

Murmurations of Arithmetic L -functions

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Simons Collaboration in Arithmetic Geometry, Number Theory, and Computation

September 14, 2023



Joint work with Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver, and Alexey Pozdnyakov,
with thanks to Nina Zubrilina and Peter Sarnak, and to Eran Assaf,
and also to Jonathan Bober, Andrew Booker, Min Lee, and David Lowry-Duda.

Elliptic curves and their L -functions

Let E/\mathbb{Q} be an elliptic curve, say $E: y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$.

For primes $p \nmid \Delta(E) := -16(4A^3 + 27B^2)$ this equation defines an elliptic curve E/\mathbb{F}_p .

For all such primes p we have the **trace of Frobenius** $a_p(E) := p + 1 - \#E(\mathbb{F}_p) \in \mathbb{Z}$.

One can also define $a_p(E)$ for $p \mid \Delta(E)$, and then construct the **L -function**

$$L(E, s) := \prod_p (1 - a_p p^{-s} + \chi(p) p^{1-2s})^{-1} = \sum a_n n^{-s}$$

where $\chi(p) = 0$ for $p \mid N(E)$ and $\chi(p) = 1$ otherwise and $N(E) \mid \Delta(E)$ is the **conductor**.

But in fact the a_p for $p \nmid \Delta(E)$ determine $L(E, s)$ (via **strong multiplicity one**), and also the conductor and **root number** $w(E) = \pm 1$, which appear in the **functional equation**

$$\Lambda(E, s) = w(E) N(E)^{1-s} \Lambda(E, 2-s)$$

where $\Lambda(s) := \Gamma_{\mathbb{C}}(s) L(E, s)$. The L -function $L(E, s)$ determines the **isogeny class** of E .

Arithmetic statistics of Frobenius traces of elliptic curves over \mathbb{Q}

Three conjectures from the 1960s and 1970s (the first is now a theorem):

1. **Sato–Tate:** The sequence $x_p := a_p(E)/\sqrt{p}$ is equidistributed with respect to the pushforward of the Haar measure of the Sato-Tate group of E (typically $SU(2)$).
2. **Birch and Swinnerton-Dyer:**

$$\lim_{x \rightarrow \infty} \frac{\log x}{2\sqrt{x}} \sum_{p \leq x} \frac{a_p(E)}{\sqrt{p}} = \frac{1}{2} - r(E).$$

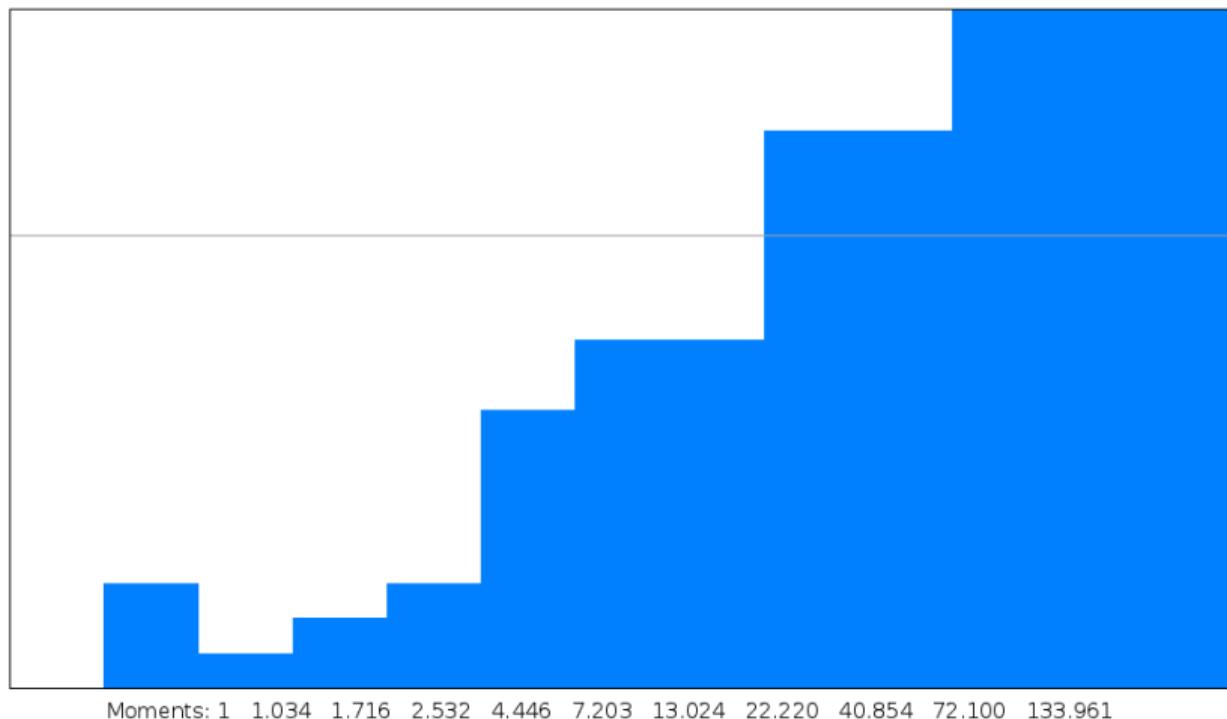
3. **Lang–Trotter:** For every nonzero $t \in \mathbb{Z}$ there is a real number $C_{E,t}$ for which

$$\#\{p \leq x : a_p(E) = t\} \sim C_{E,t} \frac{\sqrt{x}}{\log x}.$$

These conjectures depend only on $L(E, s)$ and generalize to other L -functions.

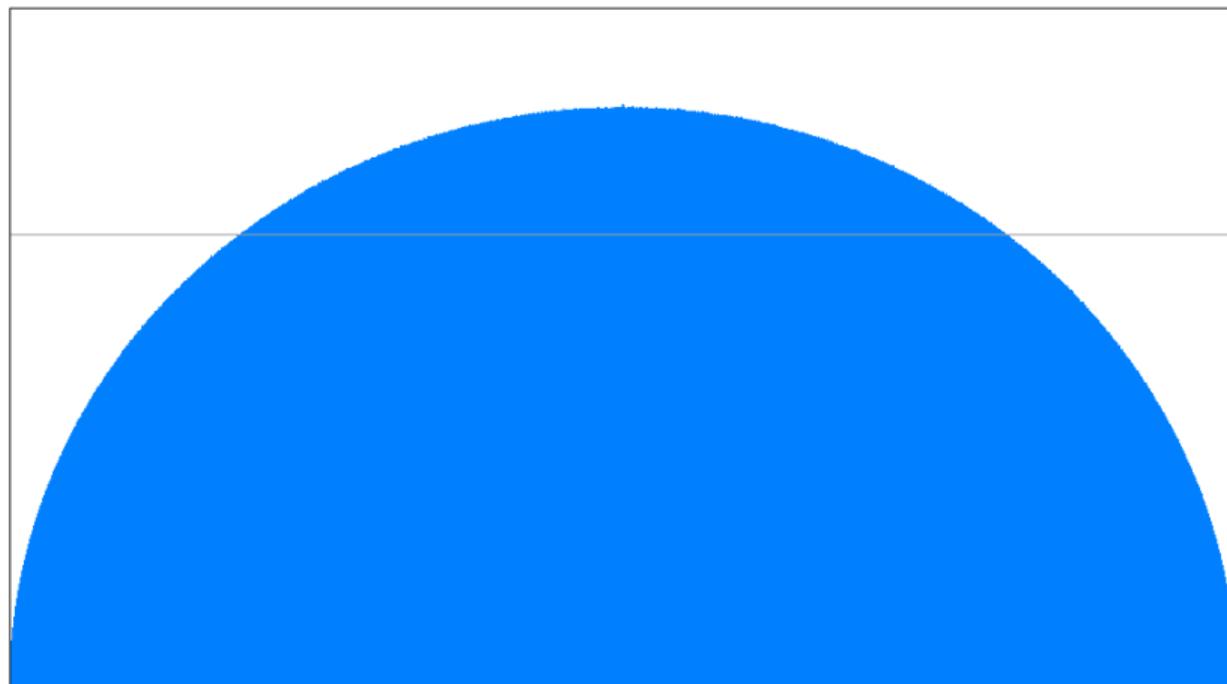
Example: Elkies' curve of rank ≥ 28 ($= 28$ under GRH).

a1 histogram of $y^2 + xy + y = x^3 - x^2 - 20067762415575526585033208209338542750930230312178956502x + 34481611795030556467032985690390720374855944359319180361266008296291939448732243429$ for $p \leq 2^{10}$
172 data points in 13 buckets, $z_1 = 0.023$, out of range data has area 0.250



Example: Elkies' curve of rank ≥ 28 ($= 28$ under GRH).

a1 histogram of $y^2 + xy + y = x^3 - x^2 - 20067762415575526585033208209338542750930230312178956502x$
+ 34481611795030556467032985690390720374855944359319180361266008296291939448732243429 for $p \leq 2^{40}$
41203088796 data points in 202985 buckets



Moments: 1 0.000 1.000 0.000 2.000 0.000 5.000 0.001 14.000 0.003 42.000

How rank effects trace distributions

One formulation of the BSD conjecture implies that

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{p \leq x} \frac{a_p(E) \log p}{p} = -r + \frac{1}{2}, \quad (1)$$

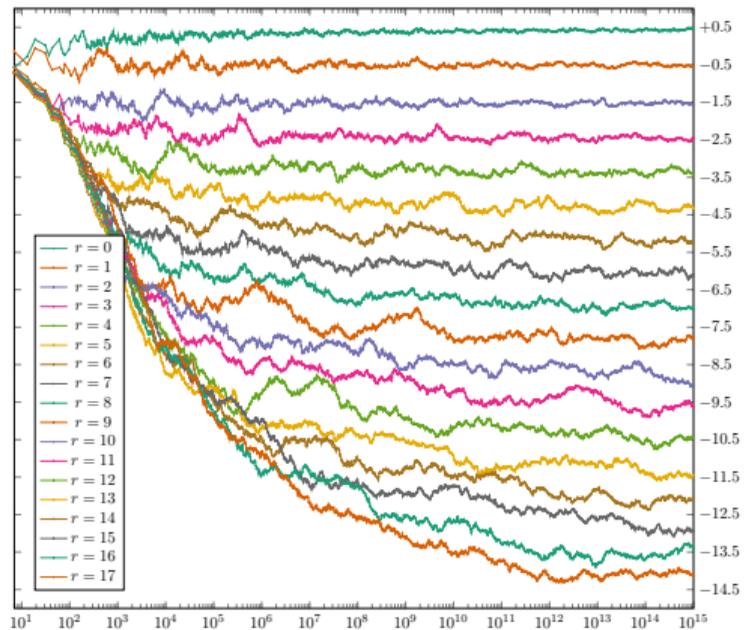
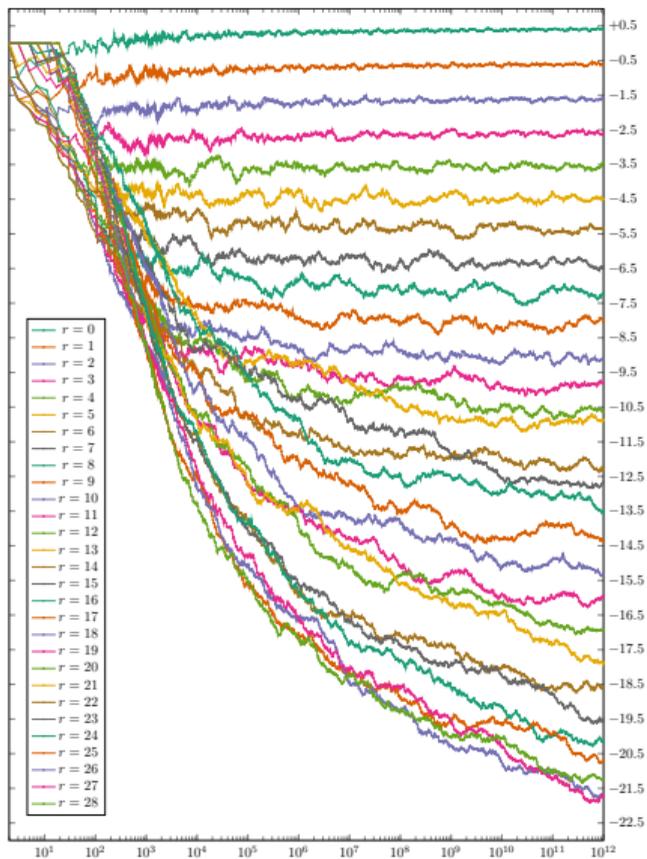
and sums of this form (Mestre-Nagao sums) are often used as a tool when searching for elliptic curves of large rank (which necessarily have large conductor N).^{1 2}

Theorem (Kim-Murty 2023)

If the limit on the LHS of (1) exists then it equals the RHS with r the analytic rank, and the L -function of E satisfies the Riemann hypothesis.

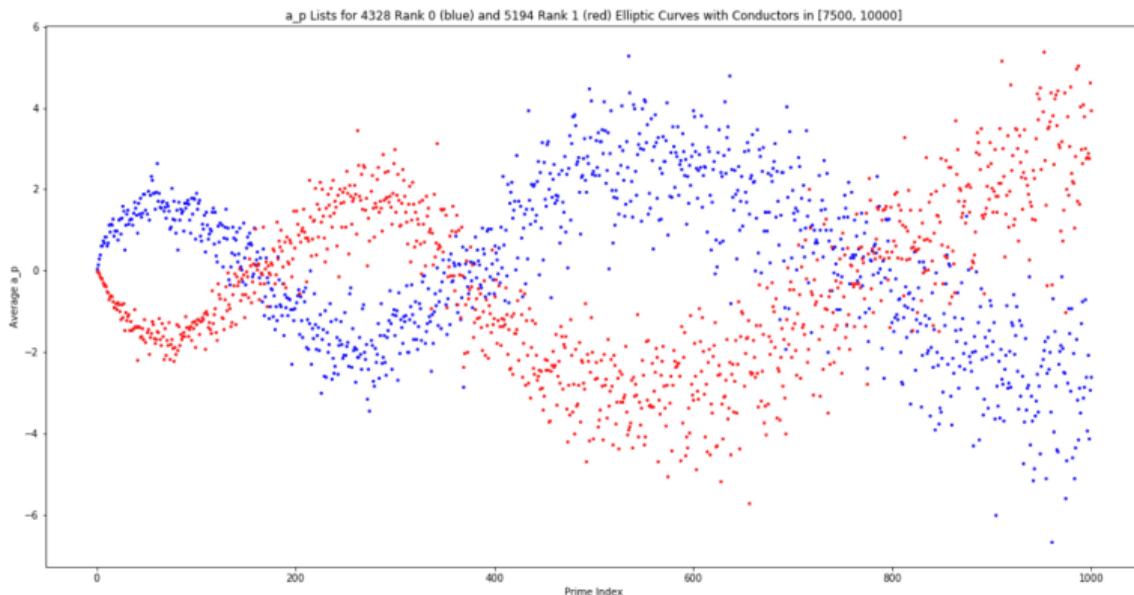
¹See [Sarnak's 2007 letter to Mazur](#).

²See the recent paper of [Kazalicki-Vlah](#) for some recent machine-learning work on this topic.



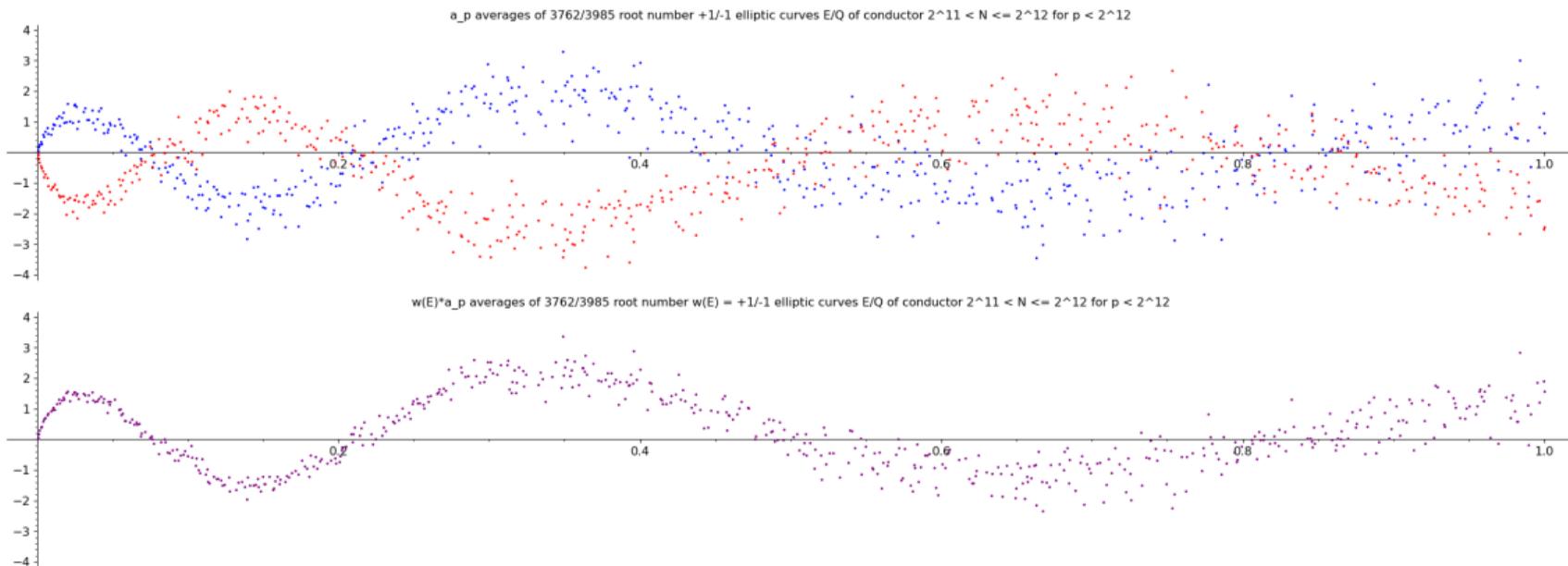
Murmurations of elliptic curves

In their 2022 preprint *Murmurations of elliptic curves*, Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver, and Alexey Pozdnyakov observed a curious fluctuation in average Frobenius traces of elliptic curves in a given conductor interval depending on the rank.



Murmurations of elliptic curves

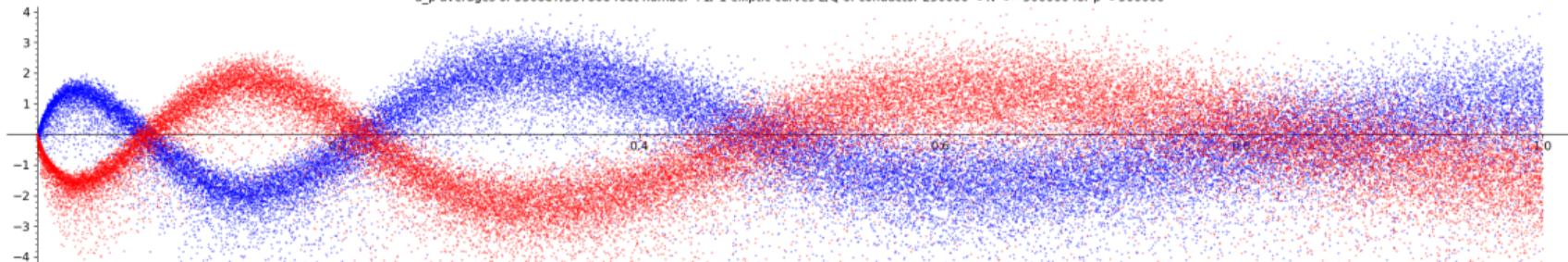
Elliptic curve L -functions of conductor $N \in (M, 2M]$ for $M = 2^{11}, 2^{12}, \dots, 2^{17}, 250000$. The x -axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or $m_p := w(E)a_p(E)$ over even/odd or all E/\mathbb{Q} with $N_E \in (M, 2M]$.



