Computing L-series of low genus curves

Andrew V. Sutherland

Massachusetts Institute of Technology

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joint work with David Harvey

The problem

Given a smooth projective curve $X/\mathbb{Q}$ and a bound N, we wish to compute $L_p(T)$ for all primes $p \le N$ where X has good reduction.

Here $L_p(T)$ is the L-*polynomial* of the reduction $X_p/\mathbb{F}_p$ of X at p. It is an integer polynomial of degree $2g$ that satisfies:

- $L(X;s) = \prod_p \boldsymbol{L_p}(p^{-s})^{-1}$;
- $Z(X_p;T) = \exp\left(\sum_{n=1}^{\infty} \#X_p(\mathbb{F}_{p^n})T^n/n\right) = \frac{\boldsymbol{L_p}(T)}{(1-T)(1-pT)}$;
- $\chi(X_p;T) = T^{2g}\boldsymbol{L_p}(T^{-1})$.

Applications: computing L-functions and Sato-Tate distributions.

Some existing solutions

Four methods were analyzed in [Kedlaya-S, 2008]:

	genus 1	genus 2	genus 3
enumerate $X_p(\mathbb{F}_p),\ldots,X_p(\mathbb{F}_{p^g})$	$p\log^{1+\epsilon}p$	$p^2\log^{1+\epsilon}p$	$p^3\log^{1+\epsilon}p$

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Within the feasible range of $p \leq N$, it *never* makes sense to use the polynomial-time algorithm that is asymptotically the best choice.

For practical purposes, group algorithms work best in genus 1 and 2, and a combination of group algorithms and p-adic methods works best in genus 3.

At least this was the situation until the fall of last year. . .

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An average polynomial-time algorithm

All of the methods above perform separate computations for each prime p. But we want to compute $L_p(T)$ for all good $p \leq N$ using reductions of *the same curve* in each case.

Is their a way to take advantage of this fact?

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Theorem (Harvey, 2012)

Let $y^2 = f(x)$ be a hyperelliptic curve over $\mathbb{Q}$, with $\deg f = 2g + 1$ odd. There is an algorithm to compute $L_p(T)$ for all good primes $p \le N$ in

$$O\big(g^{8+\epsilon} N \log^{3+\epsilon} N\big)$$

time, using $O(g^3 N \log^2 N)$ space (assuming g and $\|f\|$ are suitably bounded).

This yields an average time of $O\big(g^{8+\epsilon} \log^{4+\epsilon} p\big)$ per prime $p \le N$.

But how practical is it for feasible values of N?

The Hasse-Witt matrix

Harvey's algorithm uses the same basic approach as Kedlaya's algorithm: compute the action of Frobenius on the Monsky-Washnitser cohomology to sufficient p-adic precision. For a suitable choice of basis, this action can be described by a matrix $A_p \in \mathbb{Z}_p^{2g\times 2g}$ that satisfies

$$A_p \equiv \left(\begin{array}{c|c} W_p & 0 \\ \hline 0 & 0 \end{array}\right) \bmod p,$$

where $W_p \in (\mathbb{Z}/p\mathbb{Z})^{g\times g}$ is the *Hasse-Witt matrix* of X.

For hyperelliptic curves $y^2 = f(x)$, the matrix W_p is given by

$$W_p = \begin{pmatrix} c_{p-1} & c_{p-2} & \cdots & c_{p-g} \\ c_{2p-1} & c_{2p-2} & \cdots & c_{2p-g} \\ \vdots & \vdots & \vdots & \vdots \\ c_{gp-1} & c_{gp-2} & \cdots & c_{gp-g} \end{pmatrix},$$

where c_k is the coefficient of x^k in the expansion of $f(x)^{(p-1)/2}$ modulo p.

Computing the Hasse-Witt matrix

Theorem (Harvey-S, 2013)

Let $X/\mathbb{Q}$ be a hyperelliptic curve. The matrices W_p can be computed for all good primes $p \le N$ in $O(g^{2+\omega} N \log^{3+\epsilon} N)$ time using $O(gN)$ space.
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For primes p of good reduction the identity

$$L_p(T) \equiv \det(I - TW_p) \bmod p$$

determines $O(1)$ possibilities for $L_p(T)$ in genus 2, and $O(p^{1/2})$ in genus 3.

