from p-adic G_p-representations to $\mathbb{Q}_p$-vector spaces with additional structure (such as Frobenius operator, filtration, monodromy operator). These functors transform p-adic étale cohomology into de Rham type cohomologies (and vice versa if possible). The condition "crystalline" (resp. "semi-stable") can then be intuitively described as "is as cohomology of a projective smooth (semi-stable) $\mathbb{Z}_p$-scheme. For precise statements see [4].

In the case $k(\overline{\rho})=p+1$ we can give an explicit description of the minimality condition at p (see section 2 of [10]). In that case, $\overline{\rho}|_{G_p}$ is a wildly ramified extension of an unramified character ε by $\varepsilon\overline{\chi}_p$. A lift ρ is then semi-stable at p iff $\rho|_{G_p}$ is an extension of an unramified character $\widetilde{\varepsilon}$ by $\widetilde{\varepsilon}\chi_p$. A lift ρ is crystalline of Hodge-Tate weights $(0,p)$ if its restriction to intertia at p is an extension of 1 by χ_p^p.

The first step in the proof of Thm. 1.1 is the following result, Theorem 2.1 of [10].

2.1 Theorem. *Assume that $k(\overline{\rho})\neq p$. Then a minimal lift exists. If $k(\overline{\rho})=p+1$ then minimal lifts of both types at p exist.*

The proof of this theorem starts by considering the minimal deformation theory for $\overline{\rho}$. First, one fixes a finite subfield $\mathbb{F}\subset\overline{\mathbb{F}}_p$ such that $\overline{\rho}G_{\mathbb{Q}}$ is contained in $\mathrm{GL}_2(\mathbb{F})$, and one lets W be the ring of Witt-vectors of $\mathbb{F}$. Then (as we will see in the lecture by Bart de Smit in this seminar) there exists a *universal deformation* $\rho^{\mathrm{univ}}\colon G_{\mathbb{Q}}\to\mathrm{GL}_2(R^{\mathrm{univ}})$ that has the property that for any artinian W-algebra R with residue field $\mathbb{F}$ and for any minimally ramified $\rho\colon G_{\mathbb{Q}}\to\mathrm{GL}_2(R)$

that induces $\overline{\rho}$ via $R \to \mathbb{F}$ there exists a unique $R^{\mathrm{univ}} \to R$ such that ρ^{univ} induces a conjugate of ρ via $R^{\mathrm{univ}} \to R$. (In the case $k(\overline{\rho}) = p + 1$ such a ring exists for each of the two types of minimal lifts.) The W-algebra R^{univ} is known to be a quotient of some $W[[x_1, \ldots, x_r]]$.

In order to prove Theorem 2.1 it suffices to know that R^{univ} is flat as W-module. Gebhard Böckle has shown in [2] (using techniques that are more or less standard since Wiles) that R^{univ} has a presentation of the form:

$$R^{\mathrm{univ}} = W[[x_1, \ldots, x_r]]/(f_1, \ldots, f_s), \quad \text{with } s \leq r.$$

In view of this, it then suffices to prove that $R^{\mathrm{univ}}/pR^{\mathrm{univ}}$ is finite, because in that case $(f_1, \ldots, f_s)$ gives a regular sequence in $\mathbb{F}[[x_1, \ldots, x_r]]$, hence also in $W[[x_1, \ldots, x_r]]$, and so the quotient R^{univ} is finite and free as W-module.

To explain what is happening here it is a good moment to say a few words on the so-called Fontaine-Mazur conjecture from [7]. According to one of their conjectures, every continuous $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(O)$ that is unramified outside a finite set of primes and that has some good behaviour at p such as "crystalline of different Hodge-Tate weights" or "semi-stable" comes from "geometry", which means in this case that up to a twist by a power of χ_p it comes from a modular form. This conjecture then implies that the W-algebra R^{univ} above is then isomorphic to a suitable Hecke algebra (the algebra generated by Hecke operators on a suitable space of modular forms as we have seen in Johan Bosman's lecture). In particular, one always expects for R as above $R[1/p]$ to be finite dimensional as $W[1/p]$-vector space. It was Wiles and Taylor-Wiles who first gave important evidence for this conjecture by Fontaine and Mazur by establishing "$R = T$-theorems". The point now is that for those "$R = T$-theorems" (also called modularity lifting theorems) one always has to assume modularity of the $\overline{\rho}$ that is being deformed.