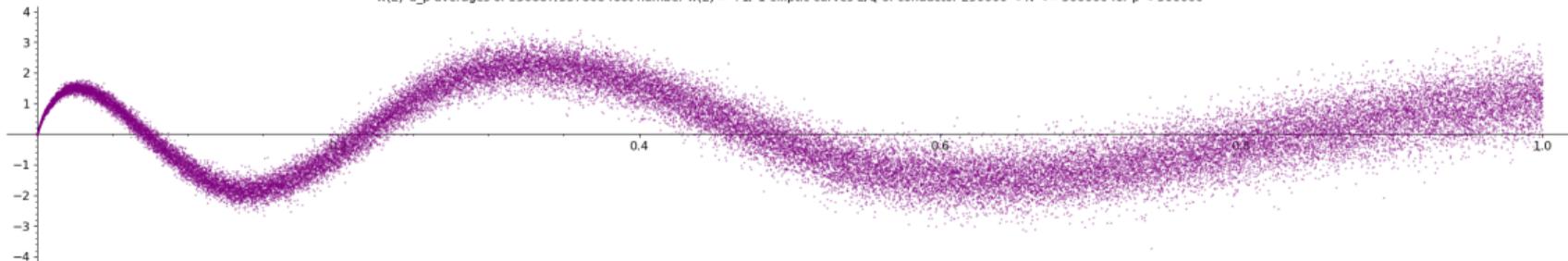
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a_p averages of 530887/537808 root number +1/-1 elliptic curves E/Q of conductor 250000 < N <= 500000 for p < 500000



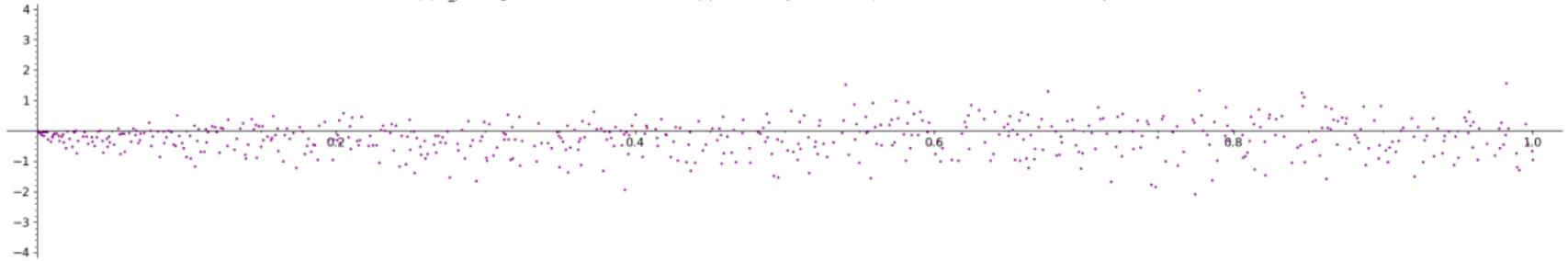
w(E)*a_p averages of 530887/537808 root number w(E) = +1/-1 elliptic curves E/Q of conductor 250000 < N <= 500000 for p < 500000



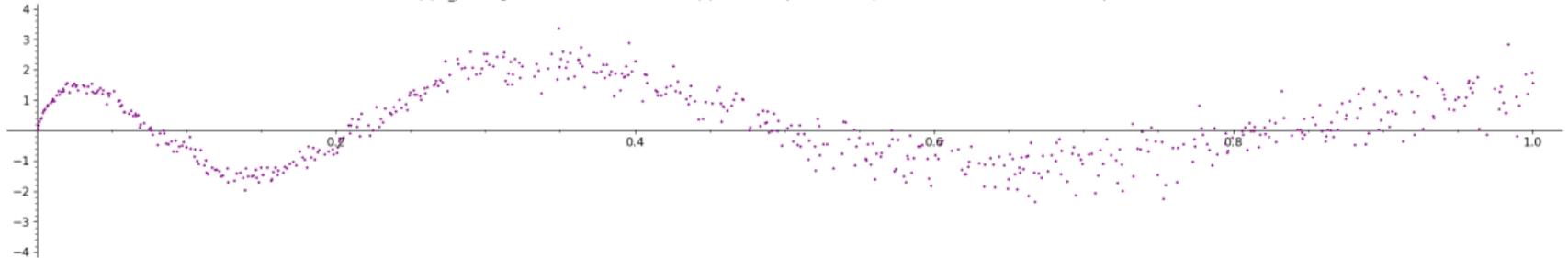
Bias cancellation

There is a negative bias in \bar{a}_p that is parity-independent and disappears in \bar{m}_p .
This is especially noticeable at primes $p \equiv 1 \pmod{24}$ and $p \equiv \square \pmod{5, 7}$.

$w(E) \cdot a_p$ averages of 3762/3985 root number $w(E) = +1/-1$ elliptic curves E/\mathbb{Q} of conductor $2^{11} < N \leq 2^{12}$ for $p < 2^{12}$

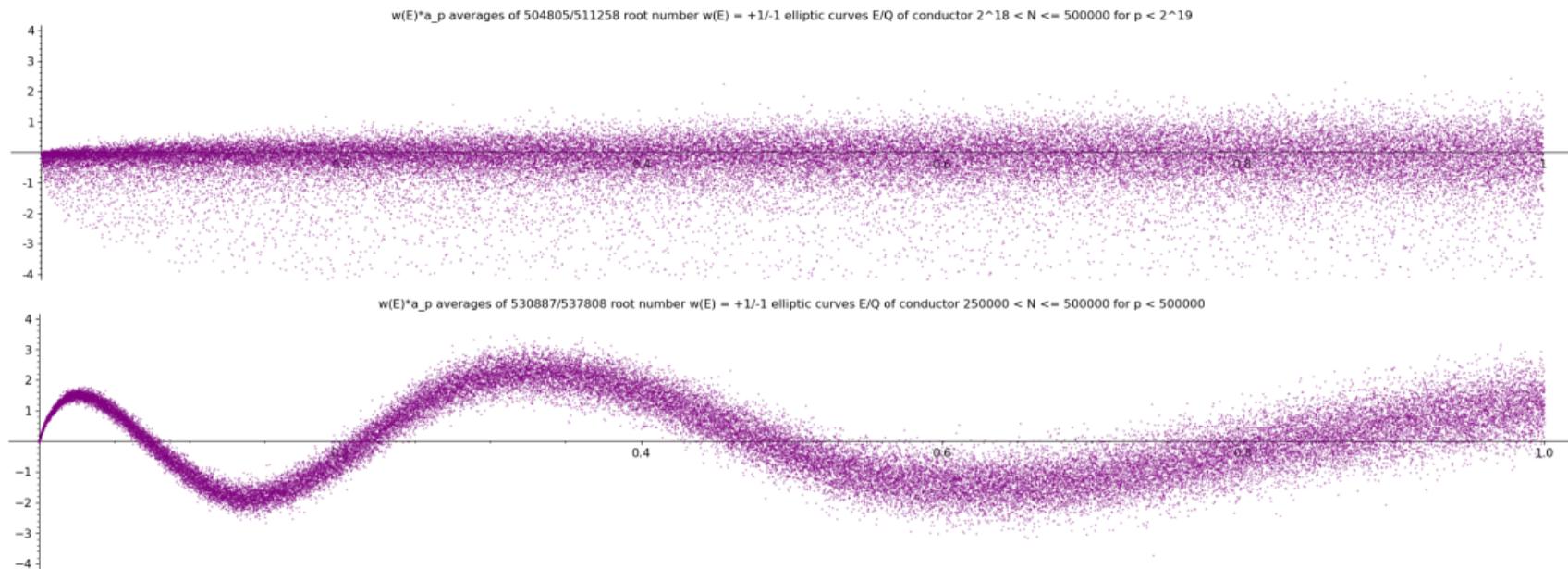


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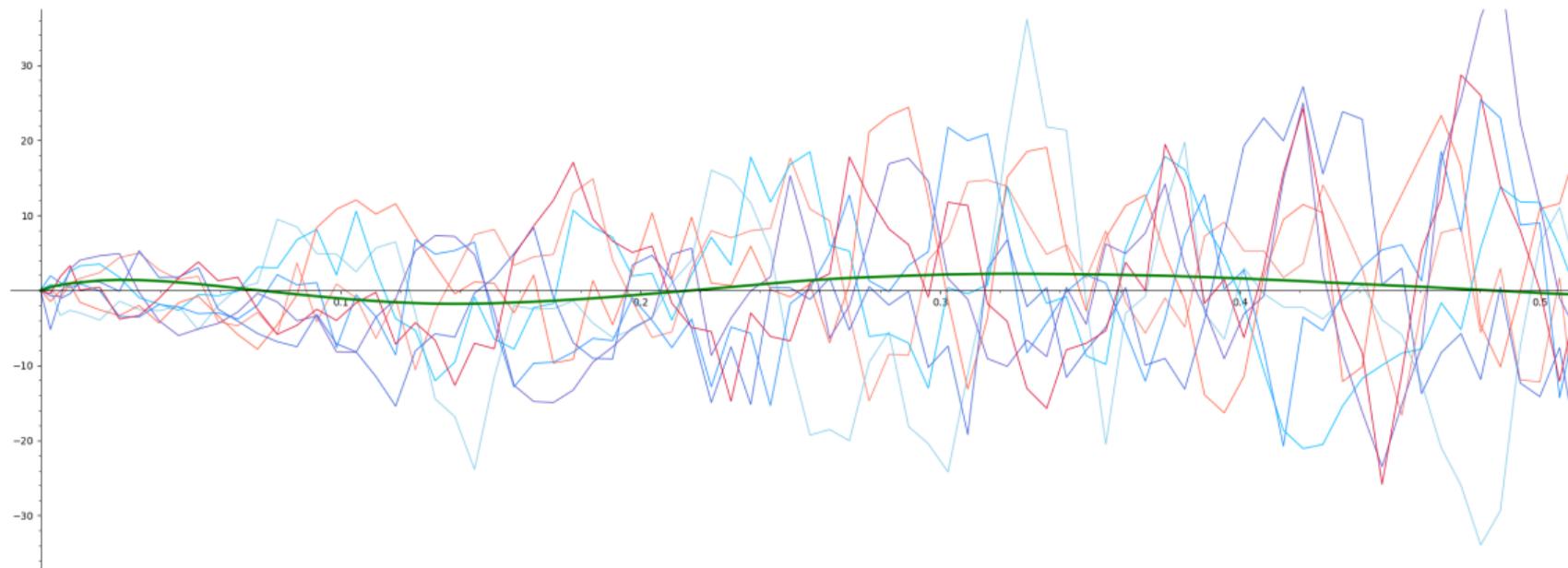
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Moving averages

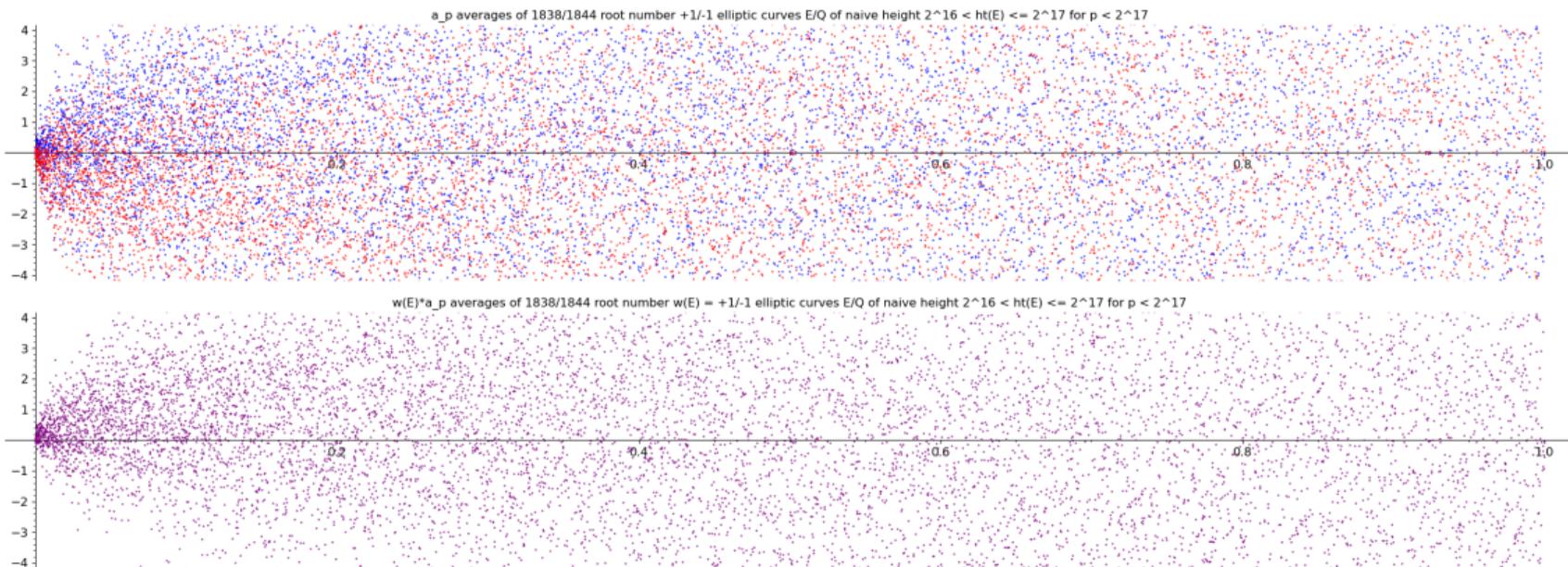
Moving average line plots of \bar{m}_p for 8 individual and all E/\mathbb{Q} with $N_E \in (M, 2M]$, using subintervals of size \sqrt{M} for $p \leq 2M$, with $M = 2^{17}$.



147455.b2, 163839.a1, 180222.be2, 196606.b1, 212990.11, 229374.a1, 245758.a1, 262143.d1

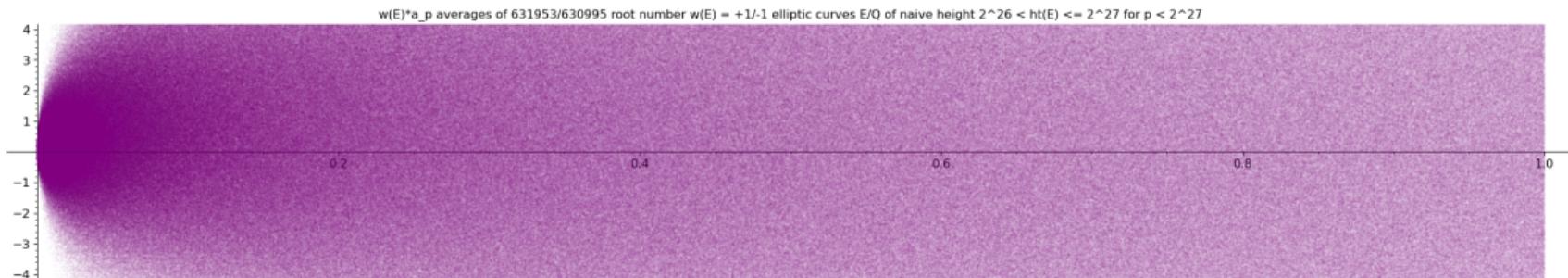
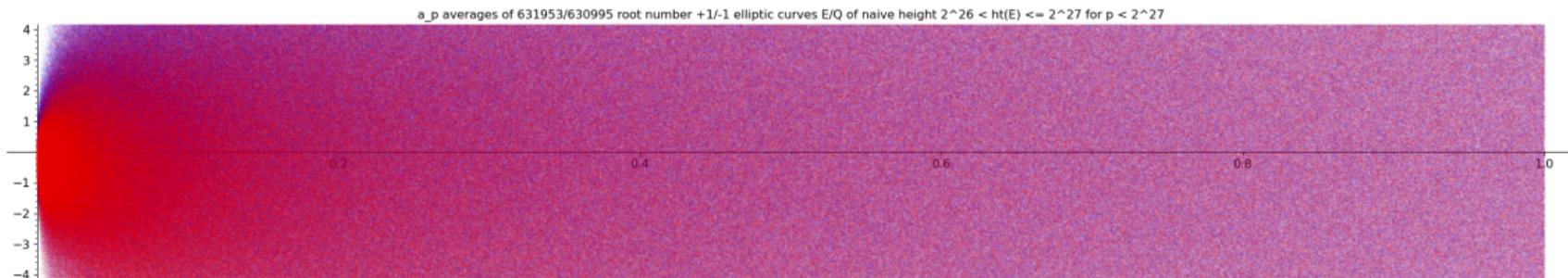
Ordering by naive height

Elliptic curves with $\text{ht}(E) := \max(4|A|^3, 27B^2)$ in $(M, 2M]$ for $M = 2^{16}, \dots, 2^{25}$. The x -axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or m_p over even/odd or all E/\mathbb{Q} with $N_E \in (M, 2M]$.



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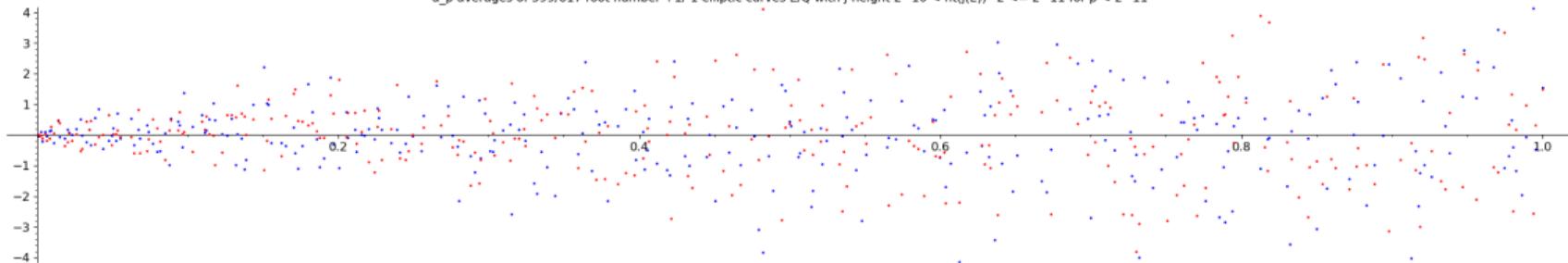


Ordering by j -invariant

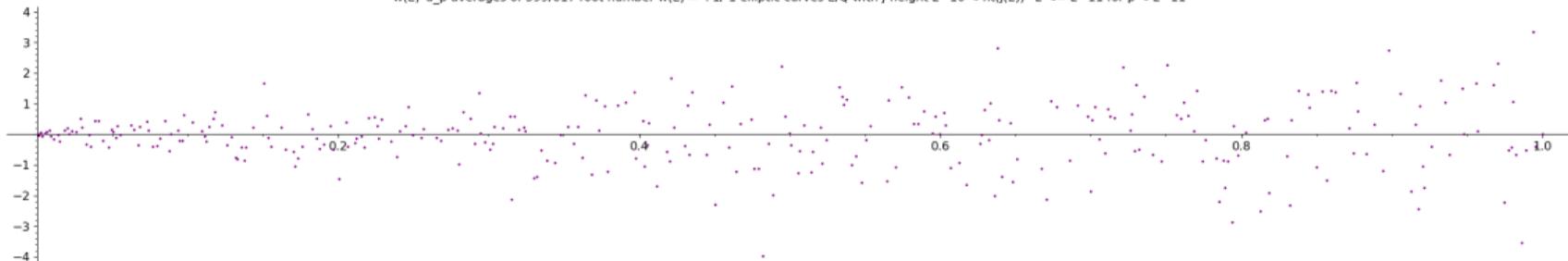
Elliptic curves with $\text{ht}(j(E))^{12/5}$ in $(M, 2M]$ for $M = 2^{14}, \dots, 2^{24}$.

The x -axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or m_p over even/odd or all E/\mathbb{Q} with $N_E \in (M, 2M]$.

a_p averages of 599/617 root number +1/-1 elliptic curves E/\mathbb{Q} with j -height $2^{10} < \text{ht}(j(E))^2 \leq 2^{11}$ for $p < 2^{11}$



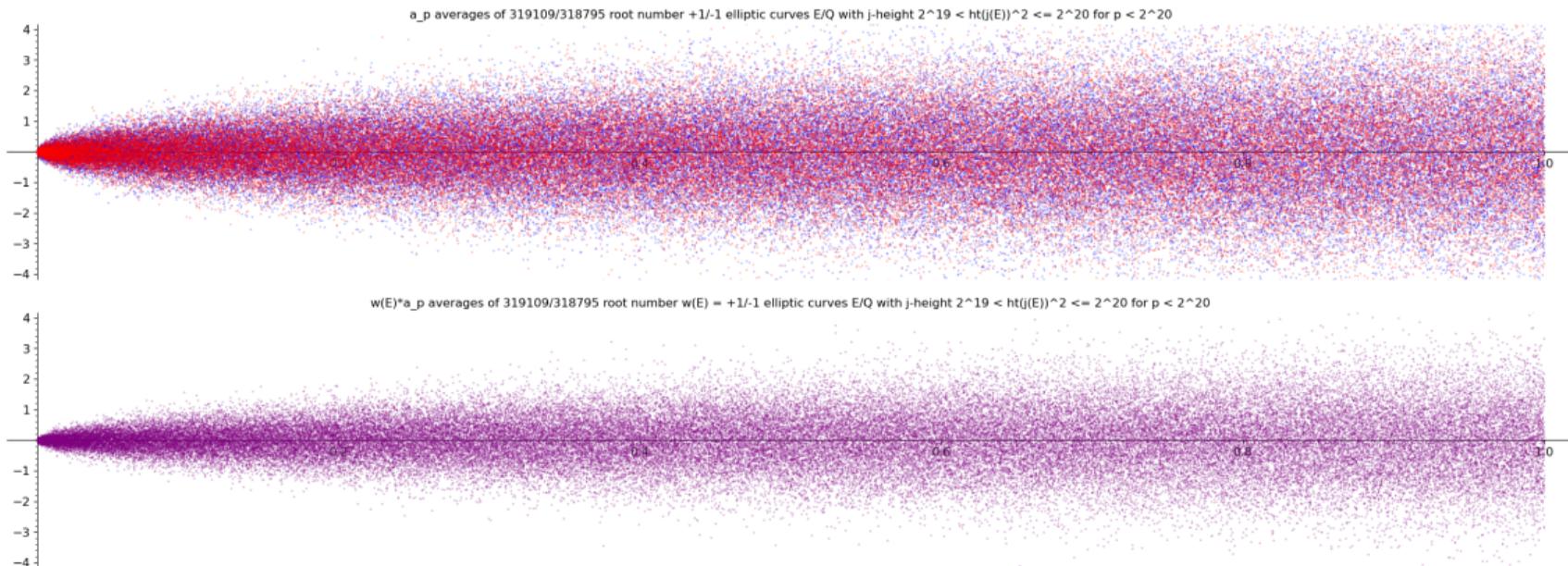
$w(E) \cdot a_p$ averages of 599/617 root number $w(E) = +1/-1$ elliptic curves E/\mathbb{Q} with j -height $2^{10} < \text{ht}(j(E))^2 \leq 2^{11}$ for $p < 2^{11}$



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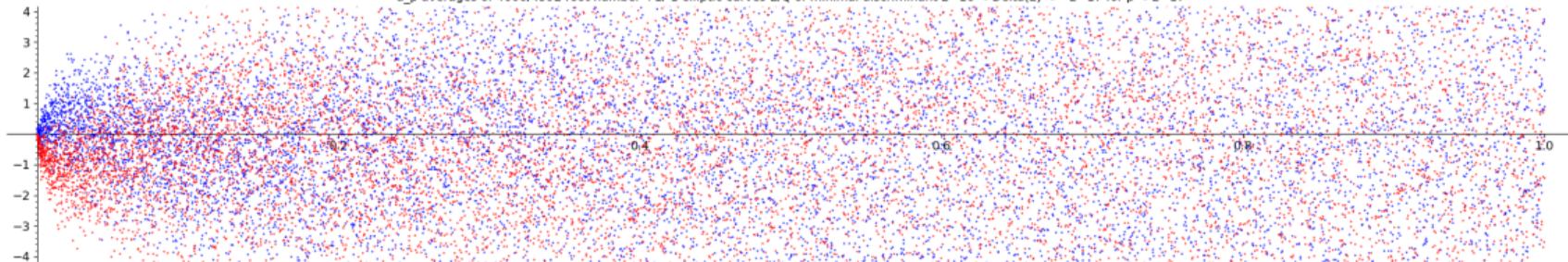
The x -axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or m_p over even/odd or all E/\mathbb{Q} with $N_E \in (M, 2M]$.