In the latter case, these can be distinguished in $O(p^{1/4} \log^{1+\epsilon} p)$ time, which is negligible for the feasible range of $p \leq N$.

In any case, W_p determines the trace of Frobenius for all sufficiently large p. When approximating $L(X; s)$, only the trace is needed for $p \geq N^{1/2}$.

The algorithm in genus 1

Let $X/\mathbb{Q}$ be the elliptic curve $y^2 = f(x) = x^3 + ax + b$.
Then $L_p(T) = pT^2 - t_pT + 1$, where $t_p \equiv c_{p-1} \bmod p$ is the trace of Frobenius.

We wish to compute the coefficient c_{p-1} of x^{p-1} in $f(x)^{(p-1)/2} \bmod p$ for $p \le N$.
Equivalently, the coefficient of x^{2n} in $f(x)^n \bmod 2n + 1$ for $n \le (N-1)/2$.

Naïve approach: iteratively compute $f, f^2, f^3, \ldots, f^{(N-1)/2}$ in $\mathbb{Z}[x]$ and reduce the x^{2n} coefficient of $f(x)^n$ modulo $2n + 1$.

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But we don't need all the coefficients of f^n, we only need one,
and we only need to know its value modulo $2n+1$.

A better approach

Let f_k^n denote the coefficient of x^k in $f(x)^n$.
Using $f^n = ff^{n-1}$ and $(f^n)' = nf'f^{n-1}$, one obtains the relations

$$\begin{aligned}
(n+2)f_{2n-2}^n &= n\left(2af_{2n-3}^{n-1} + 3bf_{2n-2}^{n-1}\right),\\
(2n-1)f_{2n-1}^n &= n\left(3f_{2n-4}^{n-1} + af_{2n-2}^{n-1}\right),\\
2(2n-1)bf_{2n}^n &= (n+1)af_{2n-4}^{n-1} + 3(2n-1)bf_{2n-3}^{n-1} - (n-1)a^2f_{2n-2}^{n-1}.
\end{aligned}$$

Letting $D_n = 2(n+2)(2n-1)b$ and

$$M_n = \begin{pmatrix} 0 & 4n(2n-1)ab & 6n(2n-1)b^2 \\ 6n(n+2)b & 0 & 2n(n+2)ab \\ (n+1)(n+2)a & 3(n+2)(2n-1)b & (1-n)(n+2)a^2 \end{pmatrix},$$

we can compute $v_n = (f_{2n-2}^n, f_{2n-1}^n, f_{2n}^n)$ from v_{n-1} via $v_n = v_{n-1}M_n^{\text{tr}}/D_n$.

A better approach

If we set $D_0 = 1$, $M_0 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, and define

$$\mathcal{M}_n = \prod_{i=0}^{n} M_i^{\mathrm{tr}} \qquad \text{and} \qquad \mathcal{D}_n = \prod_{i=0}^{n} D_i,$$

then we can compute v_n as the bottom row of $\mathcal{M}_n/\mathcal{D}_n$. Thus it suffices to compute the partial products $\mathcal{M}_n$ and $\mathcal{D}_n$ modulo $2n+1$.

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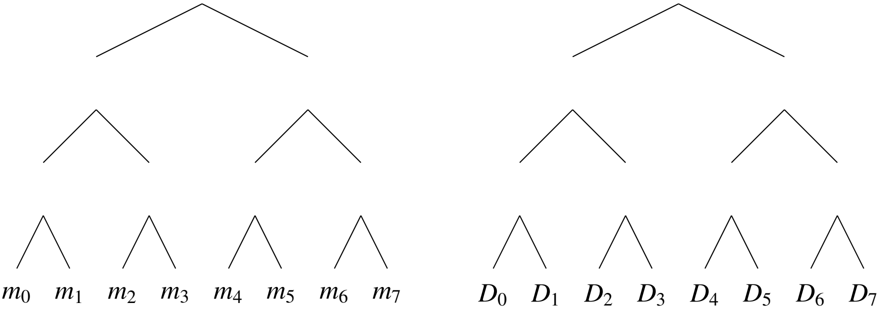
Considering just the $\mathcal{D}_n$, we want to compute

$$\begin{aligned} D_0D_1 &\bmod 3 \\ D_0D_1D_2 &\bmod 5 \\ D_0D_1D_2D_3 &\bmod 7 \\ &\vdots \\ D_0D_1D_2D_3\cdots D_{(N-1)/2} &\bmod N \end{aligned}$$