It was Taylor who realised, in [14] and [15], that Wiles's trick using the primes 3 and 5 could be generalised in a very useful way by accepting to get a result only over a totally real field of which the degree is not under control. More on this will be said later in the seminar.

Concerning the proof of Theorem 2.1 Taylor shows that for an appropriate totally real extension $\mathbb{Q} \to F$ the analogously defined ring R_F^{univ} is isomorphic to a Hecke algebra and hence has the property that $R_F^{\mathrm{univ}}/pR_F^{\mathrm{univ}}$ is finite. From the fact that G_F has finite index in $G_{\mathbb{Q}}$ it then easily follows (a result of Johan de Jong) that $R^{\mathrm{univ}}/pR^{\mathrm{univ}}$ is finite. (It follows from the fact that the image of G_F in $\mathrm{GL}_2(R^{\mathrm{univ}}/pR^{\mathrm{univ}})$ is finite, plus the fact that W-algebra R^{univ} is generated by the traces of elements of $G_{\mathbb{Q}}$.)

3 Hilbert modular forms

In order to give the audience an idea of what is going on, I just want to mention that what we have seen in Johan Bosman's lecture about Galois representations associated to modular forms, can be generalised from $\mathbb{Q}$ to totally real fields F. The action of the group $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ is then replaced by the action of $\mathrm{GL}_2(\mathbb{R} \otimes F) = \mathrm{GL}_2(\mathbb{R}^d) = \mathrm{GL}_2(\mathbb{R})^d$ on $(\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}))^d$, where $d = \dim_{\mathbb{Q}} F$. The congruence subgroups of $\mathrm{GL}_2(\mathbb{Z})$ are then replaced by congruence subgroups of $GL_2(O_F)$. All formulas are similar, but more complicated, and in order to get a comprehensible description of Hecke operators, it becomes necessary to work with the adeles of F. The modular forms arising in this context are called Hilbert modular forms, and the varieties that generalise the modular curves are called Hilbert modular varieties. More on this in two of the lectures on October 14. On November 11 two lectures are devoted to the construction of Galois representations associated to Hilbert modular forms, and some automorphic theory necessary for that construction. For f a normalised eigenform, and λ a prime of the ring of integers of the field of coefficients, the associated representation of G_F is denoted $\rho_{f,\lambda}$.

4 Taylor's potential modularity results

We just state here the results relevant to our situation: the proof of Thm. 1.1. So, the assumptions are as before: $p \geq 5$, $\overline{\rho} G_{\mathbb{Q}}$ is not solvable, $2 \leq k(\overline{\rho}) \leq p+1$ and $k(\overline{\rho}) \neq p$. The reference for the result stated here is [10, Prop. 2.5].

4.1 Theorem. (Taylor) *There exists a totally real field F and a Hilbert modular eigenform f "over F" of level one and weight $k(\overline{\rho})$ with the following properties. The extension $\mathbb{Q} \to F$ is Galois and unramified at p, and even split at p if $\overline{\rho}|_{G_p}$ is irreducible. The representation $\overline{\rho}|_{G_F}$ has nonsolvable image. There is a place $\underline{p}$ over p of the coefficient field E of f such that $\overline{\rho}_{f,\underline{p}} = \overline{\rho}|_{G_F}$. If $k(\overline{\rho}) = p+1$ then there is also a form f with level p and weight 2 such that $\overline{\rho}_{f,\underline{p}} = \overline{\rho}|_{G_F}$.*

Proof. We just say a few words on the proof. More details should be given in the 2 lectures on this subject later in the seminar. The fact that $\overline{\rho} G_F$ is nonsolvable is a consequence of F being totally real: $\overline{\rho} G_F$ contains the image of complex conjugation (which has eigenvalues 1 and -1), as well as all its conjugates.