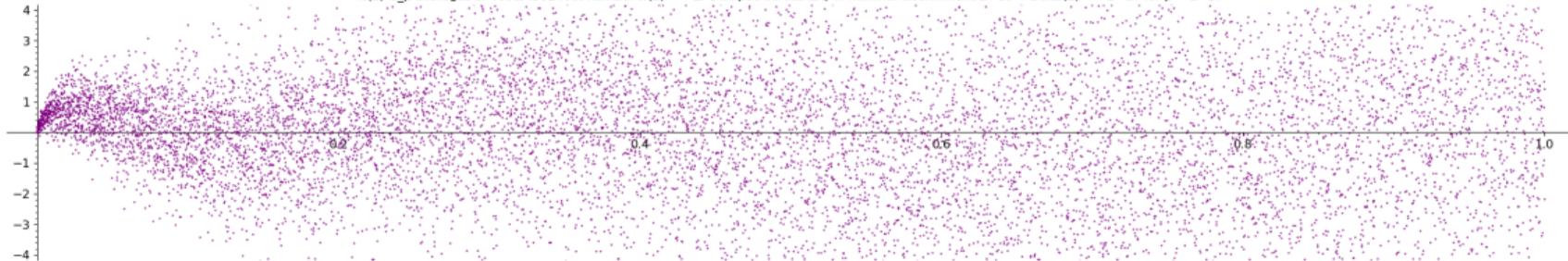
Ordering by minimal discriminant

Elliptic curves with minimal discriminant $\Delta(E)$ in $(M, 2M]$ for $M = 2^{16}, \dots, 2^{25}$. The x -axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or m_p over even/odd or all E/\mathbb{Q} with $N_E \in (M, 2M]$.

a_p averages of 4606/4592 root number $+1/-1$ elliptic curves E/\mathbb{Q} of minimal discriminant $2^{16} < \Delta(E) \leq 2^{17}$ for $p < 2^{17}$

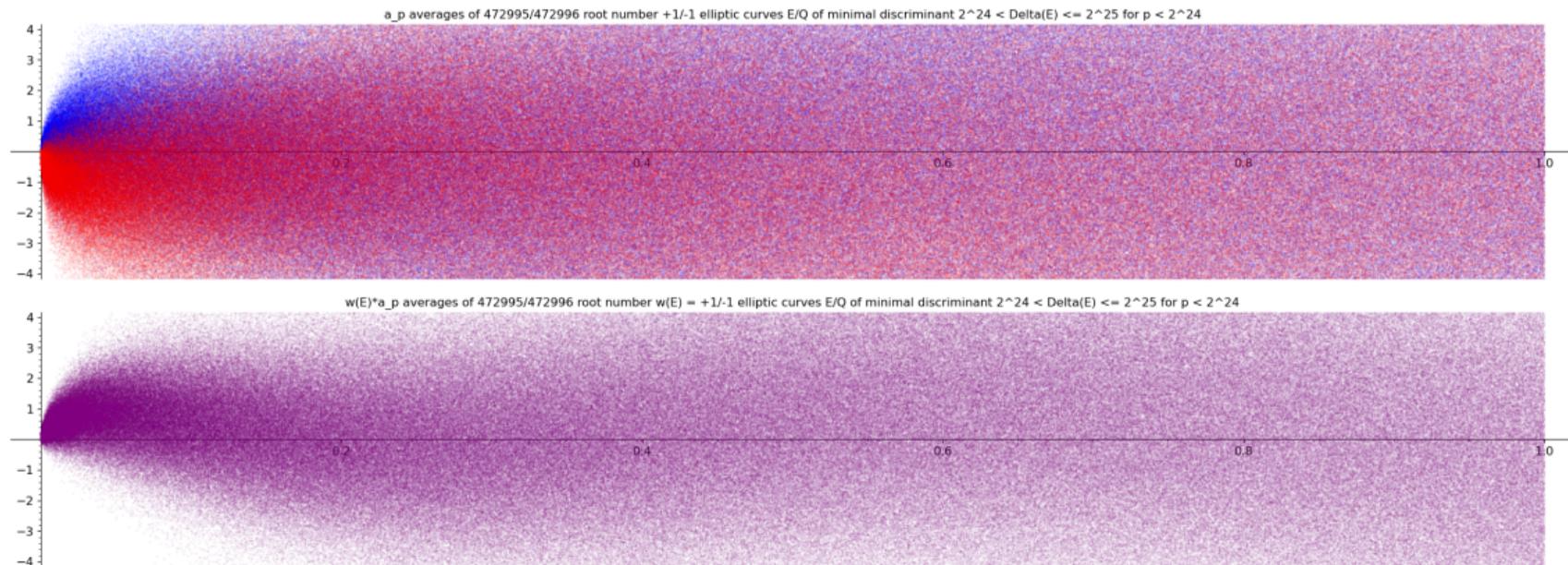


$w(E) \cdot a_p$ averages of 4606/4592 root number $w(E) = +1/-1$ elliptic curves E/\mathbb{Q} of minimal discriminant $2^{16} < \Delta(E) \leq 2^{17}$ for $p < 2^{17}$



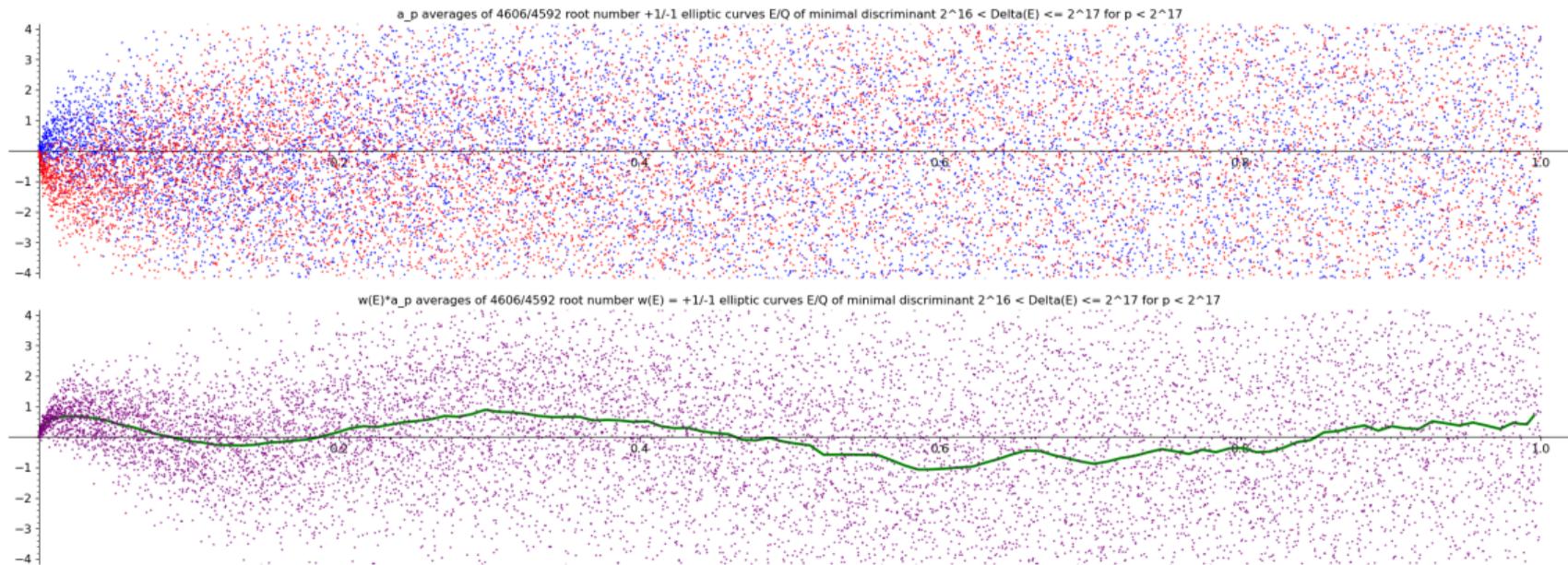
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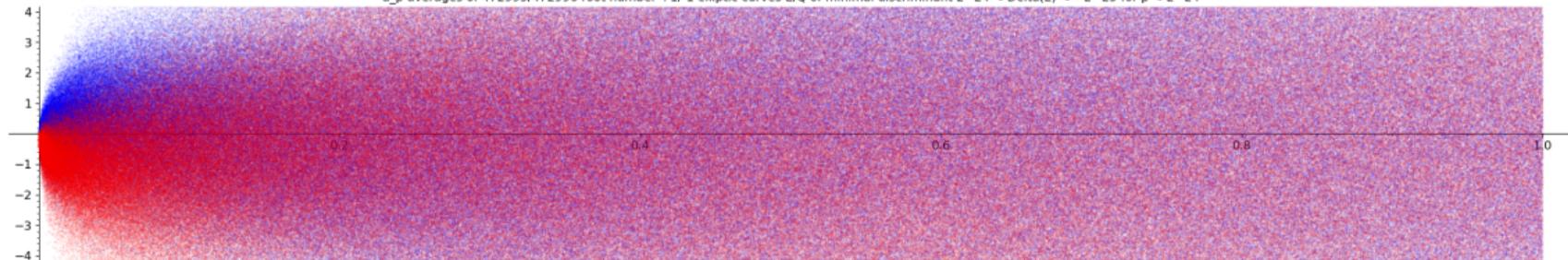
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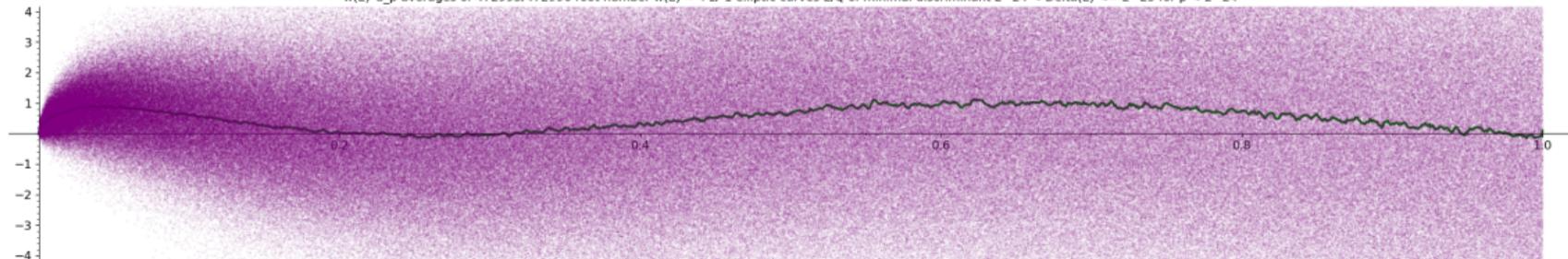
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a_p averages of 472995/472996 root number +1/-1 elliptic curves E/Q of minimal discriminant $2^{24} < \Delta(E) \leq 2^{25}$ for $p < 2^{24}$

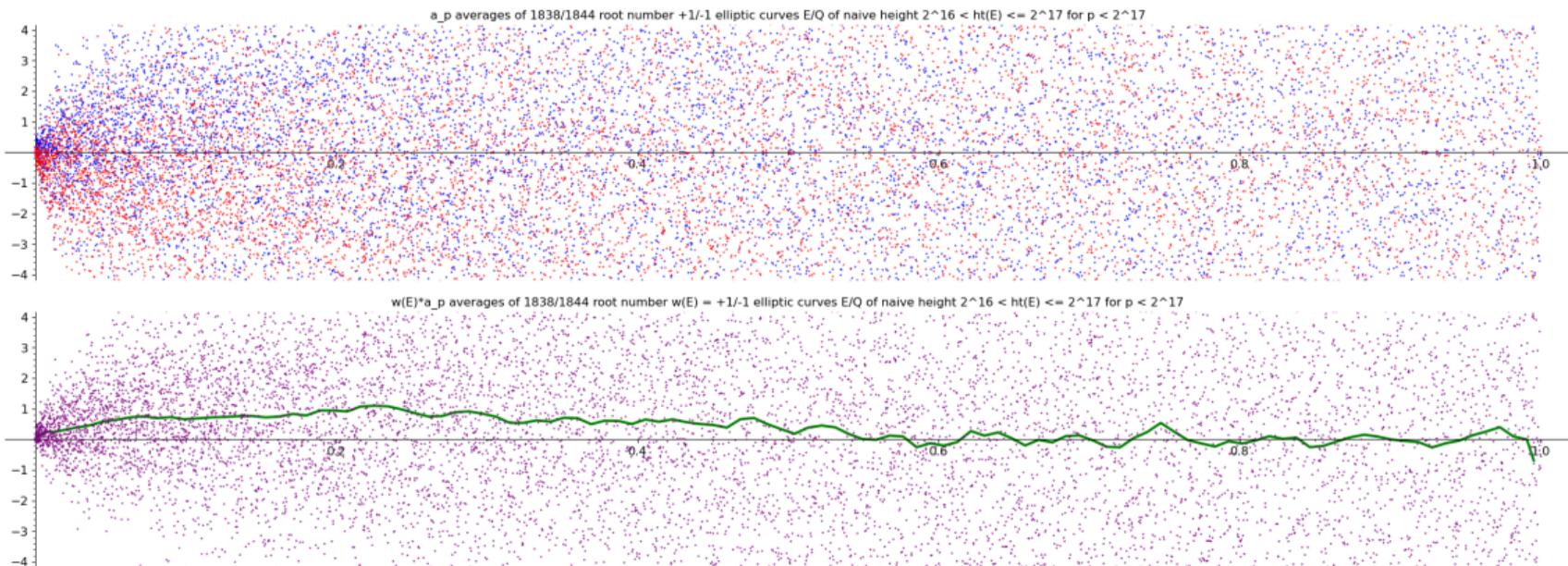


w(E)*a_p averages of 472995/472996 root number w(E) = +1/-1 elliptic curves E/Q of minimal discriminant $2^{24} < \Delta(E) \leq 2^{25}$ for $p < 2^{24}$



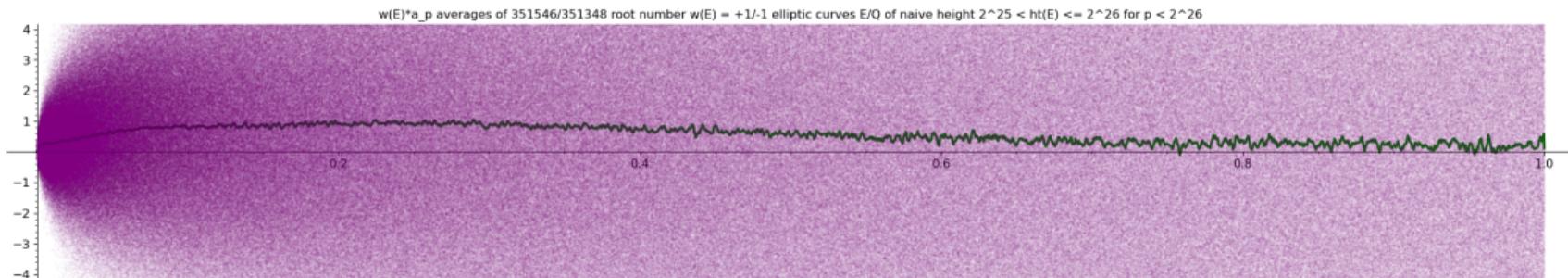
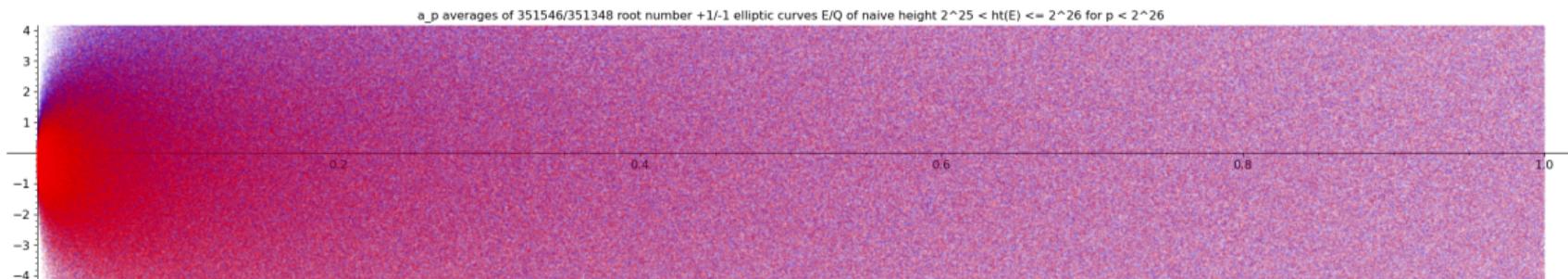
Ordering by naive height (redux)

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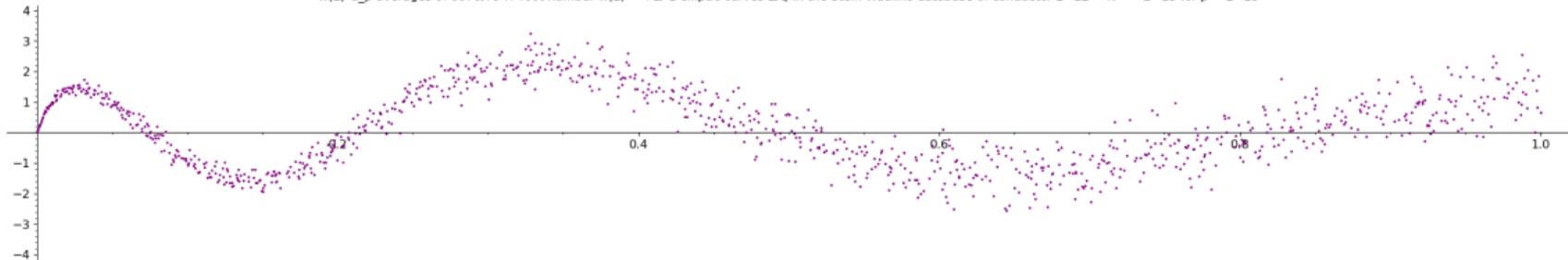


Ordering by conductor in the Stein-Watkins database (SWDB)

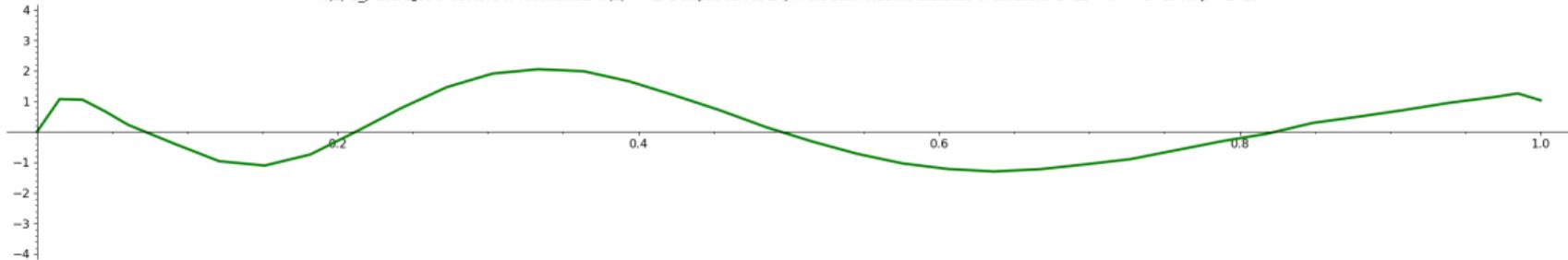
Elliptic curves in the SWDB of conductor $N \in (M, 2M]$ for $M = 2^{12}, \dots, 2^{25}$.

The x -axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or m_p over even/odd or all E/\mathbb{Q} with $N_E \in (M, 2M]$.

$w(E)*a_p$ averages of 6878/7947 root number $w(E) = +1/-1$ elliptic curves E/\mathbb{Q} in the Stein-Watkins database of conductor $2^{12} < N \leq 2^{13}$ for $p < 2^{13}$



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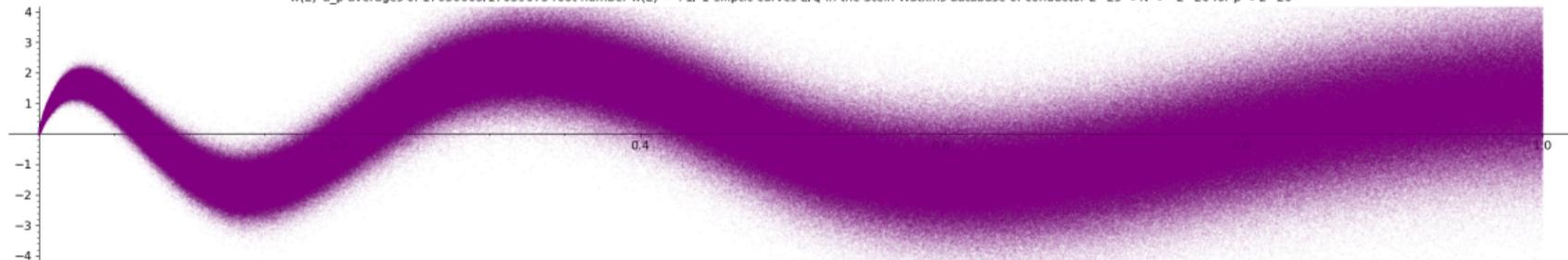


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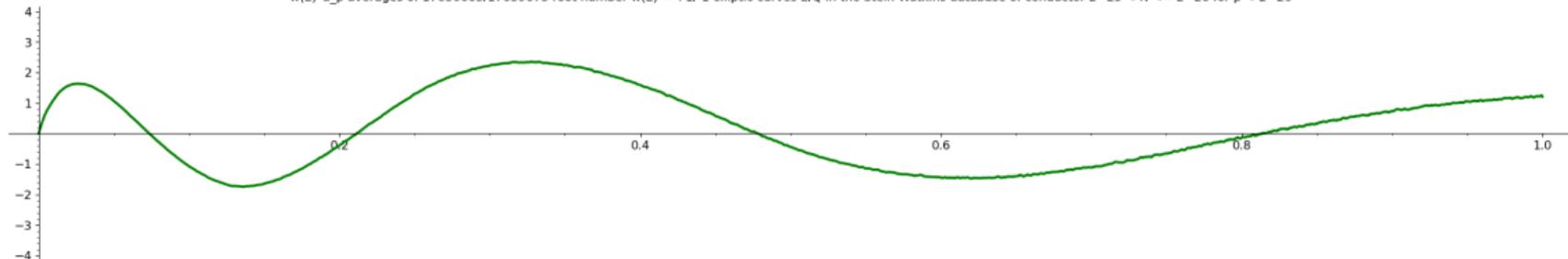
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w(E)*a_p averages of 17630665/17639675 root number w(E) = +1/-1 elliptic curves E/Q in the Stein-Watkins database of conductor $2^{25} < N \leq 2^{26}$ for $p < 2^{26}$



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Arithmetic L -functions

We call an L -function is **analytic** if it has the properties every good L -function should: analytic continuation, functional equation, Euler product, temperedness, central character; see [FPRS18](#); it is **analytically normalized** if its central value is at $s = 1/2$.