Doing this naïvely takes $O\left(N^{2+\epsilon}\right)$ time, but it can be done in $O\left(N^{1+\epsilon}\right)$ time.

Remainder trees

Let X be the elliptic curve $y^2 = x^3 + x + 1$ (so $a = b = 1$).
Let us compute $\mathcal{D}_n = \prod_{0 \le i \le n} D_n \bmod (2n+1)$ for $1 \le n < 8$,
where $D_0 = 1$ and $D_i = 2(i+2)(2i-1)b$ for $i > 0$.



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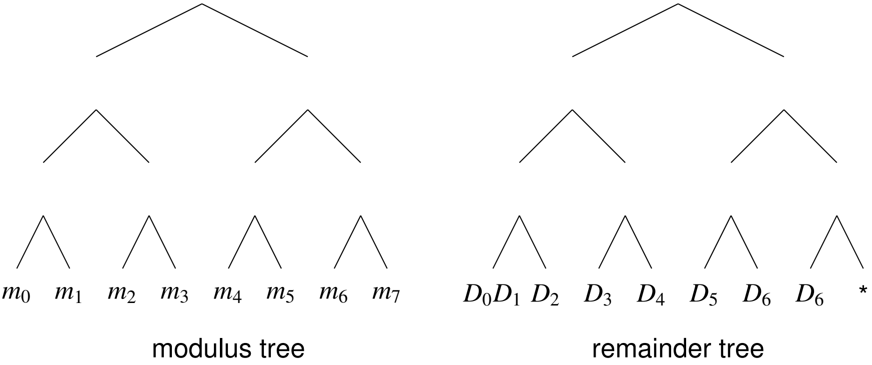

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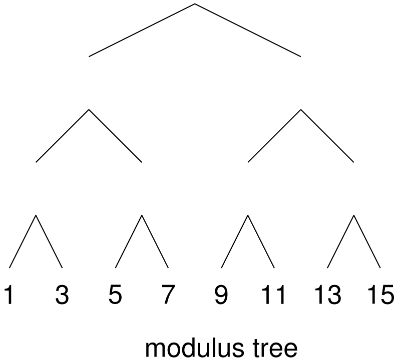

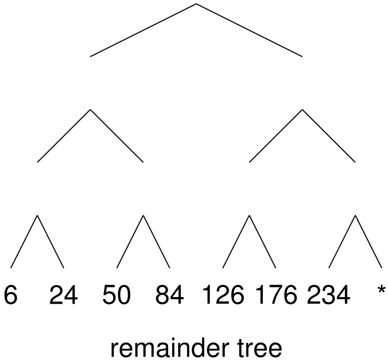

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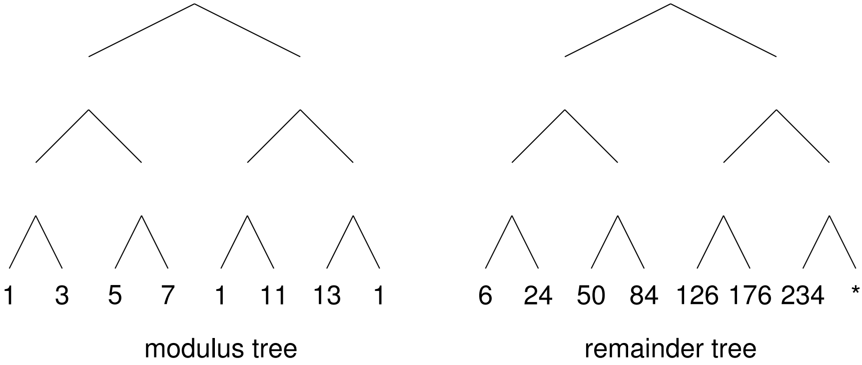