The main idea is to show that there exists an abelian variety A over some totally real number field F, with multiplications by O_M for some totally real field M such that $\dim_{\mathbb{Q}} M = \dim(A)$, such that for some place $\underline{p}$ of M over p one has $\overline{\chi}_p^i \otimes \overline{\rho}|_{G_F} \cong A(\overline{\mathbb{Q}})[\underline{p}]$ for a suitable i such that

$\det(\overline{\chi}_p^i \otimes \overline{\rho}|_{G_F}) = \overline{\chi}_p$ (this is possible because $\det\overline{\rho}$ is an odd power of $\overline{\chi}_p$), and with $A(\overline{\mathbb{Q}})[\lambda]$ dihedral (i.e., induction of a character of a quadratic extension) for some place λ of M over a prime l different from p and 2. Such an abelian variety A/F is proven to exist by the theorem explained to us in the lecture by Gunther Cornelissen, applied to a Hilbert-Blumenthal moduli space associated to M, defined over $\mathbb{Q}$, of a suitable level, and suitably twisted. We recall that we have no control on $\dim_{\mathbb{Q}} F$, but the choice of M depends only on the choice of a finite field $\mathbb{F}$ such that $\overline{\rho}G_{\mathbb{Q}} \subset \mathrm{GL}_2(\mathbb{F})$.

Then a generalisation of the results of Wiles and Taylor-Wiles, by Wiles, Skinner, Diamond, Fujiwara, Taylor, Jarvis (and probably others that I forget), shows that $A[\lambda^\infty]$ is modular, from which follows existence of f. This last part also involves level lowering in the context of Hilbert modular forms. $\square$

As said above, we do not mention the other results of Taylor on this subject concerning L-functions; let us suffice to say that he extended Artin's results on complex representations (hence with finite image) to a lot of 2-dimensional p-adic Galois representations, in particular, meromorphic continuation with the expected functional equation.

5 Compatible families of Galois representations

We now know that $\overline{\rho}|_{G_F}$ is modular. This then gives an "$R = T$-theorem" over F. In particular, any minimally ramified lift ρ of $\overline{\rho}$ as in Thm. 2.1 is modular over F.

Let now ρ be a minimally ramified lift as in Thm. 2.1, and let f be the Hilbert modular eigenform of level 1 that gives ρ, say at a place $\underline{p}$ of its field of coefficients E. Then any finite place λ of E gives us a λ-adic representation $\rho_{f,\lambda}\colon G_F \to \mathrm{GL}_2(E_\lambda)$, and this whole system of Galois representations is what one calls a compatible system (e.g., characteristic polynomials of Frobenius elements at unramified places do not depend on λ; stronger conditions can be formulated at all places using the so-called Weil-Deligne group (this will be explained, hopefully, in the lecture by Johan de Jong on compatible families)).

Now $\rho_{f,\underline{p}} = \rho|_{G_F}$ can be extended to the 2-dimensional representation ρ of $G_{\mathbb{Q}}$, which means that f has the extra symmetry under the action of $\mathrm{Gal}(F/\mathbb{Q})$ needed to correspond to a modular form over $\mathbb{Q}$. Unfortunately, not enough has been proved on this at the moment, in order to be able to conclude that ρ is modular. But the known case of solvable base change ([1]) implies that ρ is modular over subfields of the form F^H with H a solvable subgroup of $\mathrm{Gal}(F/\mathbb{Q})$.