An analytically normalized L -function $L_{\text{an}}(s) = \sum a_n n^{-s}$ is **arithmetic** if $a_n n^{\omega/2} \in \mathcal{O}_K$ for some number field K and $\omega \in \mathbb{Z}_{\geq 0}$. The least such ω is the **motivic weight**. Its **arithmetic normalization** $L(s) := L_{\text{an}}(s + \omega/2)$ has coefficients in \mathcal{O}_K and satisfies

$$\Lambda(s) = N^{1-s} w \bar{\Lambda}(1 + \omega - s).$$

L -functions of abelian varieties have motivic weight $\omega = 1$.

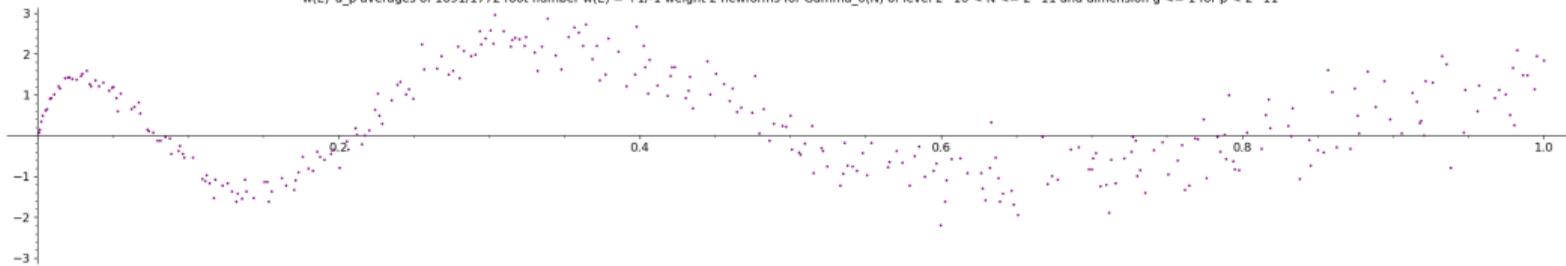
L -functions of weight- k holomorphic cuspforms have motivic weight $\omega = k - 1$.

We consider **Galois-closed** families of **self-dual** arithmetically normalized L -functions. In any such family the values of a_p and m_p are integers and $w = \pm 1$.

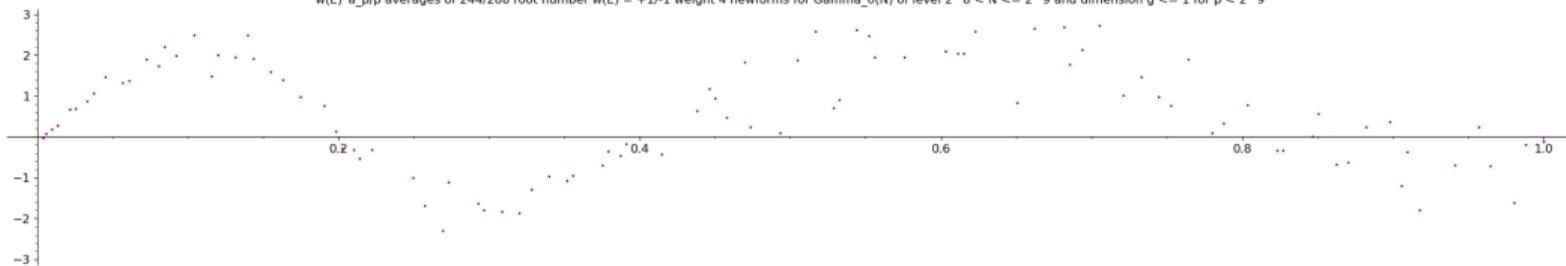
When averaging a_p 's in motivic weight $\omega > 1$ we **normalize** them via $a_p \mapsto a_p / p^{(\omega-1)/2}$. This ensures that we always have $|a_p| = O(\sqrt{p})$, as with elliptic curves.

Newforms for $\Gamma_0(N)$ of weight $k = 2, 4, 6$ with rational coefficients.

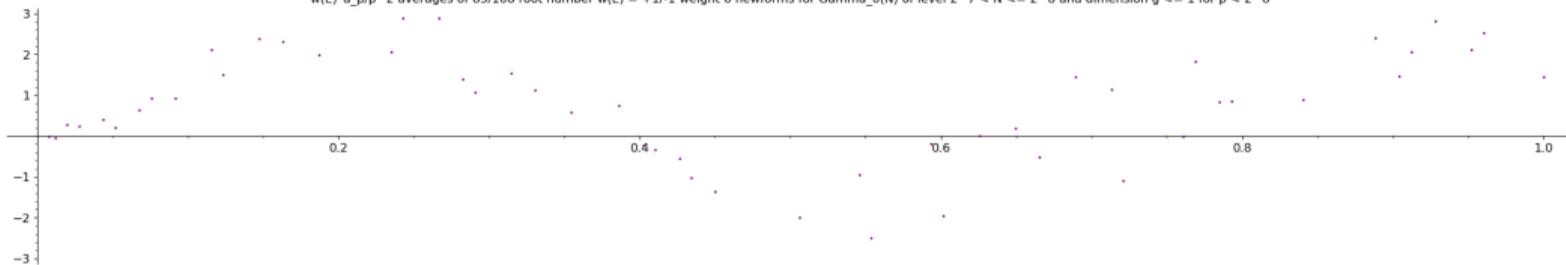
$w(E) \cdot a_p$ averages of 1691/1772 root number $w(E) = +1/-1$ weight 2 newforms for $\Gamma_0(N)$ of level $2^{10} < N \leq 2^{11}$ and dimension $g \leq 1$ for $p < 2^{11}$



$w(E) \cdot a_{p/p}$ averages of 244/288 root number $w(E) = +1/-1$ weight 4 newforms for $\Gamma_0(N)$ of level $2^8 < N \leq 2^9$ and dimension $g \leq 1$ for $p < 2^9$

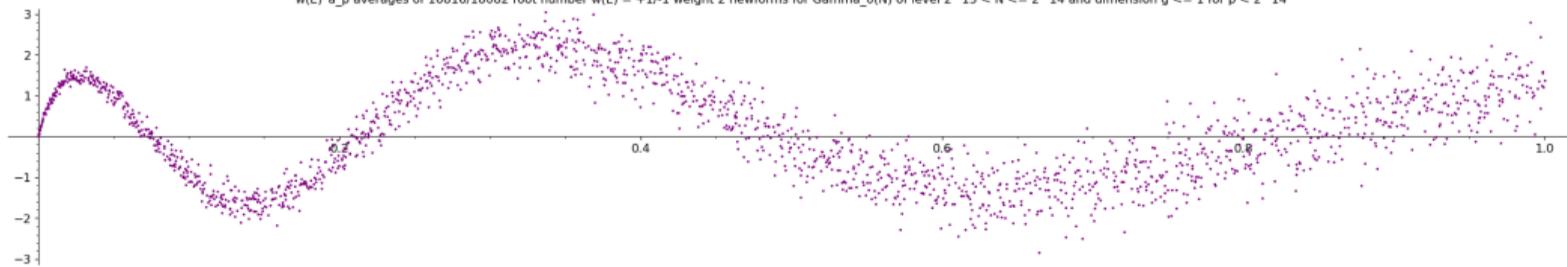


$w(E) \cdot a_{p/p^2}$ averages of 85/108 root number $w(E) = +1/-1$ weight 6 newforms for $\Gamma_0(N)$ of level $2^7 < N \leq 2^8$ and dimension $g \leq 1$ for $p < 2^8$

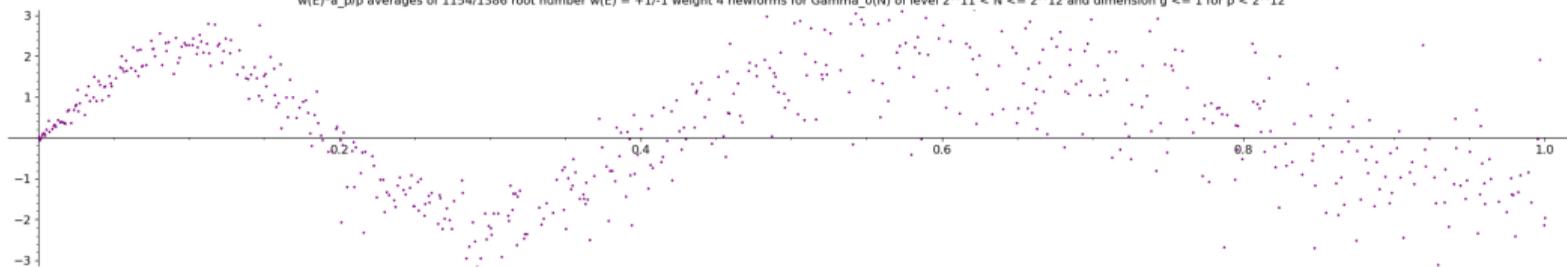


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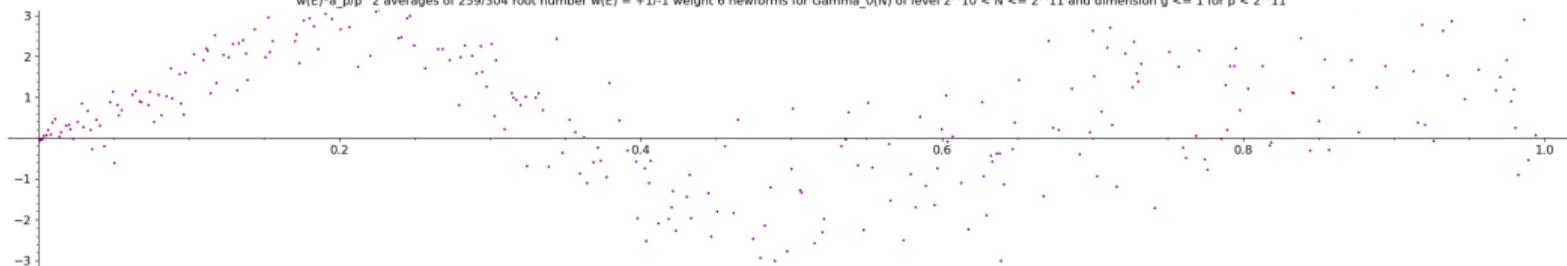
$w(E)^*a_p$ averages of 16816/18082 root number $w(E) = +1/-1$ weight 2 newforms for $\Gamma_0(N)$ of level $2^{13} < N \leq 2^{14}$ and dimension $g \leq 1$ for $p < 2^{14}$



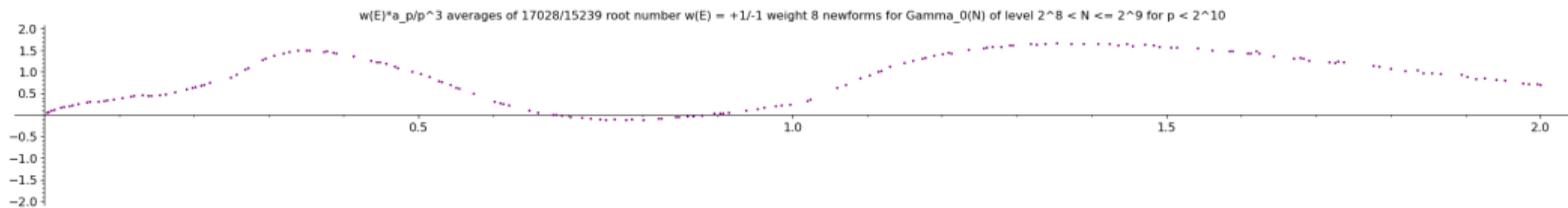
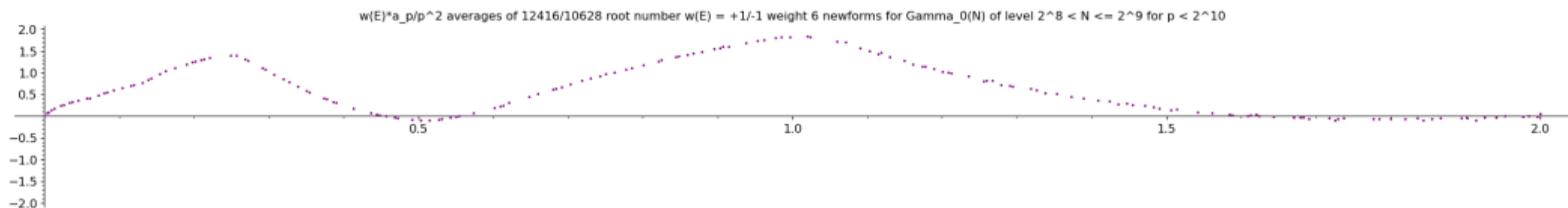
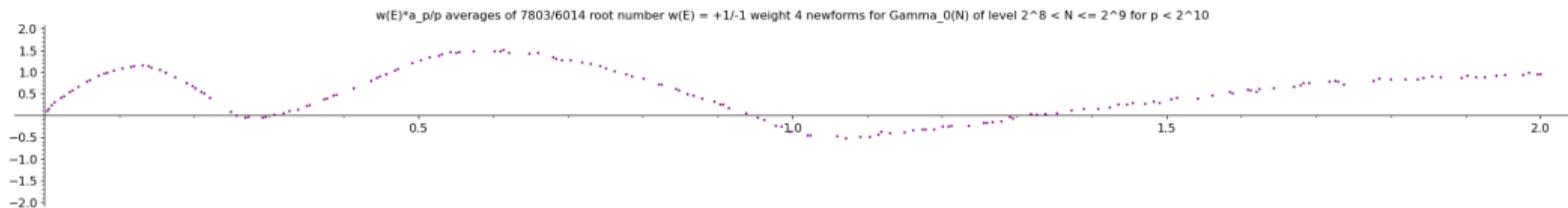
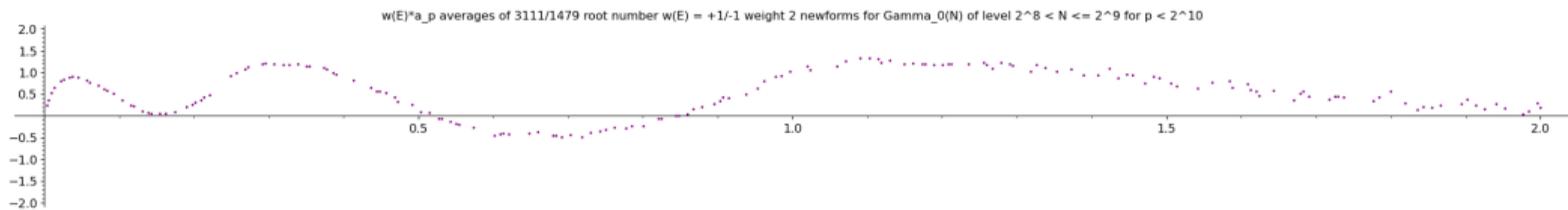
$w(E)^*a_p$ averages of 1154/1386 root number $w(E) = +1/-1$ weight 4 newforms for $\Gamma_0(N)$ of level $2^{11} < N \leq 2^{12}$ and dimension $g \leq 1$ for $p < 2^{12}$



$w(E)^*a_p/p^2$ averages of 259/304 root number $w(E) = +1/-1$ weight 6 newforms for $\Gamma_0(N)$ of level $2^{10} < N \leq 2^{11}$ and dimension $g \leq 1$ for $p < 2^{11}$

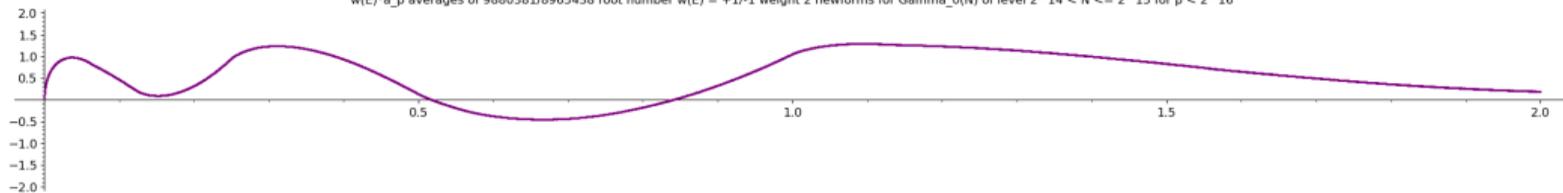


Newforms for $\Gamma_0(N)$ of weight $k = 2, 4, 6, 8$.

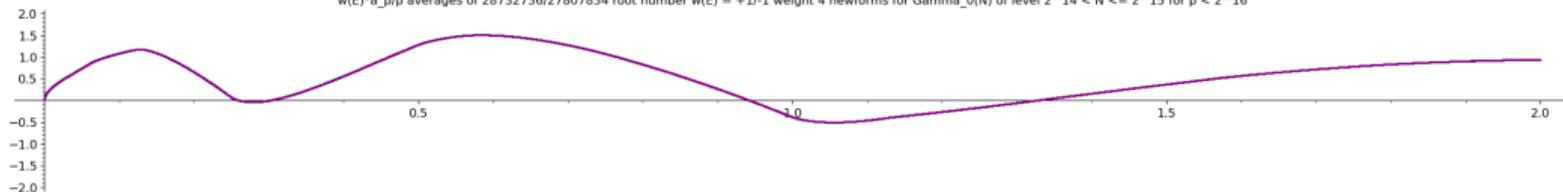


Newforms for $\Gamma_0(N)$ of weight $k = 2, 4, 6, 8$.

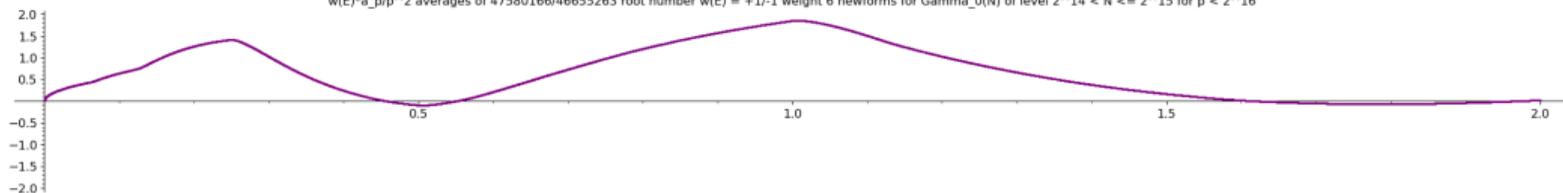
$w(E)^*a_p$ averages of 9880381/8965438 root number $w(E) = +1/-1$ weight 2 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$



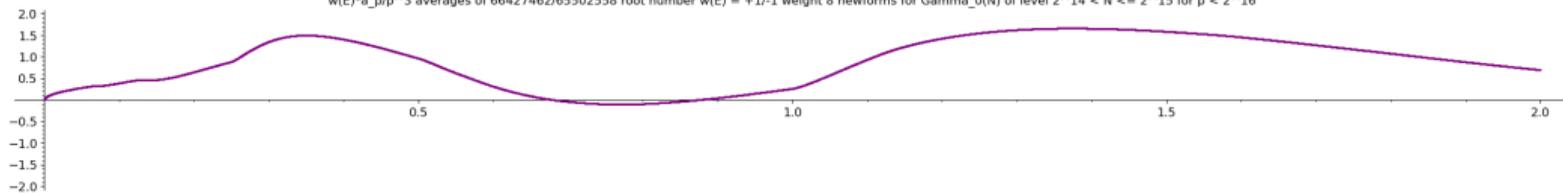
$w(E)^*a_p/p$ averages of 28732736/27807834 root number $w(E) = +1/-1$ weight 4 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$



