Murmurations for elliptic curves ordered by height

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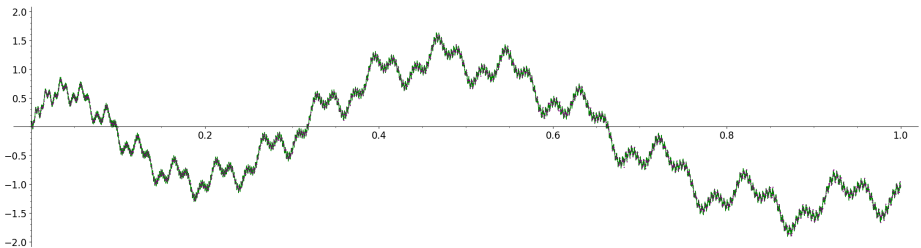


Joint work with Will Sawin (arXiv:2504.12295).

Elliptic curves and their L-functions

Let $E/\mathbb{Q}$ be an elliptic curve, say $E\colon y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$.
For primes $p \nmid \Delta(E) := -16(4A^3 + 27B^2)$ this equation defines an elliptic curve $E/\mathbb{F}_p$.
For all such primes p we have the trace of Frobenius $a_p(E) := p + 1 - \#E(\mathbb{F}_p) \in \mathbb{Z}$.

One can also define $a_p(E)$ for $p|\Delta(E)$, and then construct the L-function

$$L(E,s) := \prod_p (1 - a_p p^{-s} + \chi(p)p^{1-2s})^{-1} = \sum_{n\geq 1} a_n n^{-s}$$

where $\chi(p) = \begin{cases} 0 & p|N(E) \\ 1 & \text{otherwise} \end{cases}$ and the conductor $N(E)$ divides $\Delta(E)$.

But in fact the a_p for $p \nmid \Delta(E)$ determine $L(E,s)$ (via strong multiplicity one), as well as the conductor and root number $w(E) = \pm 1$ which appear in the functional equation

$$\Lambda(E,s) = w(E)N(E)^{1-s}\Lambda(E, 2-s),$$

where $\Lambda(s) := \Gamma_{\mathbb{C}}(s)L(E,s)$. The L-function $L(E,s)$ determines the isogeny class of E.

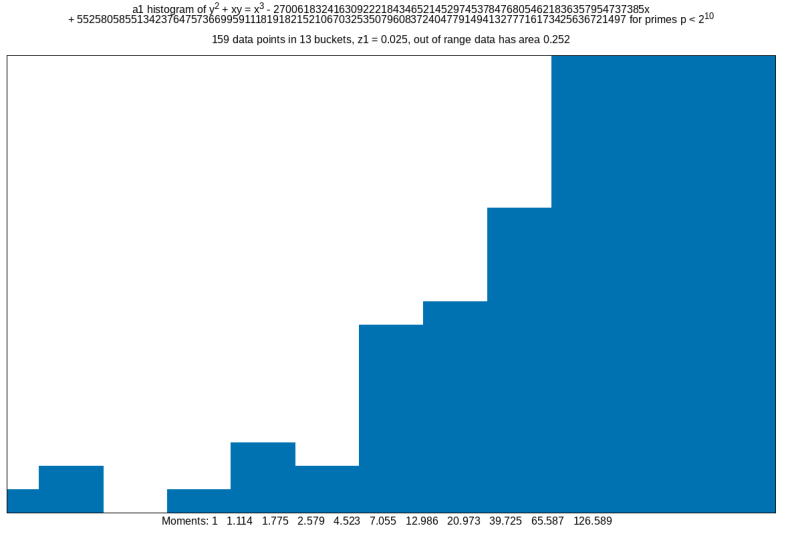
Arithmetic statistics of Frobenius traces of elliptic curves $E/\mathbb{Q}$

Three conjectures from the 1960s and 1970s (the first is now a theorem):

1. **Sato–Tate**: The sequence $x_p := a_p(E)/\sqrt{p}$ is equidistributed with respect to the pushforward of the Haar measure of $\mathrm{ST}(E)$ ($= \mathrm{SU}(2)$ if E does not have CM).
2. **Birch and Swinnerton-Dyer**:
$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{p\le x} \frac{a_p(E)\log p}{p} = \frac{1}{2} - r,$$
3. **Lang–Trotter**: For every nonzero $t \in \mathbb{Z}$ there is a real number $C_{E,t}$ for which
$$\#\{p \le x : a_p(E) = t\} \sim C_{E,t}\frac{\sqrt{x}}{\log x}.$$

These conjectures depend only on $L(E, s)$ and generalize to other L-functions.