Using Brauer's result that says that every representation of $\mathrm{Gal}(F/\mathbb{Q})$ over $\overline{\mathbb{Q}}$ is a finite linear combination with $\mathbb{Z}$-coefficients of representations of the form $\mathrm{Ind}_H^{\mathrm{Gal}(F/\mathbb{Q})}\chi$ with H a solvable subgroup of $\mathrm{Gal}(F/\mathbb{Q})$ and $\chi\colon H \to \overline{\mathbb{Q}}^\times$ a one-dimensional representation, one can show that there exists a compatible family $\rho_\lambda\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(E_\lambda)$. For a detailed statement, we refer to Thm. 3.1 of [10]. This result of existence of compatible families was first proved by Dieulefait in [6], and, independently, by Wintenberger in [17]. At this point we would also like to note that Dieulefait had all necessary ideas to prove Serre's conjecture in the case of level one and weight two, *except* for the use of Boeckle's result mentioned above; see [5] and the references therein.

6 Khare's induction on primes

At this point we have enough ingredients to explain Khare's strategy for proving Thm. 1.1. We note that already some parts of this were already achieved in [10].

A way to think about what has to be done is to consider the pairs (p,k) in $\mathbb{Z}^2$ with p prime and $2 \leq k \leq p^2-1$ together with $(2,4)$ (the range of possible weights for $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$). For each such pair (p,k) we must check that every odd irreducible $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ that is unramified outside p with $k(\overline{\rho}) = p$ is modular. This unramifiedness implies that $\det \overline{\rho} = \overline{\chi}_p^i$ for some i, and the oddness of $\overline{\rho}$ then implies that $k(\overline{\rho})$ is even. So we only need to consider even k.

For $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ the property of being modular is invariant under twisting by powers of $\overline{\chi}_p$ (this holds in fact for twisting p-adic representations with characters with finite image, because modular forms can also be twisted). As we already said, for each $\overline{\rho}$ there is an i such that $2 \leq k(\overline{\rho} \otimes \overline{\chi}_p^i) \leq p+1$. This means that in order to prove the conjecture for all (p,k) for a given p, it suffices to prove the result for the even k with $2 \leq k \leq p+1$.

As explained in the two preceding lectures, we know the conjecture to be true for p in $\{2,3\}$. Let us now start working in the horizontal direction, i.e., varying p. We first do this for $k=2$. So let $p > 3$ and assume that $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is odd, irreducible and of weight 2 (actually, according to Serre's conjecture, these should not exist, as the space $S_2(\Gamma_1(1))$ is zero, so we should arrive at a contradiction). Let now ρ be a p-adic lift of $\overline{\rho}$ as given by Thm. 2.1: $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(O_E)$ is unramified outside p and is given by a p-divisible group over $\mathbb{Z}_p$ (i.e, ρ is crystalline of weight 2). Let (ρ_λ) be the compatible system as in the previous section. Then consider the residual representation $\overline{\rho}_3$ at some place of E over 3. By what we already know, $\overline{\rho}_3$ is reducible and therefore modular, and a modularity lifting result by Skinner and Wiles proves that the 3-adic representation ρ_3 is modular. But then the p-adic representation ρ is modular, and therefore $\overline{\rho}$ is.

The same argument shows that if we know the conjecture for (p,k) with $k \leq p+1$, then we know it for all (q,k) with $k \leq q+1$. This depends of course on the modularity lifting results that are available as of today, so let us here state such results (Theorem 6.2 of [9]).

6.1 Theorem. (“many people”) *Let $p > 2$ and $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be continuous, odd,* not *necessarily irreducible, with $2 \leq k(\overline{\rho}) \leq p+1$. Let ρ be a lift of $\overline{\rho}$ to a p-adic representation.*