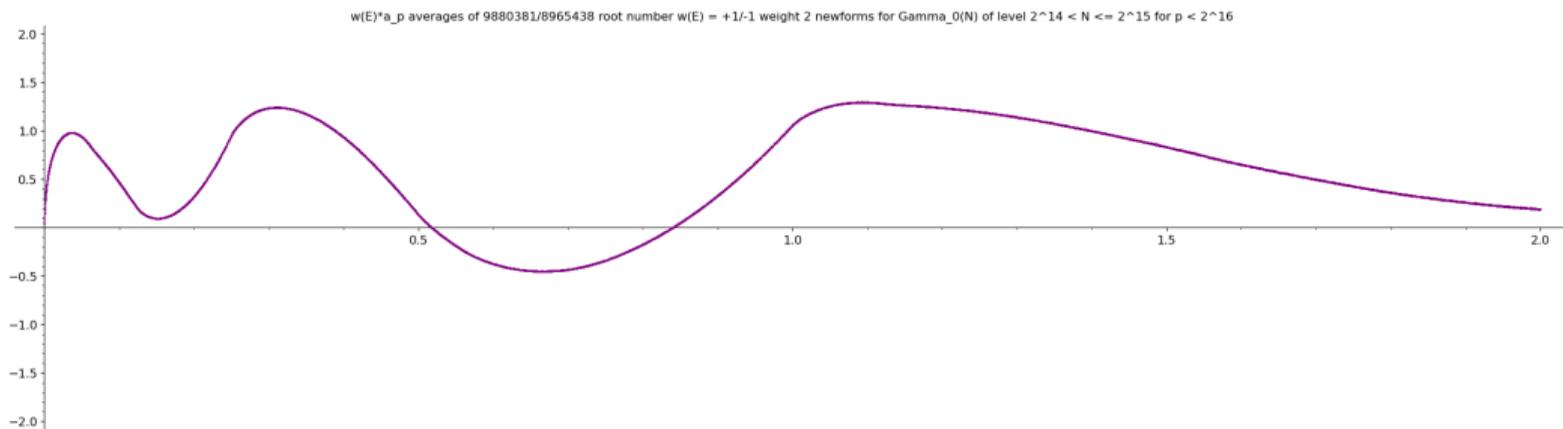
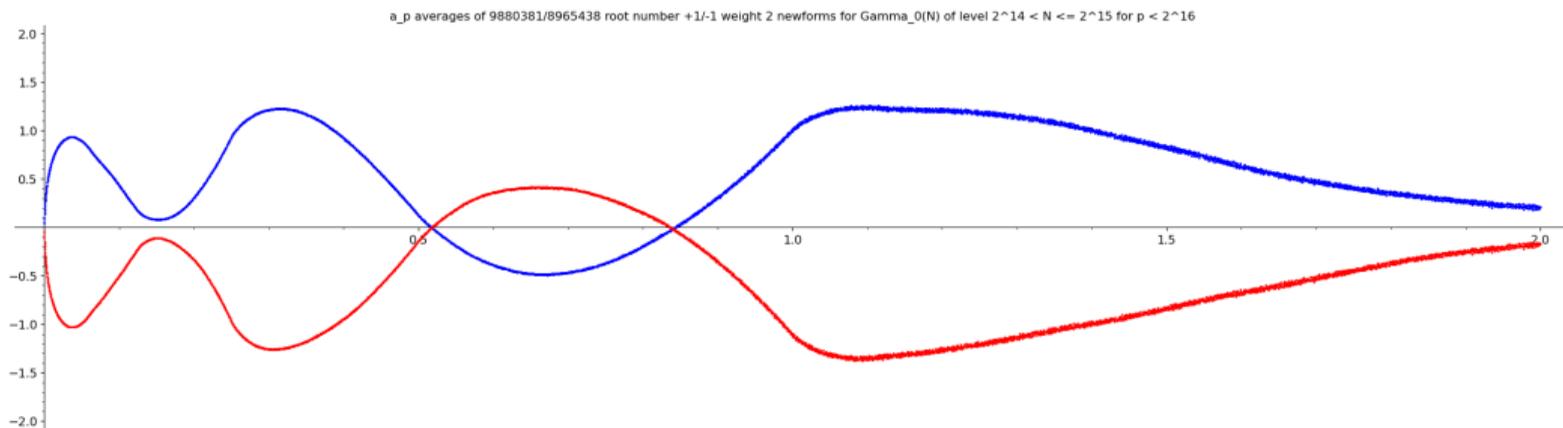
$w(E)^*a_p/p^2$ averages of 47580166/46655263 root number $w(E) = +1/-1$ weight 6 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$



$w(E)^*a_p/p^3$ averages of 66427462/65502558 root number $w(E) = +1/-1$ weight 8 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$

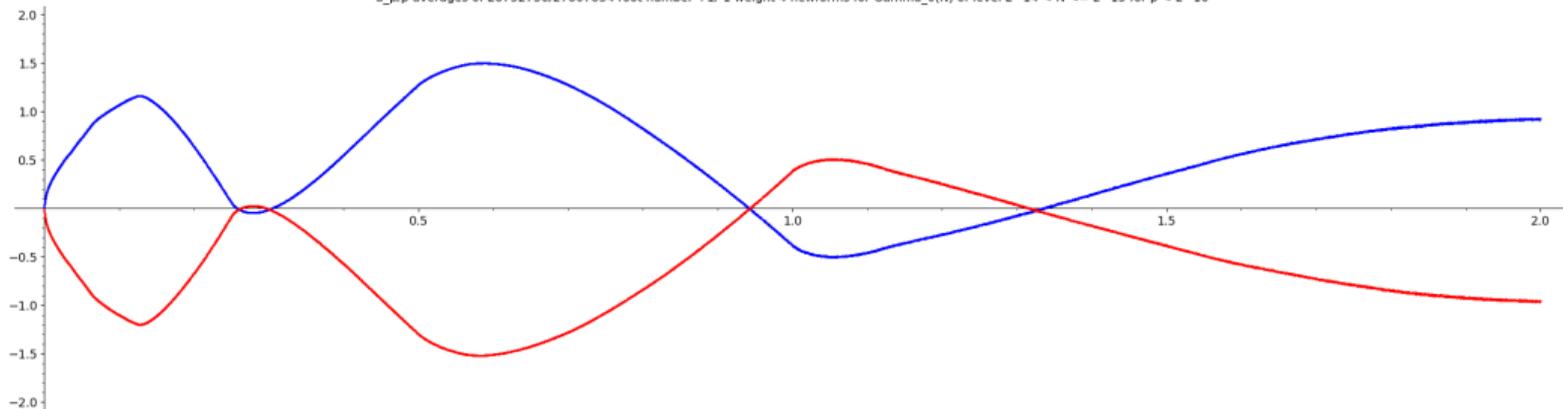


Newforms for $\Gamma_0(N)$ of weight $k = 2$.

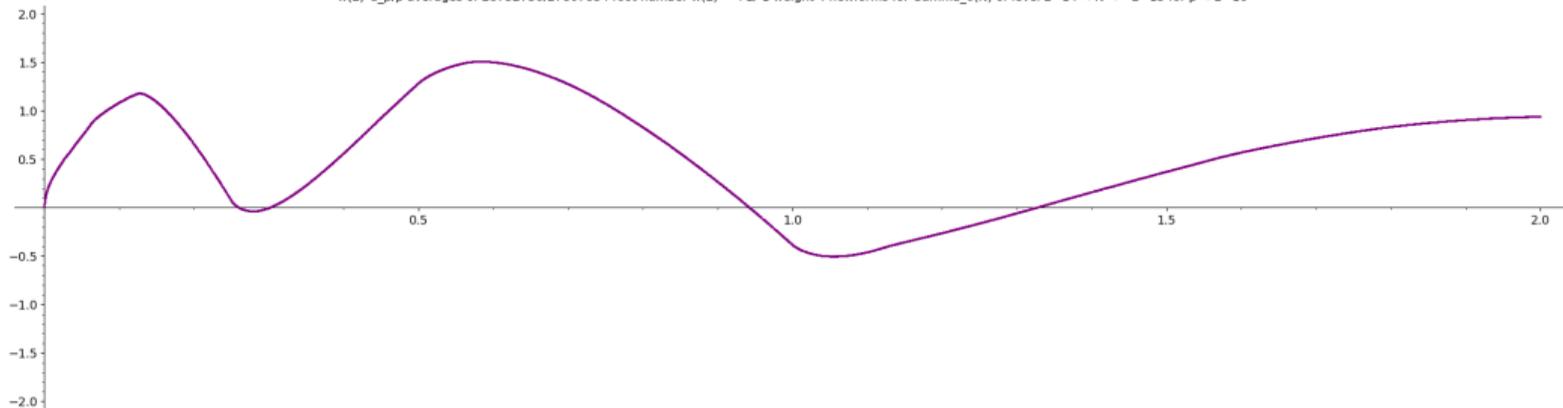


Newforms for $\Gamma_0(N)$ of weight $k = 4$.

a_p/p averages of 28732736/27807834 root number +1/-1 weight 4 newforms for Gamma_0(N) of level 2^14 < N <= 2^15 for p < 2^16

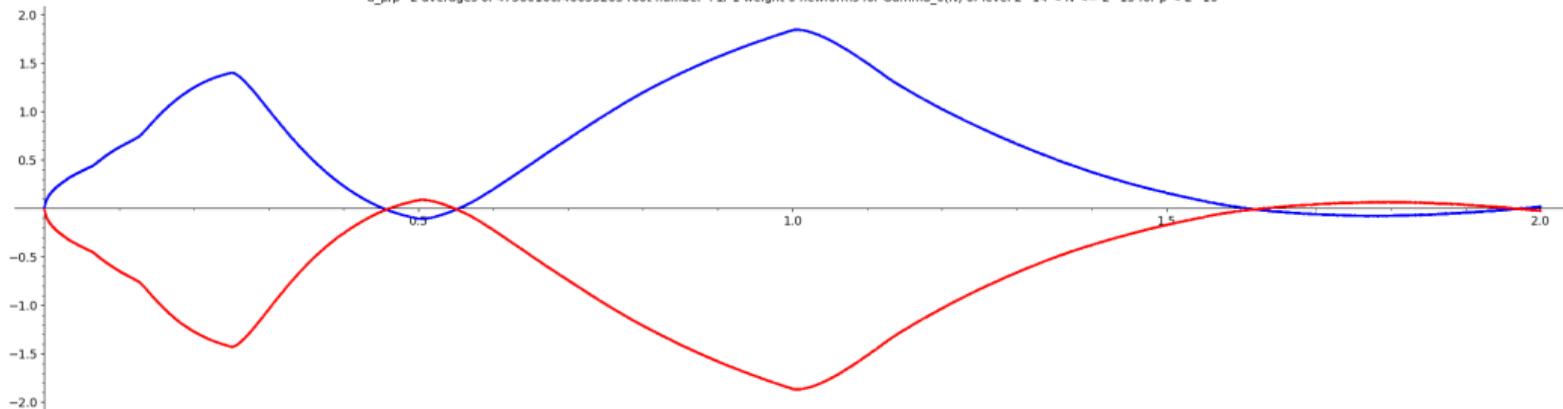


w(E)*a_p/p averages of 28732736/27807834 root number w(E) = +1/-1 weight 4 newforms for Gamma_0(N) of level 2^14 < N <= 2^15 for p < 2^16

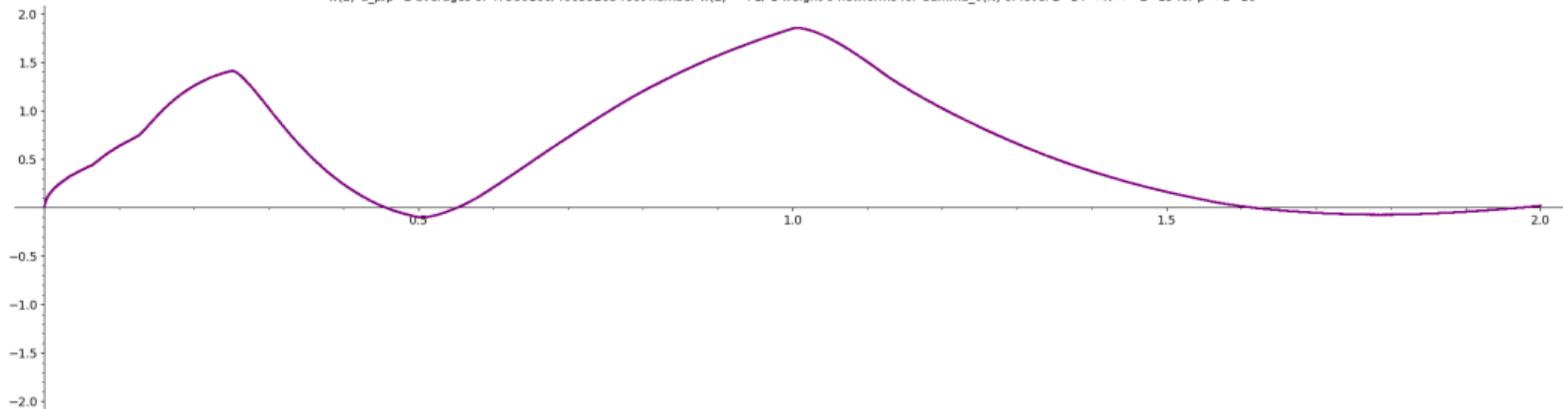


Newforms for $\Gamma_0(N)$ of weight $k = 6$.

a_p/p^2 averages of 47580166/46655263 root number +1/-1 weight 6 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$

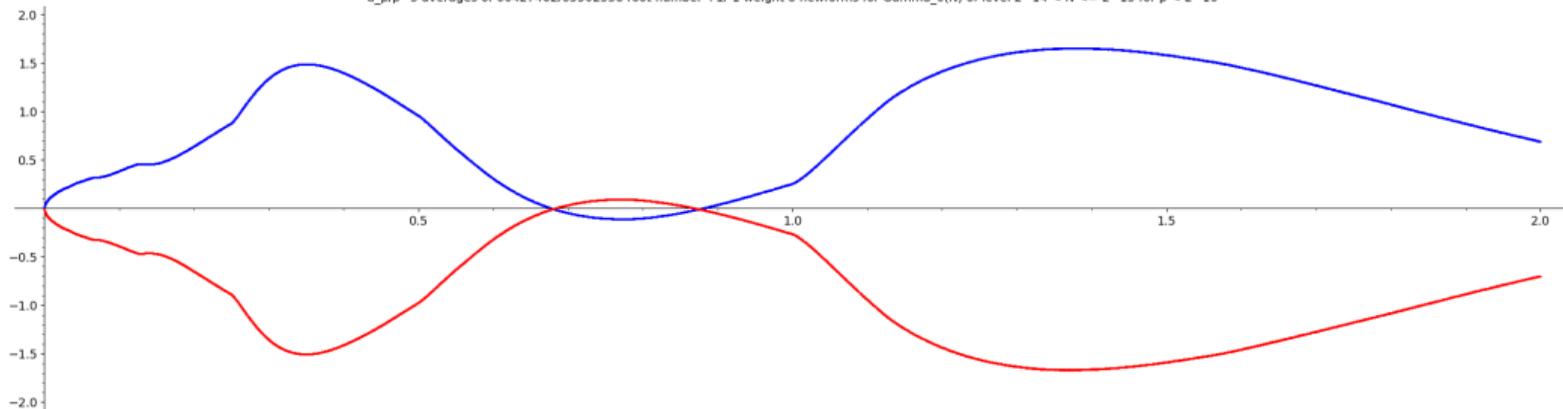


$w(E)a_p/p^2$ averages of 47580166/46655263 root number $w(E) = +1/-1$ weight 6 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$

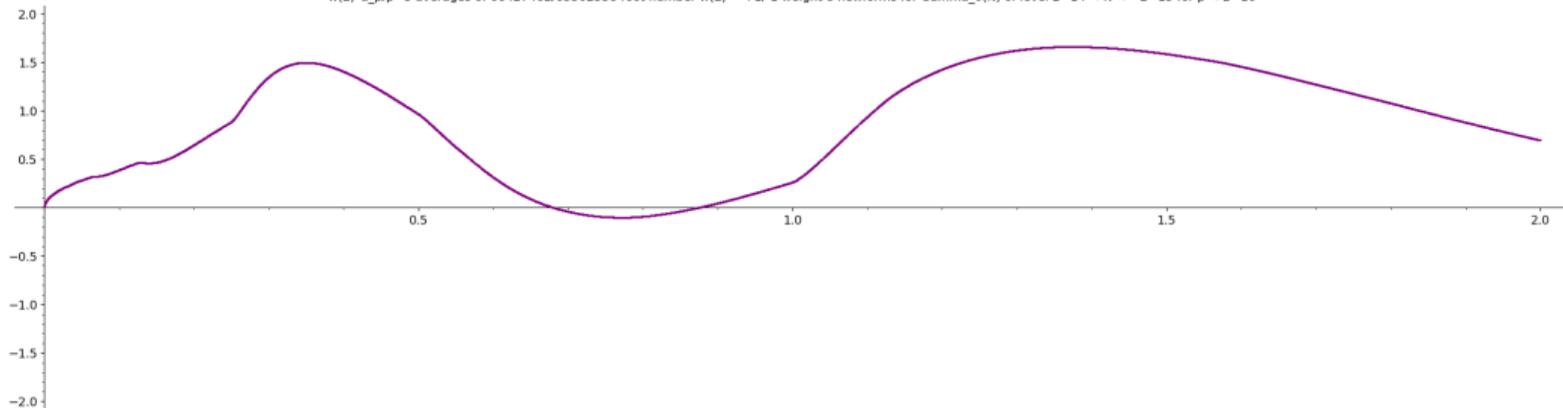


Newforms for $\Gamma_0(N)$ of weight $k = 8$.

a_p/p^3 averages of 66427462/65502558 root number +1/-1 weight 8 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$

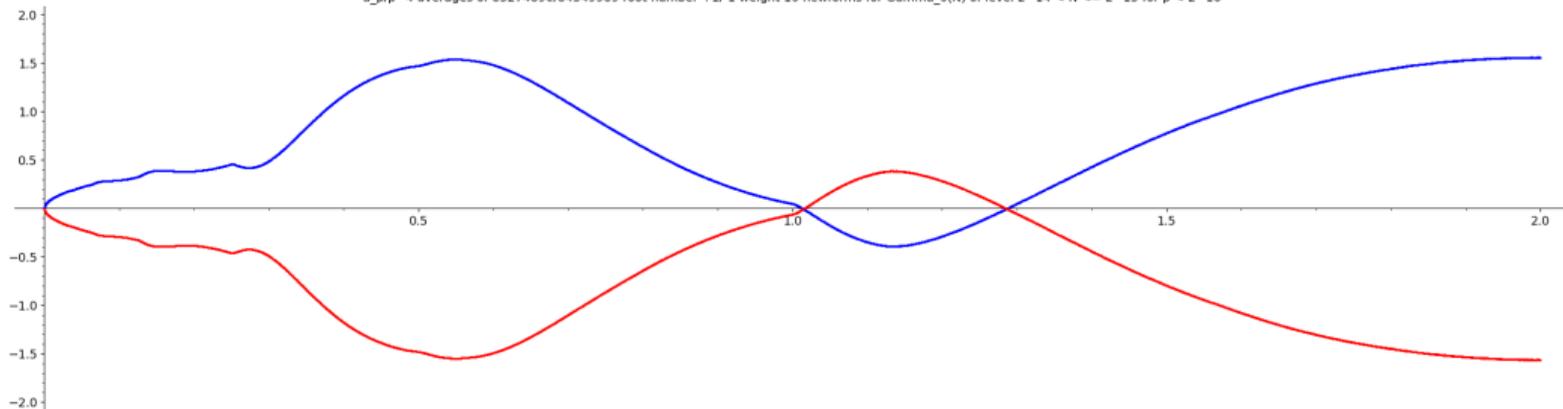


$w(E)a_p/p^3$ averages of 66427462/65502558 root number $w(E) = +1/-1$ weight 8 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$

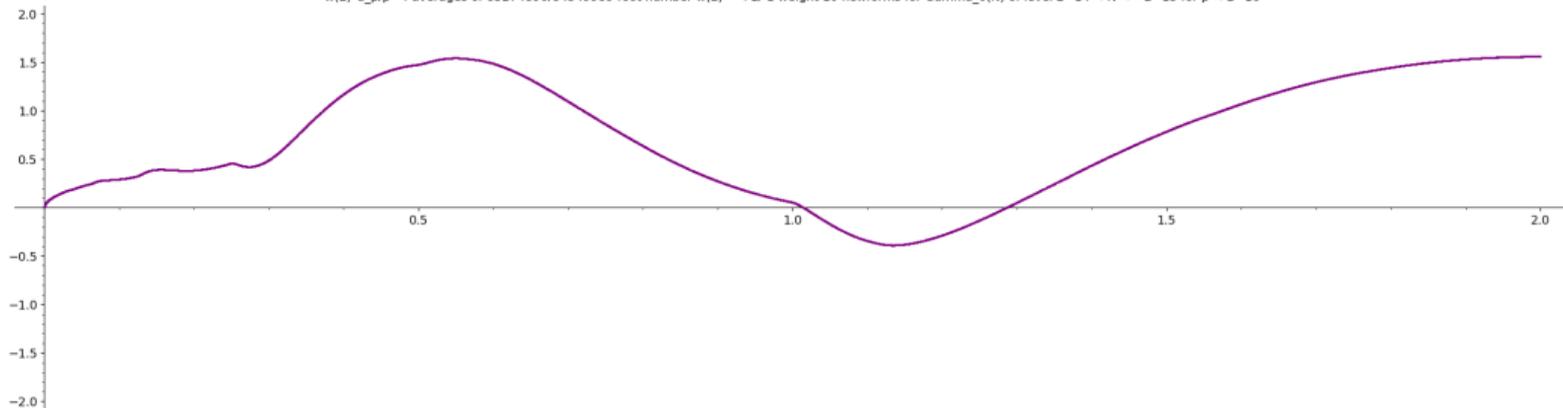


Newforms for $\Gamma_0(N)$ of weight $k = 10$.

a_p/p^4 averages of 85274890/84349989 root number $+1/-1$ weight 10 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$

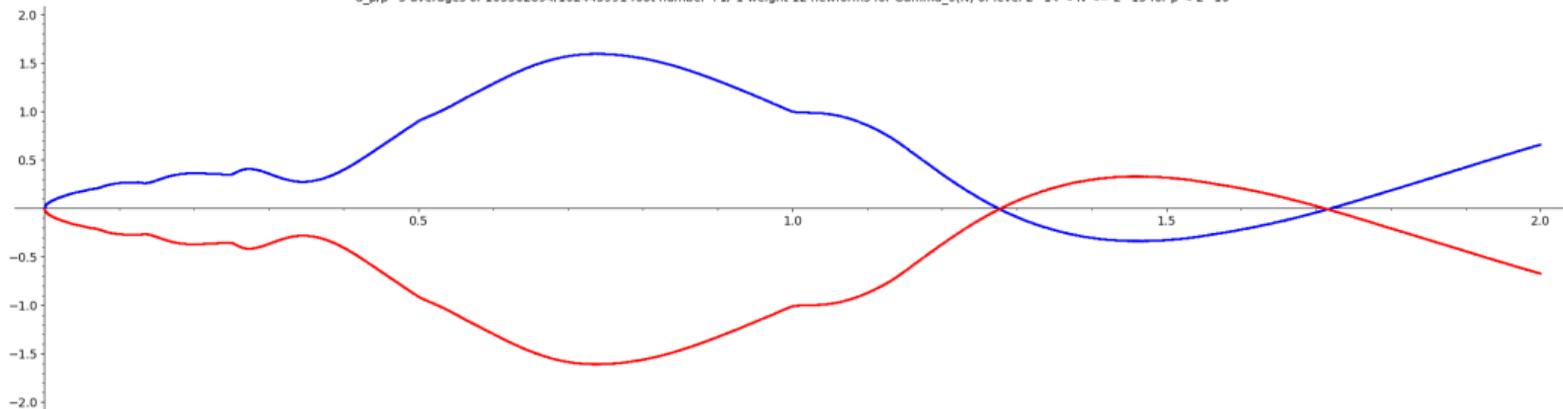


$w(E)a_p/p^4$ averages of 85274890/84349989 root number $w(E) = +1/-1$ weight 10 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$