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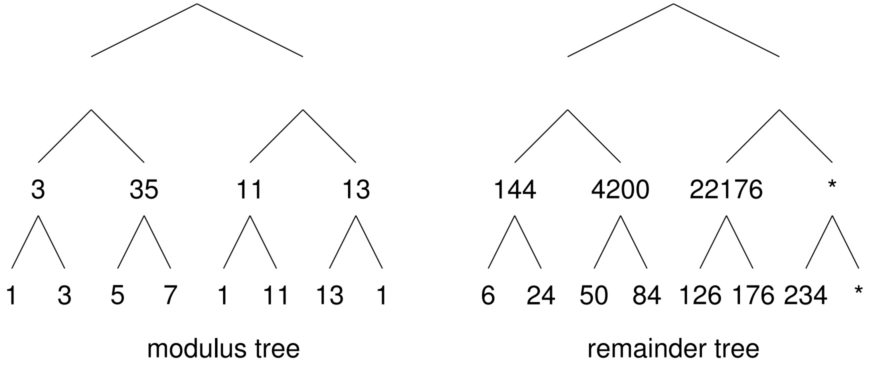

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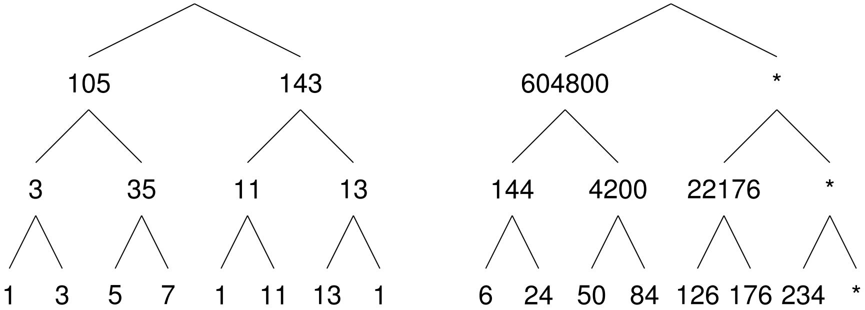

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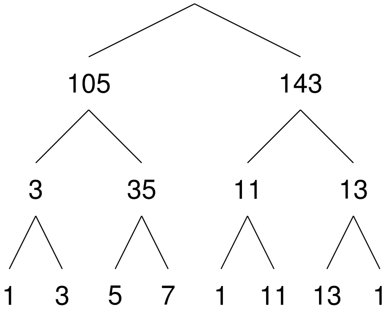

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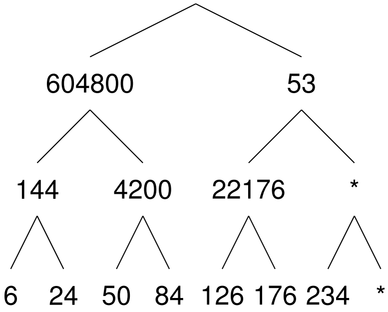

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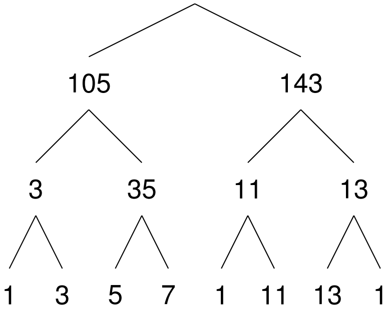

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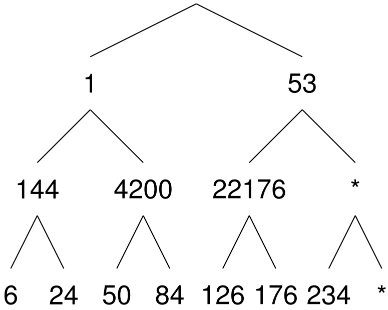