1. If ρ is unramified outside p and crystalline of weight k with $2 \leq k \leq p+1$, then ρ is modular.

2. If ρ is unramified outside p, and either semistable of weight two at p or Barsotti-Tate at p, then ρ is modular.

3. If $q \neq p$ is prime and ρ is unramified outside pq and Barsotti-Tate at p, then ρ is modular.

The next case to treat is $(5,6)$. For details, see Theorem 4.1 of [9]; this case was already treated in [10]. We also note that this is the first case that Moon and Taguchi cannot treat without assuming GRH in [12]. So assume $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_5)$ be of that type: continuous, odd, irreducible, unramified outside 5 and with $k(\overline{\rho}) = 6$. Then (using Thm. 2.1) let $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(O)$ be a 5-adic lift of $\overline{\rho}$ that is unramified outside 5 and semistable of weight 2 at 5. Then Taylor's result Thm. 4.1 implies that there is an abelian variety A over $\mathbb{Q}$, such that ρ is isomorphic to $A(\overline{\mathbb{Q}})[5^\infty]$. This A has good reduction outside 5 and semistable reduction at 5. But by a result of Brumer and Kramer [3] (see also Schoof [13]) such a non-zero abelian variety does not exist (the method of proof of those results is an extension of the method that we have seen for 2 and 3 exploiting groupscheme structures for getting better upper bounds for discriminants). This contradiction proves what we need.

The next case to treat is $(7,8)$; also this case was already treated in [10] but in a different way. This case is more complicated, but it shows very well the general strategy, so it will be the last case that we speak of (for details see [9, Section 8]); details will be given in the last two lectures of the seminar. Suppose $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_7)$ is continuous, odd, irreducible, unramified outside 7 and with $k(\overline{\rho}) = 8$. Then $\overline{\rho}|_{I_7}$ is a wildly ramified extension of 1 by $\overline{\chi}_7$. Let ρ be a 7-adic lift as in Thm. 2.1 that unramified outside 7, semistable of weight 2 at 7. Then $\rho|_{I_7}$ is an extension of 1 by χ_7. Let (ρ_λ) be a compatible family as in the previous section. Take a 3-adic ρ_3 (i.e., choose a place over 3 of the coefficient field of (ρ_λ)), and let $\overline{\rho}' := \overline{\rho}_3\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_3)$. Then $k(\overline{\rho}') = 2$ and $\overline{\rho}'$ is unramified outside $3{\cdot}7$. If $\overline{\rho}'$ has solvable image then it is modular (trivially in the reducible case and otherwise by Langlands-Tunnell) and by Thm. 6.1 ρ_3 is modular and hence ρ and $\overline{\rho}$ are. So we assume that $\overline{\rho}'$ has nonsolvable image. If $\overline{\rho}'$ is unramified at 7, then it is reducible by the results for 2 and 3 so this is not the case. Properties of compatible families show that $\overline{\rho}'|_{I_7}$ is a nontrivial extension of 1 by 1 (think of the 7 and 3-adic Tate-modules of an elliptic curve over $\mathbb{Q}$ that has multiplicative reduction at 7). We lift $\overline{\rho}'$ to a 3-adic ρ' that is Barsotti-Tate at 3 and such that $\rho'|_{I_7}$ is $\omega_7^2 \oplus 1$ (with $\omega_7\colon G_{\mathbb{Q}} \to (\mathbb{Z}/7\mathbb{Z})^\times \hookrightarrow \overline{\mathbb{Q}}^\times$ a Teichmüller lift of $\overline{\chi}_7$). This step is the main innovation in [8] with respect to [10]; see Thm. 6.1 of [9], the existence of such a lift is an analog of what is called Carayol's lemma on the modular form side that says that when one wants to lift a mod p modular form $\overline{f}$ to a characteristic zero form f, in almost all cases one can choose the character $\varepsilon(f)$ among all characters reducing to $\varepsilon(\overline{f})$ mod p. Note that in this case indeed ω_7^2 has order 3 hence reduces to the trivial character mod 3. The improvement that has been achieved now is that $\rho'|_{I_7}$ has finite image of order 3, whereas $\rho_3|_{I_7}$ has infinite image. Let (ρ'_λ) be a compatible family as in Section 5. The final miracle is now that $\overline{\rho}'_7$ either has solvable image or satisfies $k(\overline{\rho}'_7) \in \{4, 6\}$, so that in all cases we can deduce that ρ'_7 is modular.

The general method is precisely like this. In the most complicated case one lifts p-adically, gets a compatible family, reduces modulo a smaller prime p', twists and changes to a different p'-adic lift, and passes to the compatible p-adic representation that leads to a smaller weight.

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