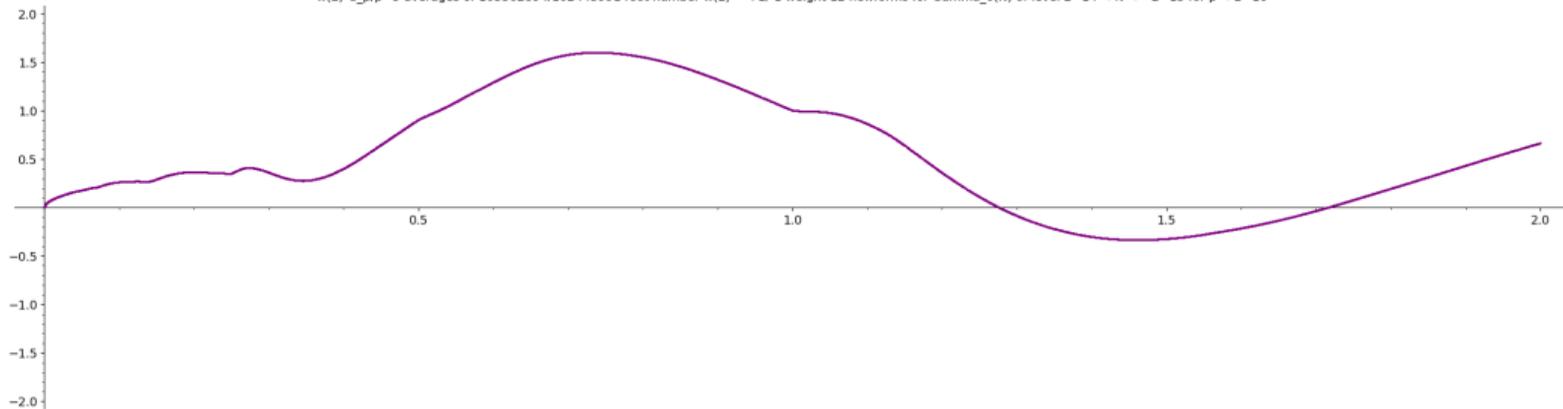


Newforms for $\Gamma_0(N)$ of weight $k = 12$.

a_p/p^5 averages of 103362894/102443991 root number +1/-1 weight 12 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$



$w(E)a_p/p^5$ averages of 103362894/102443991 root number $w(E) = +1/-1$ weight 12 newforms for $\Gamma_0(N)$ of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$



Zubrilina's theorem

Theorem (Zubrilina 2023)

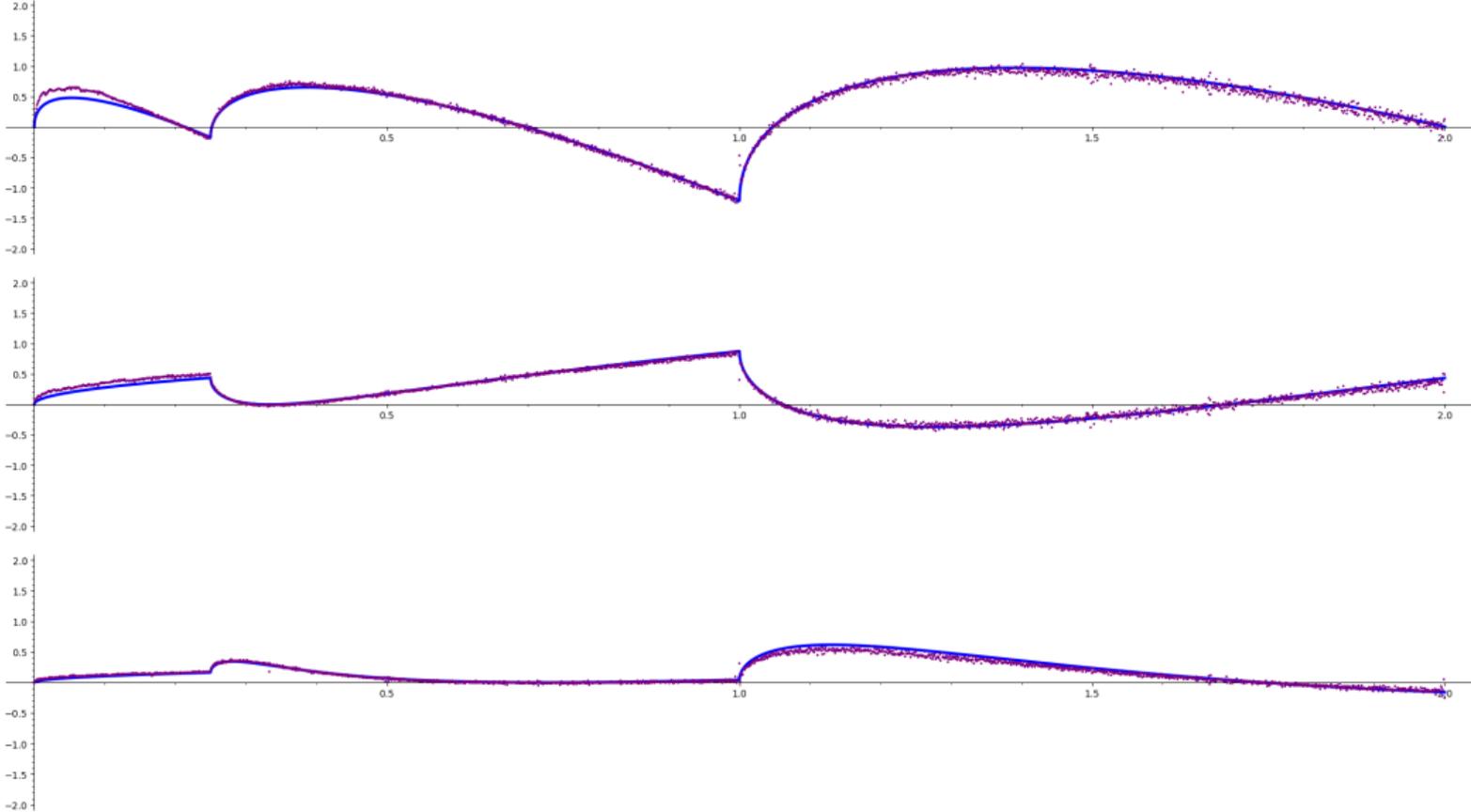
Let $f = \sum_{n \geq 1} a_n q^n \in S_k^{\text{new}}(N)$ denote a newform of weight $k \geq 2$ for $\Gamma_0(N)$ with root number $\varepsilon(f)$. Let $X, Y, P \rightarrow \infty$ with P prime, $Y = (1 + o(1))X^{1-\delta_Y}$, $P \ll X^{1+\delta_P}$, for some $\delta_Y, \delta_P > 0$ with $2\delta_P < \delta_Y < 1$. As a function of $y := P/X$ we have

$$\frac{\zeta(2)\pi}{XY} \sum_{\substack{N \in [X, X+Y] \\ N \perp P \text{ } \square\text{-free}}} \sum_f \frac{\varepsilon(f) a_P(f)}{P^{(k/2-1)}} = A\sqrt{y} + (-1)^{k/2-1} \sum_{1 \leq r \leq 2\sqrt{y}} c(r) \sqrt{4y - r^2} U_{k-2} \left(\frac{r}{2\sqrt{y}} \right) \\ - \pi y \delta_{k=2} + O_\varepsilon(X^{-\delta'+\varepsilon})$$

where U_n is the Chebyshev polynomial defined by $U_n(\cos x) = \sin(nx + x)/\sin x$, $\delta' := \max(\delta_Y/2 - \delta_P, (\delta_Y + 1)/9 - \delta_P)$, $c(r) := B \prod_{p|r} (1 + p^2/(p^4 - 2p^2 - p + 1))$, where $A = 1.450032\dots$ and $B = 0.731311\dots$ are explicit constants.

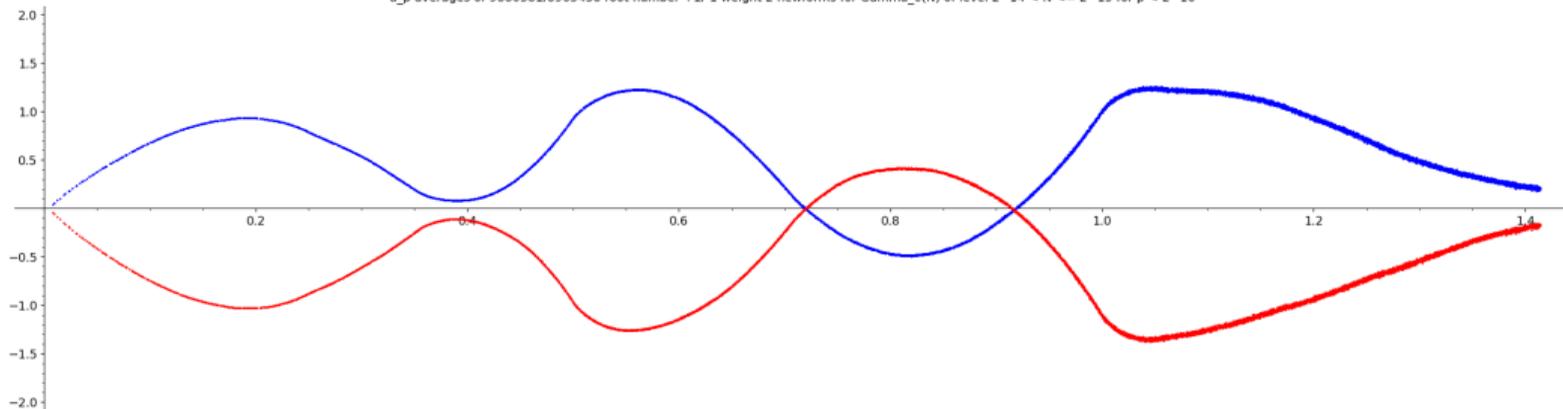
For every $\delta_P < 2/9$ one can choose δ_Y so that $\delta' > 0$.

Zubrilina's theorem

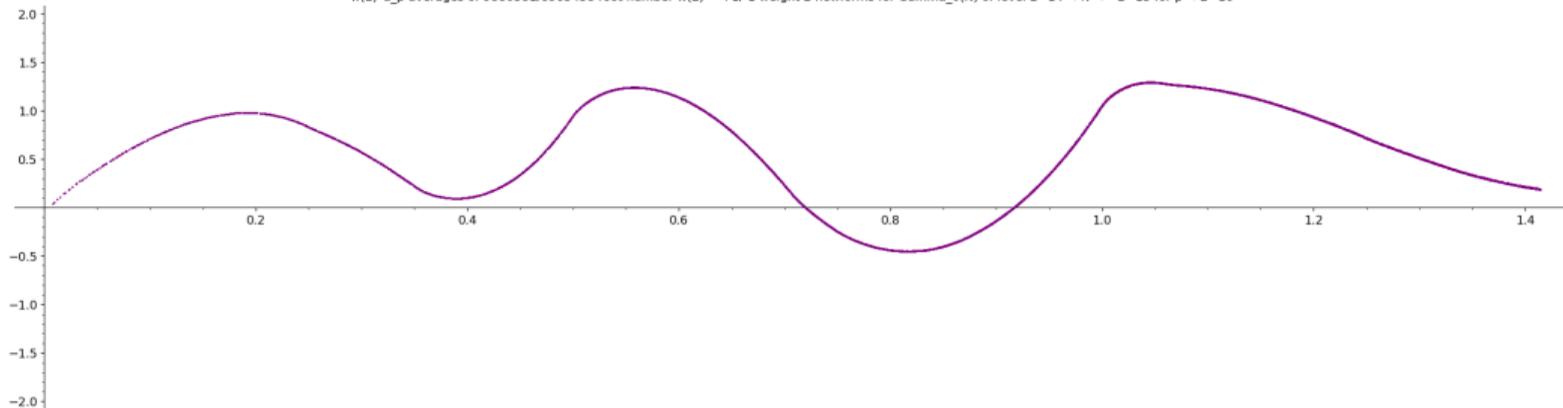


Newforms for $\Gamma_0(N)$ of weight $k = 2$ with square root normalization.

a_p averages of 9880381/8965438 root number +1/-1 weight 2 newforms for Gamma_0(N) of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$

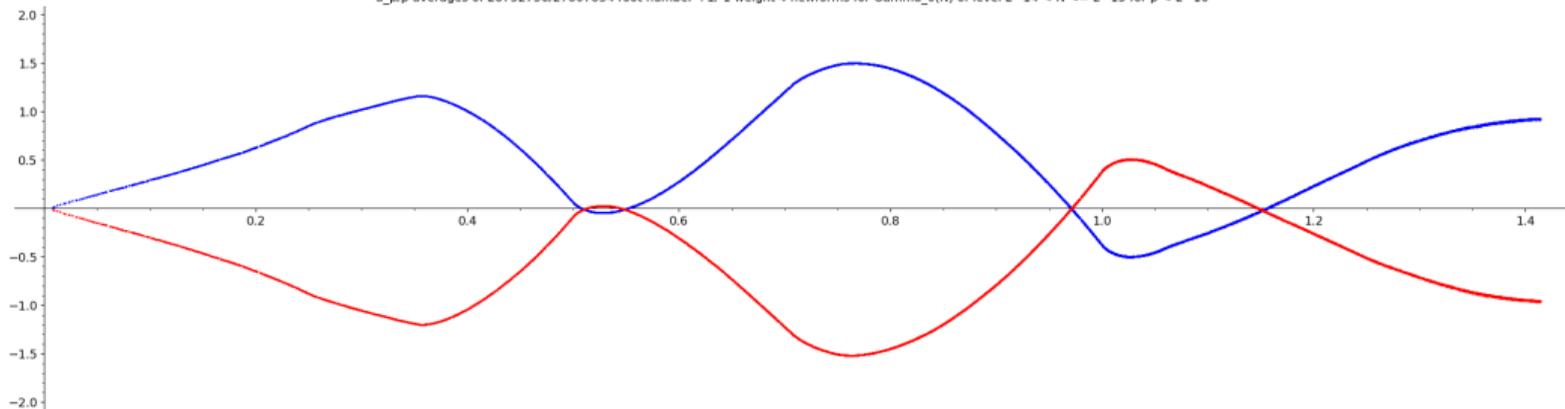


w(E)*a_p averages of 9880381/8965438 root number w(E) = +1/-1 weight 2 newforms for Gamma_0(N) of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$

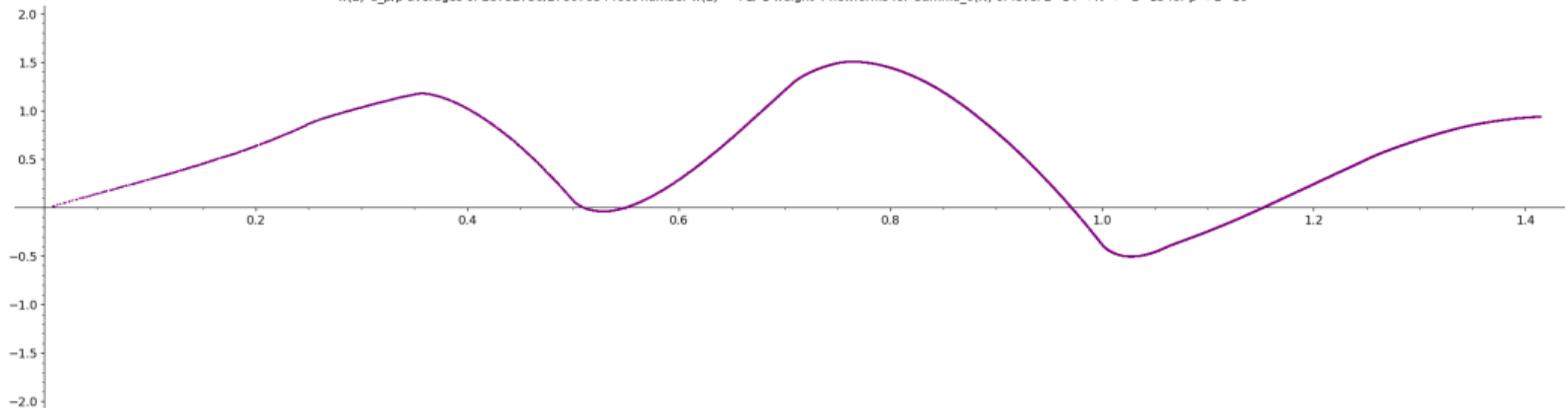


Newforms for $\Gamma_0(N)$ of weight $k = 4$ with square root normalization.

a_p/p averages of 28732736/27807834 root number +1/-1 weight 4 newforms for Gamma_0(N) of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$



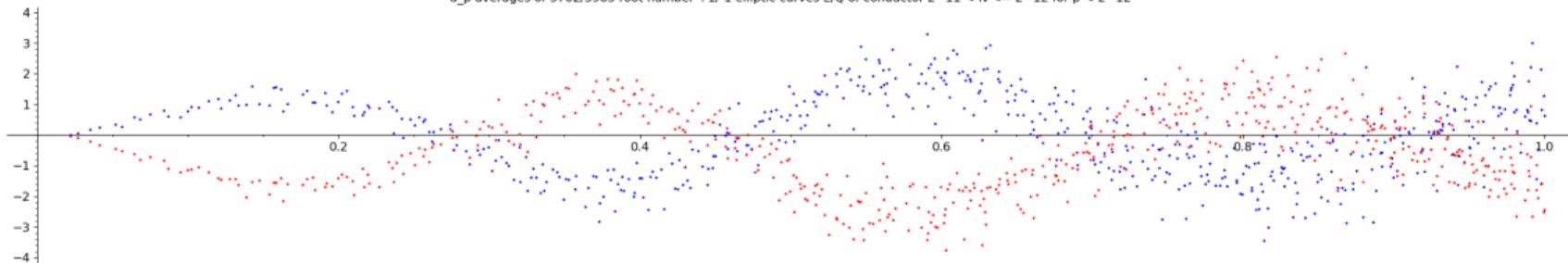
w(E)*a_p/p averages of 28732736/27807834 root number w(E) = +1/-1 weight 4 newforms for Gamma_0(N) of level $2^{14} < N \leq 2^{15}$ for $p < 2^{16}$



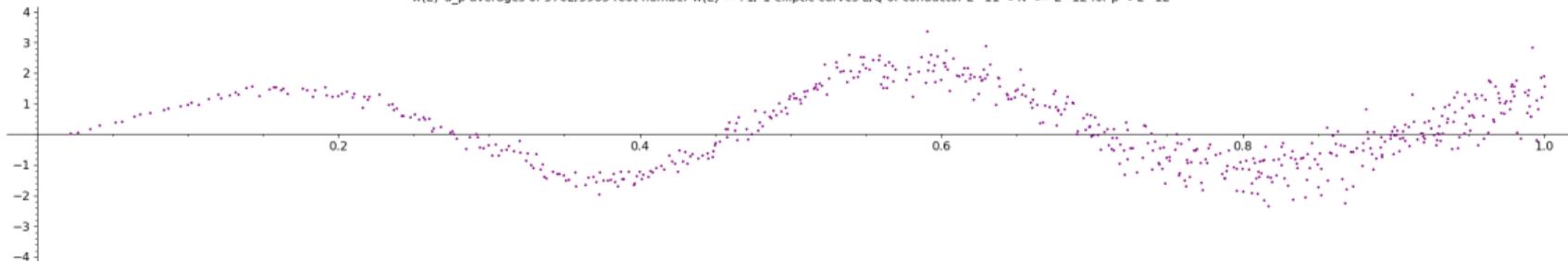
Murmurations of elliptic curves with squareroot normalization

Elliptic curve L -functions of conductor $N \in (M, 2M]$ for $M = 2^{11}, 2^{12}, \dots, 2^{17}, 250000$. The x -axis range is $[0, 2M]$. A blue/red or purple dot at $(\sqrt{p}, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or $m_p := w(E)a_p(E)$ over even/odd or all E/\mathbb{Q} with $N_E \in (M, 2M]$.

a_p averages of 3762/3985 root number $+1/-1$ elliptic curves E/\mathbb{Q} of conductor $2^{11} < N \leq 2^{12}$ for $p < 2^{12}$



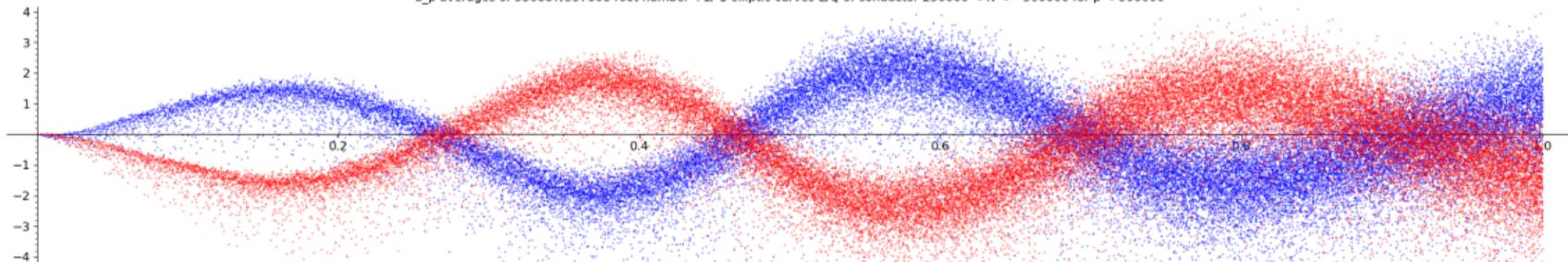
$w(E)a_p$ averages of 3762/3985 root number $w(E) = +1/-1$ elliptic curves E/\mathbb{Q} of conductor $2^{11} < N \leq 2^{12}$ for $p < 2^{12}$



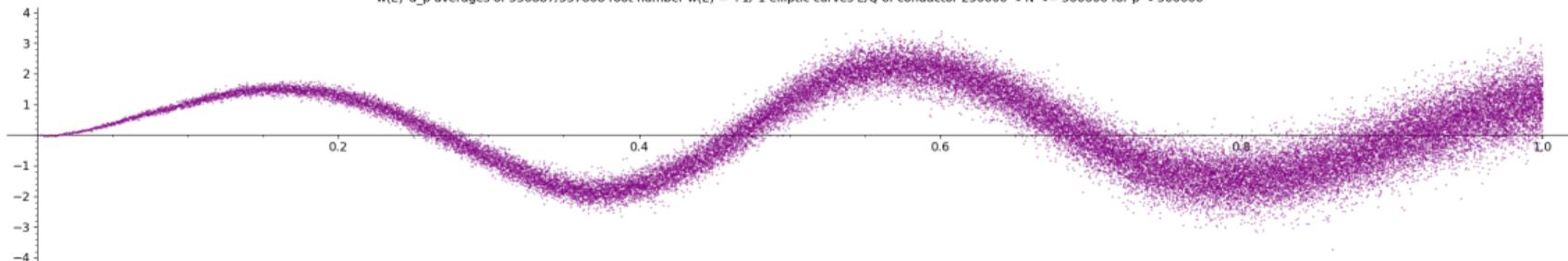
Murmurations of elliptic curves with squareroot normalization

Elliptic curve L -functions of conductor $N \in (M, 2M]$ for $M = 2^{11}, 2^{12}, \dots, 2^{17}, 250000$. The x -axis range is $[0, 2M]$. A blue/red or purple dot at $(\sqrt{p}, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or $m_p := w(E)a_p(E)$ over even/odd or all E/\mathbb{Q} with $N_E \in (M, 2M]$.

a_p averages of 530887/537808 root number +1/-1 elliptic curves E/Q of conductor 250000 < N <= 500000 for p < 500000



w(E)*a_p averages of 530887/537808 root number w(E) = +1/-1 elliptic curves E/Q of conductor 250000 < N <= 500000 for p < 500000



Murmurations in the weight aspect

Theorem (Bober, Booker, Lee, Lowry–Duda 2023)

Assume GRH for the L-functions of Dirichlet characters and modular forms.