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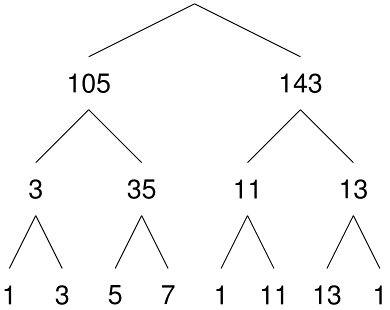

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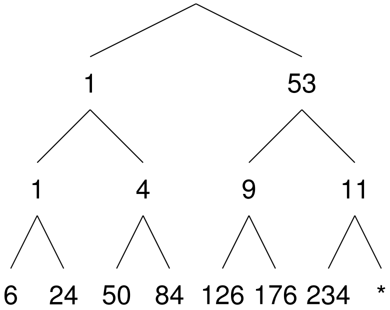

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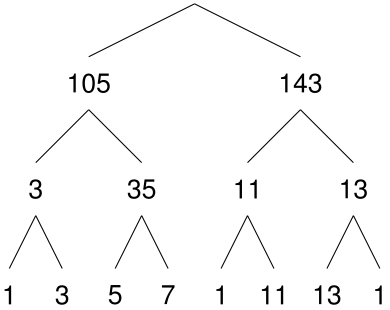

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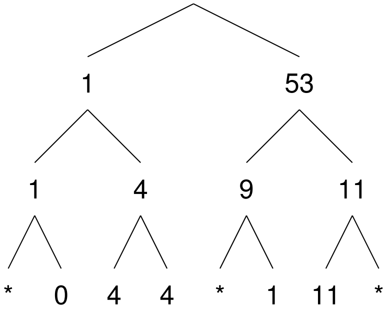

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4. When many moduli are trivial, space can be further reduced (by another log factor in our setting).
5. A time-space trade-off can be used to reduce space even more, but we do not need to do this.

Comparison

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enumerate $X_p(\mathbb{F}_p), \ldots, X_p(\mathbb{F}_{p^g})$	$p \log^{1+\epsilon} p$	$p^2 \log^{1+\epsilon} p$	$p^3 \log^{1+\epsilon} p$

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enumerate $X_p(\mathbb{F}_p), \ldots, X_p(\mathbb{F}_{p^g})$	$p \log^{1+\epsilon} p$	$p^2 \log^{1+\epsilon} p$	$p^3 \log^{1+\epsilon} p$
generic group algorithms	$p^{1/4} \log^{1+\epsilon} p$	$p^{3/4} \log^{1+\epsilon} p$	$p^{5/4} \log^{1+\epsilon}$
p-adic cohomology (Kedlaya-Harvey)	$p^{1/2} \log^{2+\epsilon} p$	$p^{1/2} \log^{2+\epsilon} p$	$p^{1/2} \log^{2+\epsilon} p$

Comparison

	genus 1	genus 2	genus 3
enumerate $X_p(\mathbb{F}_p), \ldots, X_p(\mathbb{F}_{p^g})$	$p \log^{1+\epsilon} p$	$p^2 \log^{1+\epsilon} p$	$p^3 \log^{1+\epsilon} p$
generic group algorithms	$p^{1/4} \log^{1+\epsilon} p$	$p^{3/4} \log^{1+\epsilon} p$	$p^{5/4} \log^{1+\epsilon}$
p-adic cohomology (Kedlaya-Harvey)	$p^{1/2} \log^{2+\epsilon} p$	$p^{1/2} \log^{2+\epsilon} p$	$p^{1/2} \log^{2+\epsilon} p$
ℓ-adic CRT (Schoof-Pila)	$\log^{5+\epsilon} p$	$\log^{8+\epsilon} p$	$\log^{14?+\epsilon} p$
Hasse-Witt matrices	$\log^{4+\epsilon} p$	$\log^{4+\epsilon} p$	$\log^{4+\epsilon} p +$ $p^{1/4} \log^{1+\epsilon} p$

In genus 2 the new algorithm already outperforms smalljac when $N > 2^{21}$.
The prospects in genus 3 look even better (work in progress).

Next steps: generalize to non-hyperelliptic curves of genus 3.