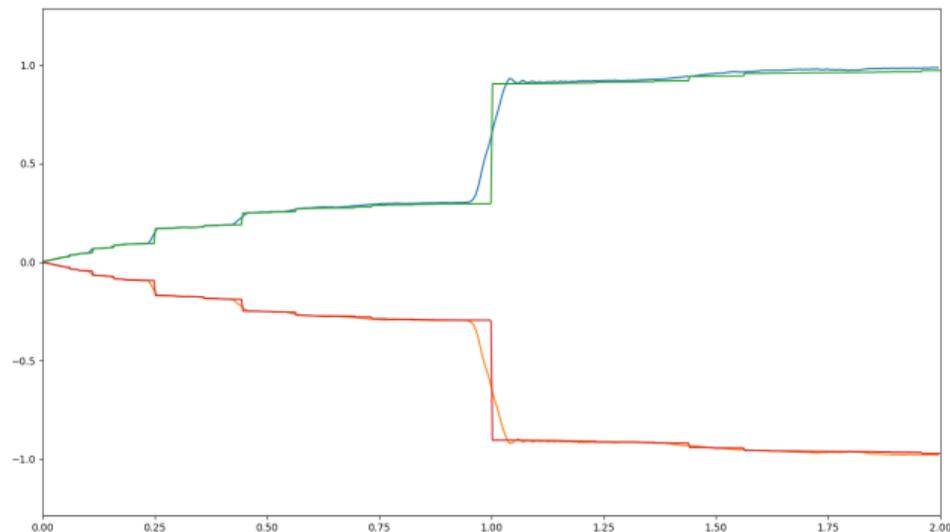
Let $E \subset \mathbb{R}_{>0}$ be a compact interval with $|E| > 0$, let $\delta \in \{0, 1\}$ and $K, H, \varepsilon \in \mathbb{R}_{>0}$, with $K^{\frac{5}{6} + \varepsilon} < H < K^{1 - \varepsilon}$ as $K \rightarrow \infty$, and put $N := \left(\frac{\exp \psi(K/2)}{2\pi}\right)^2$. We have

$$\frac{\sum_{\substack{k \equiv 2\delta \pmod{4} \\ |k-K| \leq H}} \sum_f \sum_{p/N \in E} \frac{a_p(f) \log p}{p^{k/2-1}}}{\sum_{\substack{k \equiv 2\delta \pmod{4} \\ |k-K| \leq H}} \sum_f \sum_{p/N \in E} \log p} = (-1)^\delta \left(\frac{\nu(E)}{|E|} + o_{E,\varepsilon}(1) \right), \quad \text{where}$$

$$\nu(E) := \frac{1}{\zeta(2)} \sum_{\substack{r, q \in \mathbb{Z}_{>0} \\ \gcd(r, q) = 1 \\ (r/q)^{-2} \in E}}^* \frac{\mu(q)^2}{\varphi(q)^2 \sigma(q)} \left(\frac{q}{r}\right)^4 = \frac{1}{2} \sum_{t \in \mathbb{Z}} \prod_{p|t} \frac{p^2 - p - 1}{p^2 - p} \cdot \int_E \sqrt{x} \cos\left(\frac{2\pi t}{\sqrt{x}}\right) dx,$$

and the $*$ indicates that values of $(r/q)^{-2}$ at endpoints of E have weight $\frac{1}{2}$.

Murmurations in the weight aspect

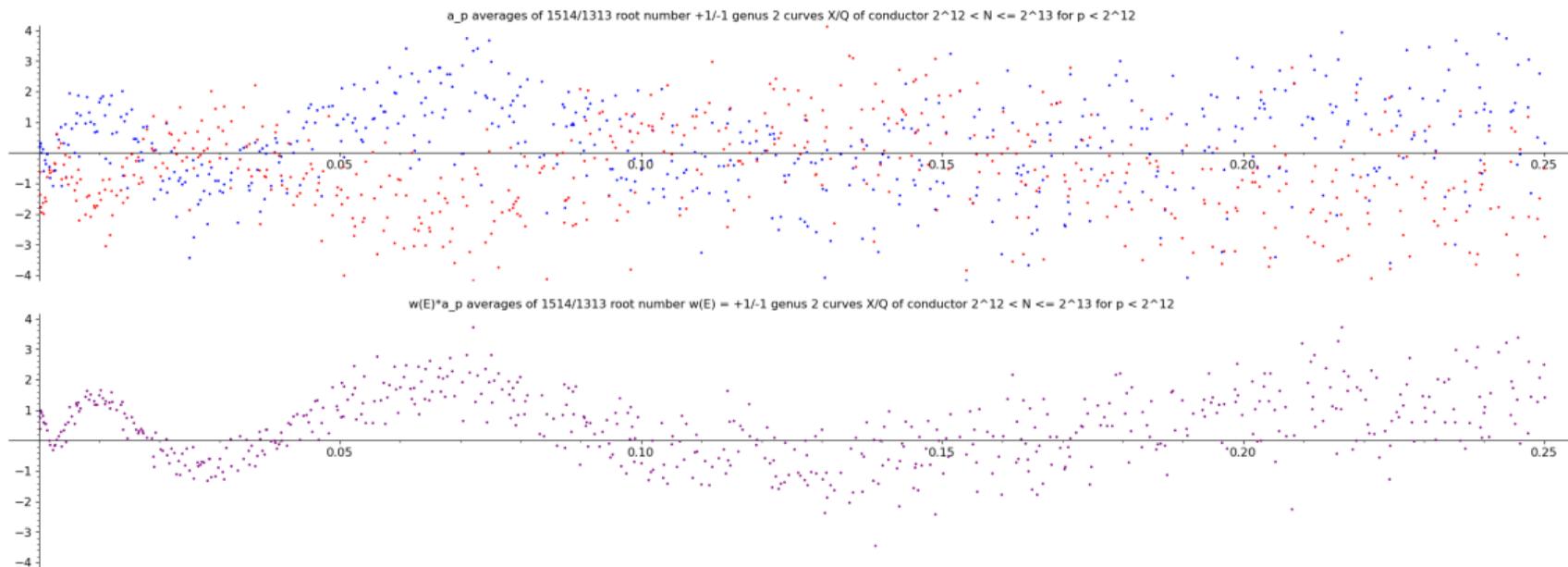


Also see [Iwaniec-Luo-Sarnak 2000] *Low lying zeros of families of L-functions*, where they consider a_p averages over p on the scale of \mathcal{N}^ϑ , for $\vartheta < 1$ and $\vartheta > 1$. They observe a phase transition at $\vartheta = 1$, which is the murmuration regime.

L -functions of genus 2 curves over \mathbb{Q} with Sato-Tate group $\mathrm{USp}(4)$.

Recently constructed database of more than 5 million genus 2 curves X/\mathbb{Q} of conductor at most 2^{20} includes 1,440,894 isogeny classes with Sato-Tate group $\mathrm{USp}(4)$.

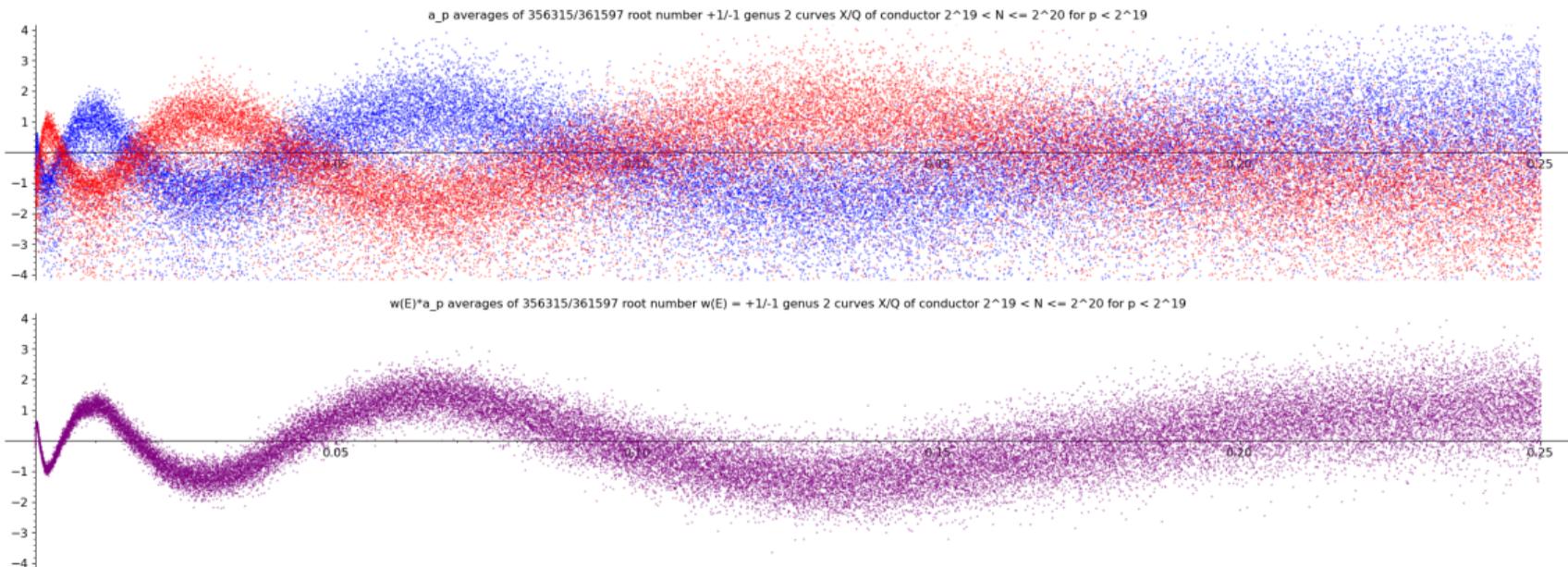
Conductor of $L(X, s)$ in $(M, 2M]$ for $M = 2^{12}, \dots, 2^{19}$ with x -axis range $[0, M/2]$.



Coming soon to the [LMFDB](#).

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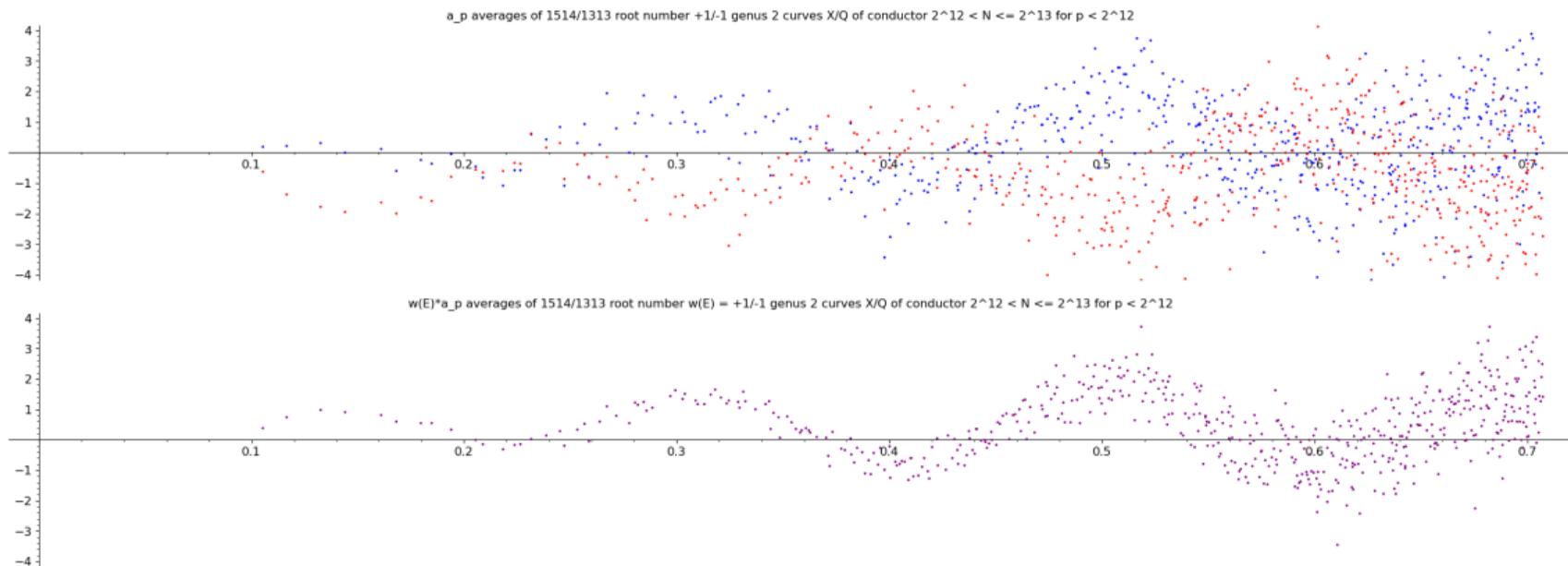


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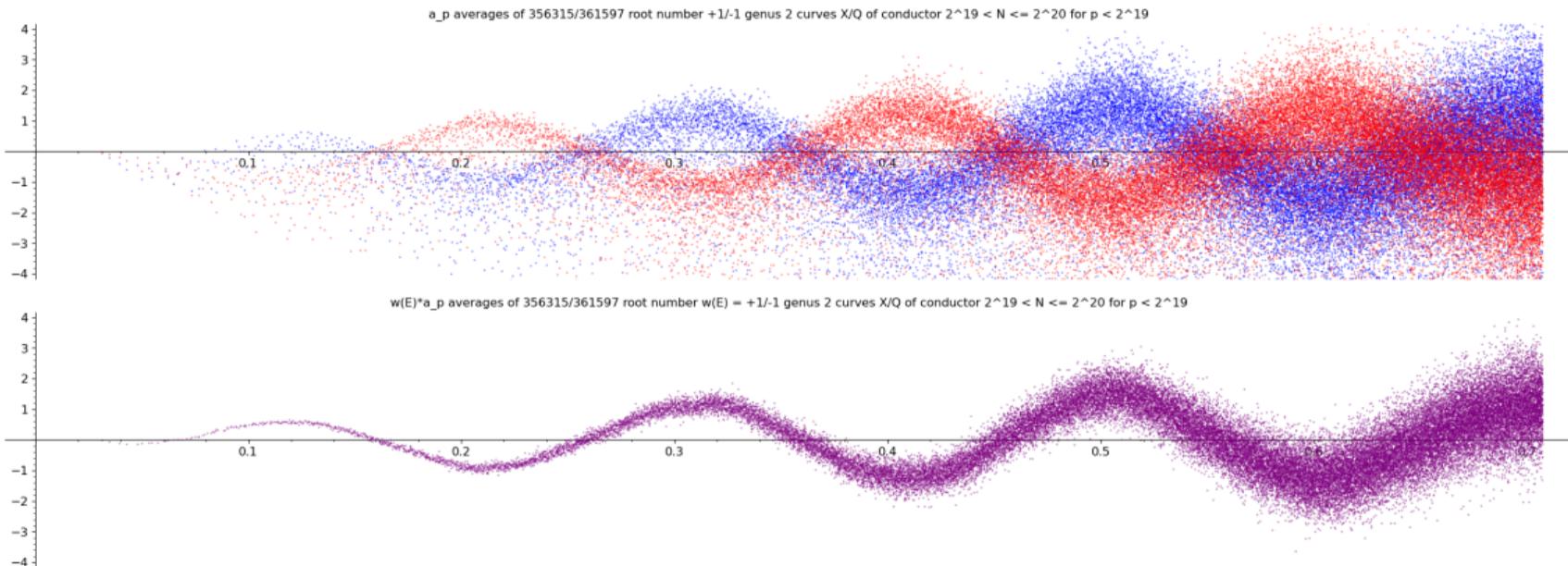
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Coming soon to the [LMFDB](#).

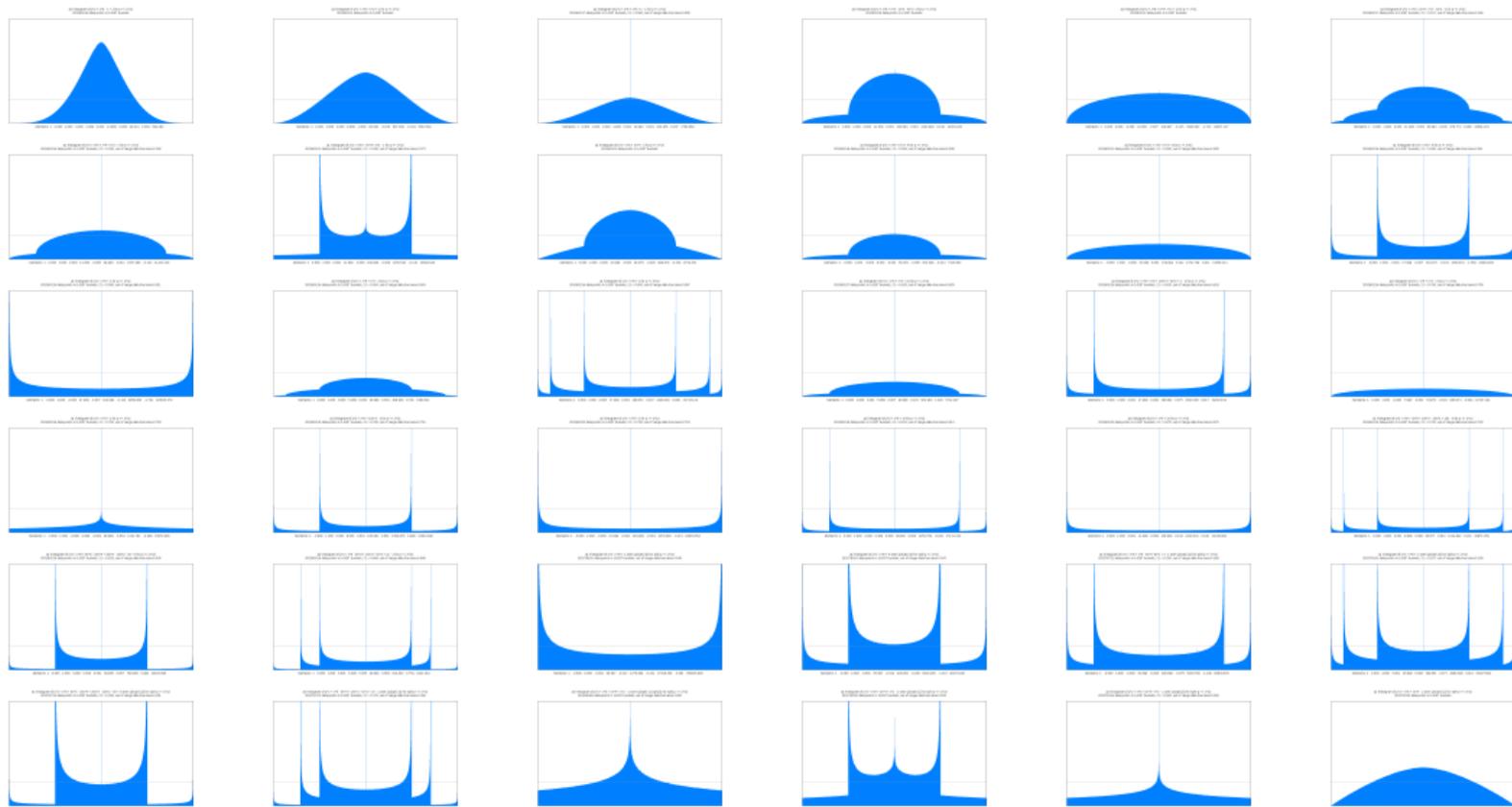
L-functions of genus 2 curves over \mathbb{Q} with Sato-Tate group $\mathrm{USp}(4)$.

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Coming soon to the [LMFDB](#).

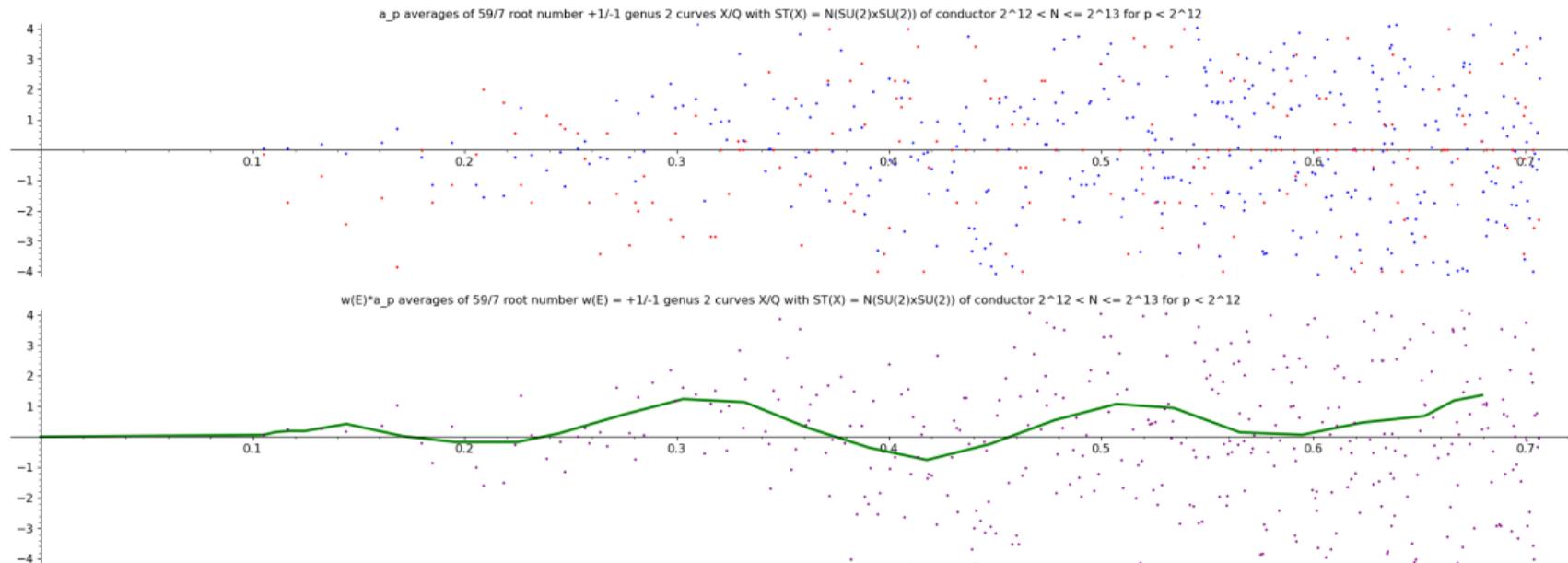
Trace distributions of genus 2 curves



L -functions of genus 2 curves over \mathbb{Q} , Sato-Tate group $N(\mathrm{SU}(2) \times \mathrm{SU}(2))$.

These are primitive L -functions arising from Hilbert or Bianchi modular forms.

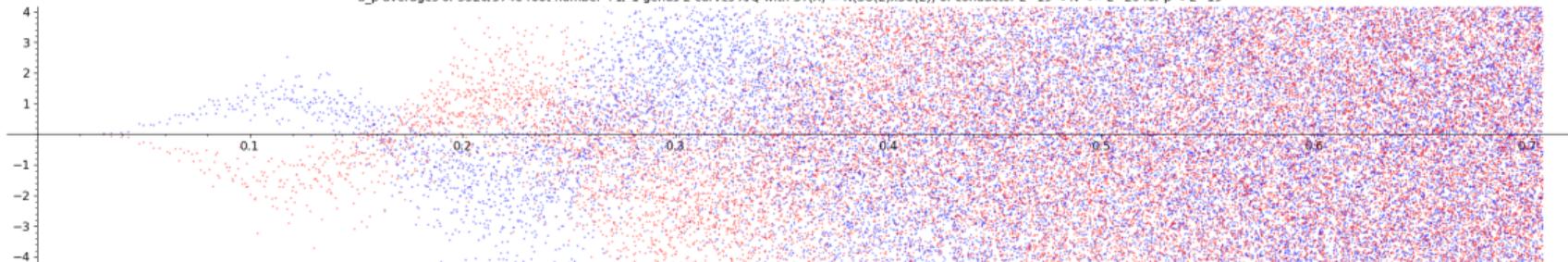
Conductor of $L(X, s)$ in $(M, 2M]$ for $M = 2^{12}, \dots, 2^{19}$ with x -axis range $[0, M/2]$.



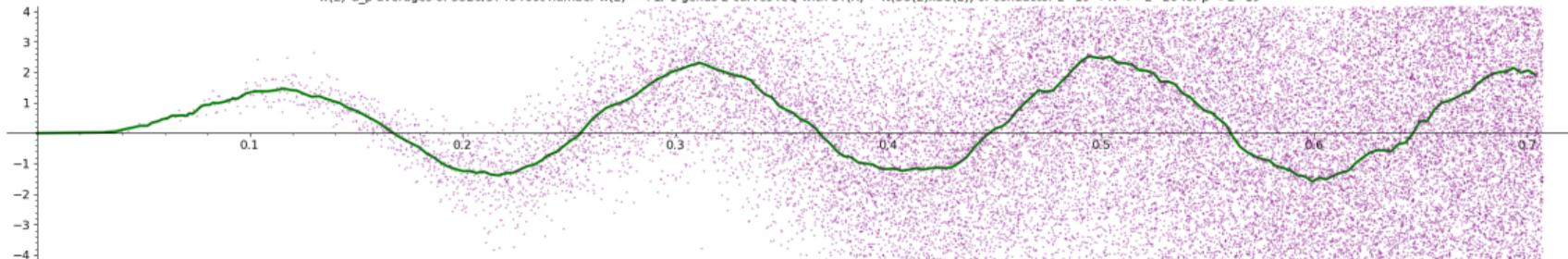
L -functions of genus 2 curves over \mathbb{Q} , Sato-Tate group $N(\mathrm{SU}(2) \times \mathrm{SU}(2))$.

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Conductor of $L(X, s)$ in $(M, 2M]$ for $M = 2^{12}, \dots, 2^{19}$ with x -axis range $[0, M/2]$.

a_p averages of 3326/3748 root number +1/-1 genus 2 curves X/\mathbb{Q} with $\mathrm{ST}(X) = N(\mathrm{SU}(2) \times \mathrm{SU}(2))$ of conductor $2^{19} < N \leq 2^{20}$ for $p < 2^{19}$

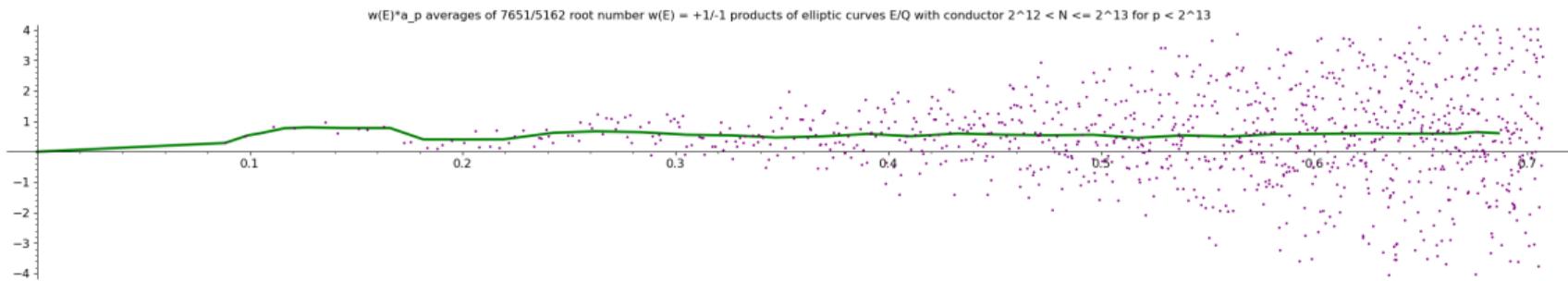
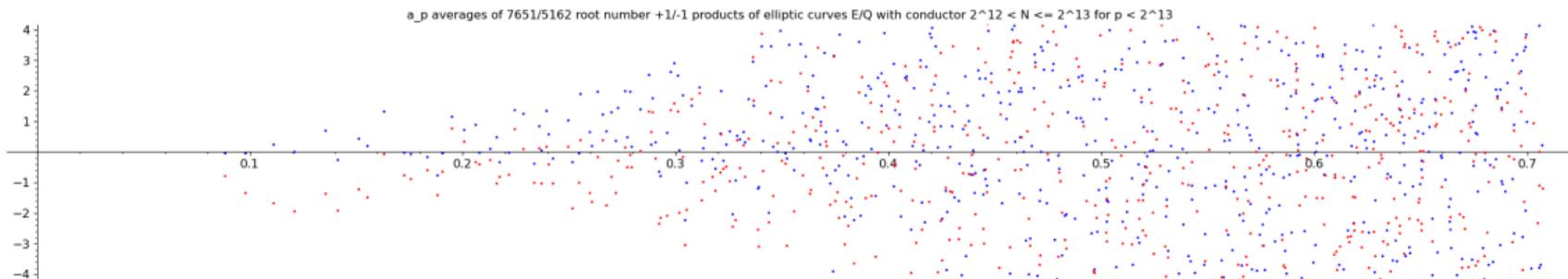


$w(E)a_p$ averages of 3326/3748 root number $w(E) = +1/-1$ genus 2 curves X/\mathbb{Q} with $\mathrm{ST}(X) = N(\mathrm{SU}(2) \times \mathrm{SU}(2))$ of conductor $2^{19} < N \leq 2^{20}$ for $p < 2^{19}$



L -functions of products of E/\mathbb{Q} , Sato-Tate group $SU(2) \times SU(2)$.

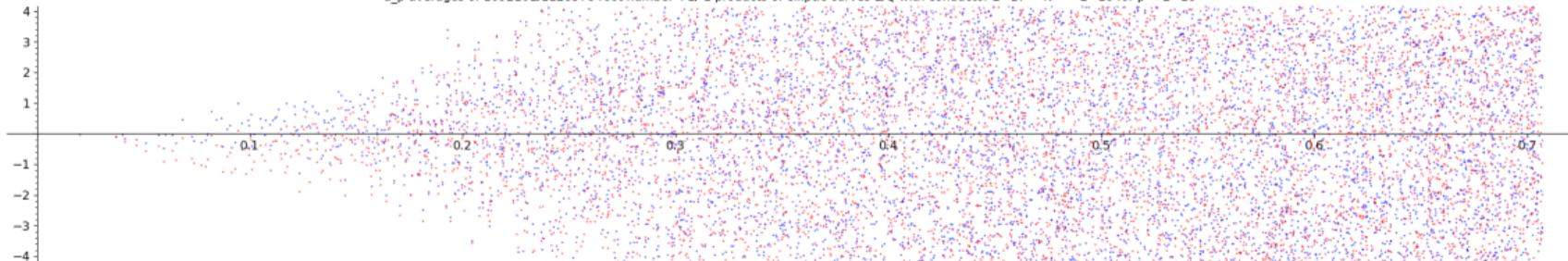
Conductor of $L(X, s)$ in $(M, 2M]$ for $M = 2^{12}, \dots, 2^{17}$ with x -axis range $[0, M/2]$.



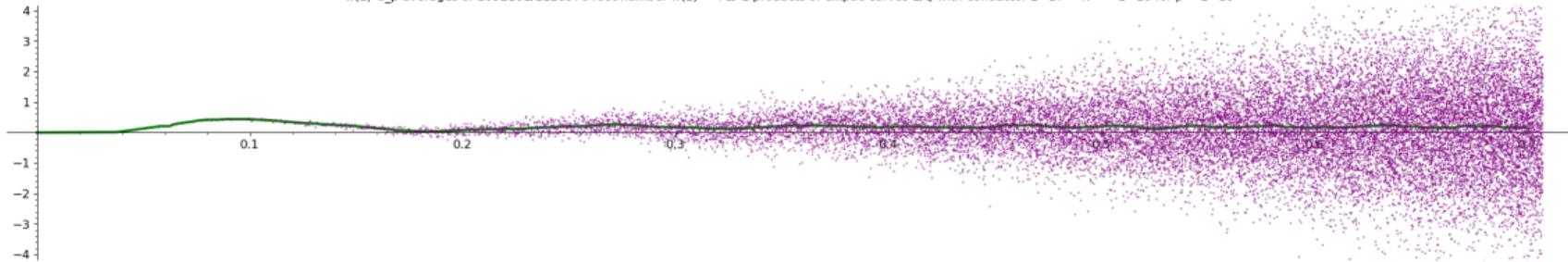
L -functions of products of E/\mathbb{Q} , Sato-Tate group $SU(2) \times SU(2)$.

Conductor of $L(X, s)$ in $(M, 2M]$ for $M = 2^{12}, \dots, 2^{17}$ with x -axis range $[0, M/2]$.

a_p averages of 1092101/1128578 root number $+1/-1$ products of elliptic curves E/\mathbb{Q} with conductor $2^{17} < N \leq 2^{18}$ for $p < 2^{18}$



$w(E) \cdot a_p$ averages of 1092101/1128578 root number $w(E) = +1/-1$ products of elliptic curves E/\mathbb{Q} with conductor $2^{17} < N \leq 2^{18}$ for $p < 2^{18}$

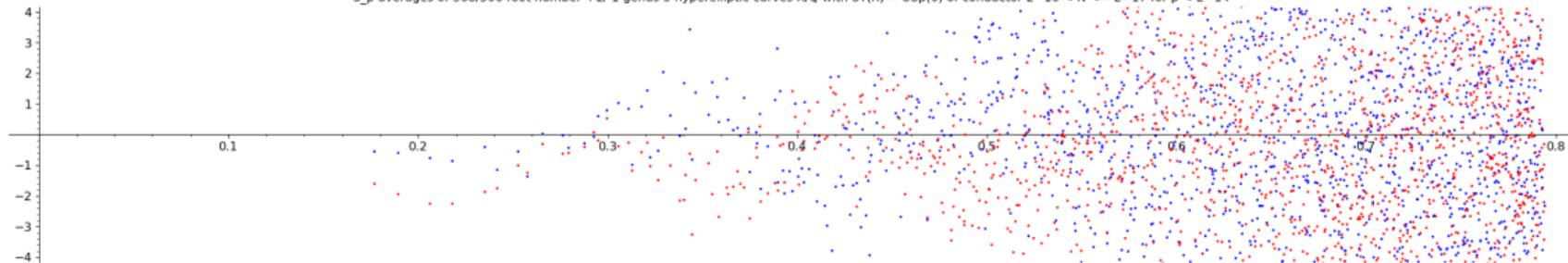


L -functions of genus 3 curves over \mathbb{Q} with Sato-Tate group $\mathrm{USp}(6)$.

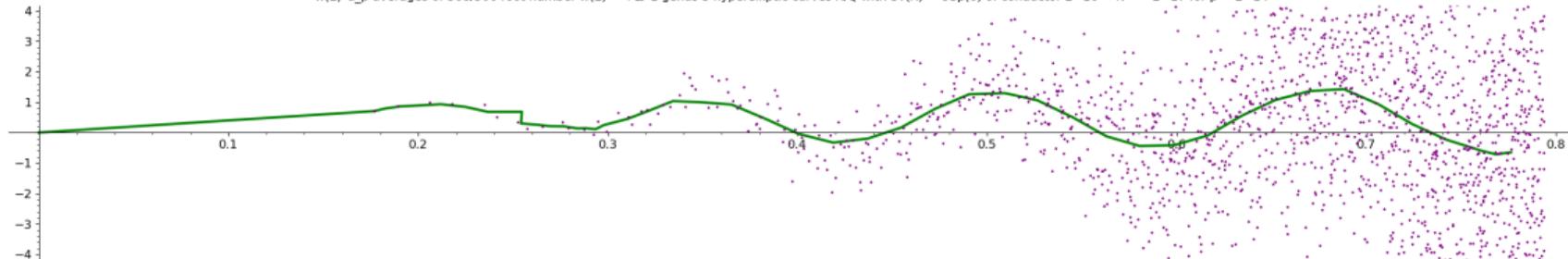
Recently constructed database of genus 3 curves X/\mathbb{Q} of conductor at most 10^7 includes 59,214 isogeny classes of hyperelliptic curves with ST group $\mathrm{USp}(6)$.

Conductor of $L(X, s)$ in $(M, 2M]$ for $M = 2^{16}, \dots, 2^{22}$ with x -axis range $[0, M/2]$.

a_p averages of 368/506 root number $+1/-1$ genus 3 hyperelliptic curves X/\mathbb{Q} with $\mathrm{ST}(X) = \mathrm{USp}(6)$ of conductor $2^{16} < N \leq 2^{17}$ for $p < 2^{14}$



$w(E)a_p$ averages of 368/506 root number $w(E) = +1/-1$ genus 3 hyperelliptic curves X/\mathbb{Q} with $\mathrm{ST}(X) = \mathrm{USp}(6)$ of conductor $2^{16} < N \leq 2^{17}$ for $p < 2^{14}$



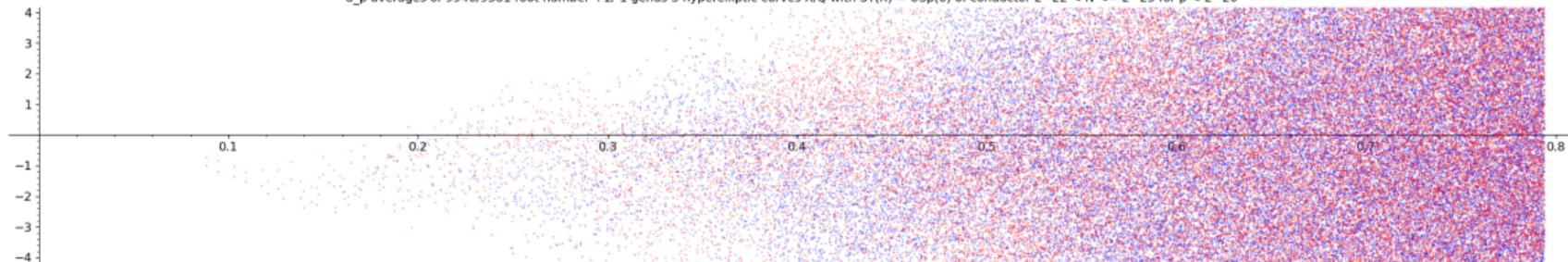
Coming soon to the [LMFDB](#).

L-functions of genus 3 curves over \mathbb{Q} with Sato-Tate group $\mathrm{USp}(6)$.

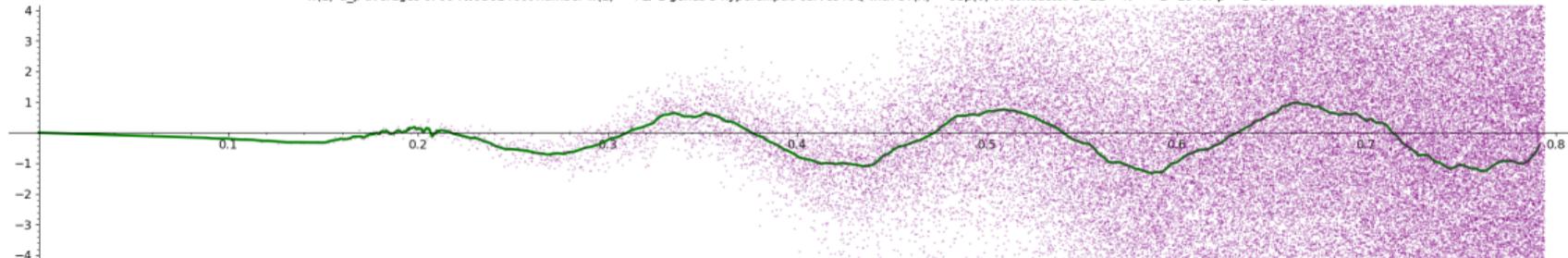
Recently constructed database of genus 3 curves X/\mathbb{Q} of conductor at most 10^7 includes 59,214 isogeny classes of hyperelliptic curves with ST group $\mathrm{USp}(6)$.

Conductor of $L(X, s)$ in $(M, 2M]$ for $M = 2^{16}, \dots, 2^{22}$ with x-axis range $[0, M/2]$.

a_p averages of 9946/9381 root number +1/-1 genus 3 hyperelliptic curves X/Q with ST(X) = USp(6) of conductor $2^{22} < N \leq 2^{23}$ for $p < 2^{20}$

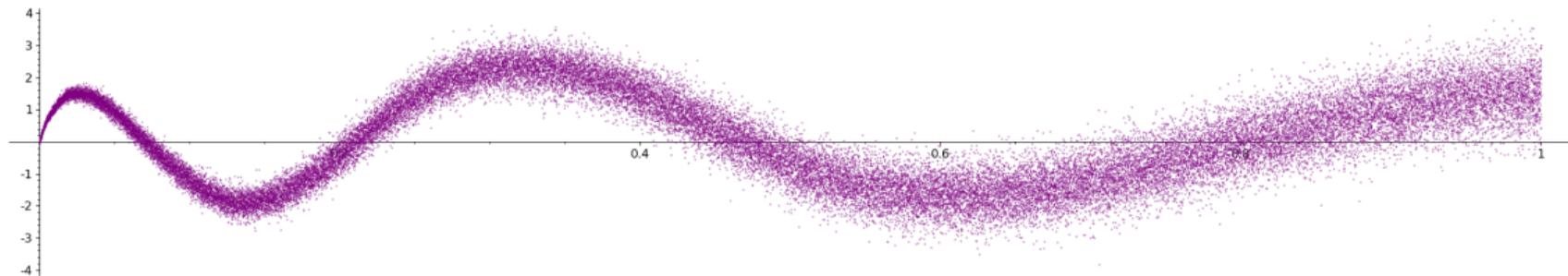


w(E)*a_p averages of 9946/9381 root number w(E) = +1/-1 genus 3 hyperelliptic curves X/Q with ST(X) = USp(6) of conductor $2^{22} < N \leq 2^{23}$ for $p < 2^{20}$



Coming soon to the [LMFDB](#).

Thank you



Animations available at <https://math.mit.edu/~drew/murmurations.html>.