Example: Elkies-Klagsbrun curve of rank ≥ 29.



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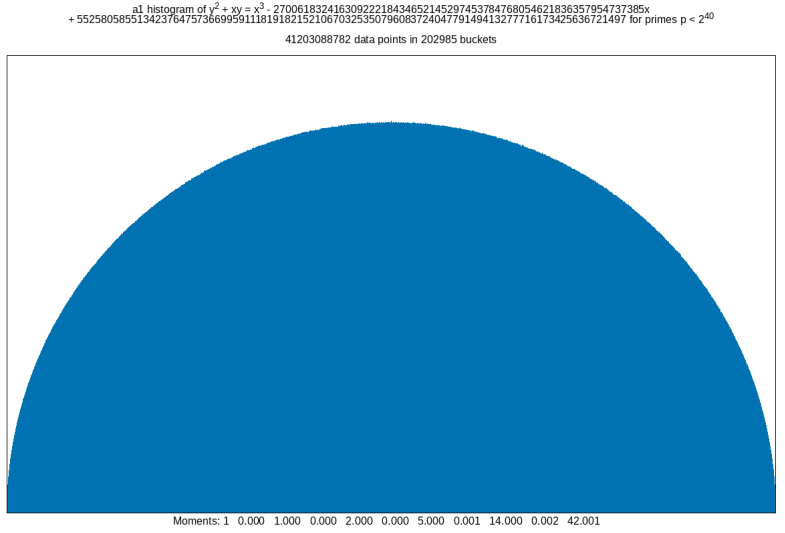


How rank effects trace distributions

An early form of the BSD conjecture implies that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{p\leq x} \frac{a_p(E)\log p}{p} = \frac{1}{2} - r. \tag{1}$$

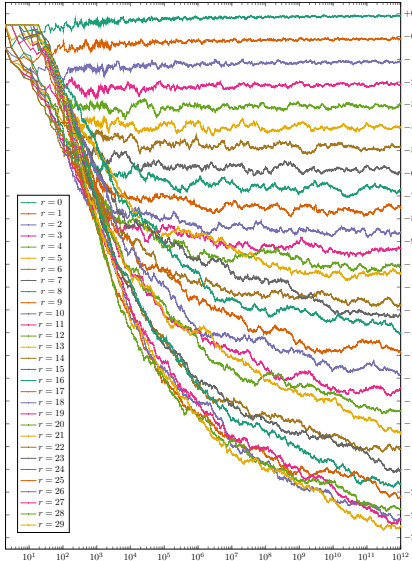
Sums of this form (Mestre-Nagao sums) are often used as a tool when searching for elliptic curves of large rank (which necessarily have large conductor N).[1] [2]

Theorem (Kim-Murty 2023)

If the limit on the LHS of (1) exists then it equals the RHS with r the analytic rank, and the L-function of E satisfies the Riemann hypothesis.

[1] See Sarnak's 2007 letter to Mazur.

[2] See Kazalicki-Vlah for some recent machine-learning work on this topic.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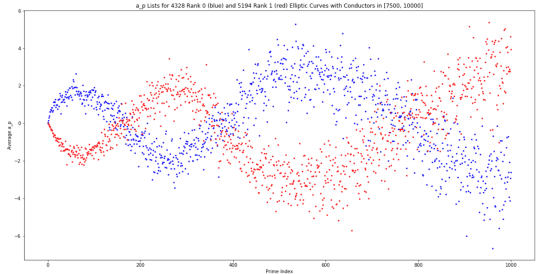
r = 1
r = 2
r = 3
r = 4
r = 5
r = 6
r = 7
r = 8
r = 9
r = 10
r = 11
r = 12
r = 13
r = 14
r = 15
r = 16
r = 17
r = 18
r = 19
r = 20
r = 21
r = 22
r = 23
r = 24
r = 25
r = 26
r = 27
r = 28
r = 29
+0.5
0.5
−1.5
−2.5
−3.5
−4.5
−5.5
−6.5
−7.5
−8.5
−9.5
−10.5
−11.5
−12.5
−13.5
−14.5
−15.5
−16.5
−17.5
−18.5
−19.5
−20.5
−21.5
−22.5
10^1
10^2
10^3
10^4
10^5
10^6
10^7
10^8
10^9
10^10
10^11
10^12

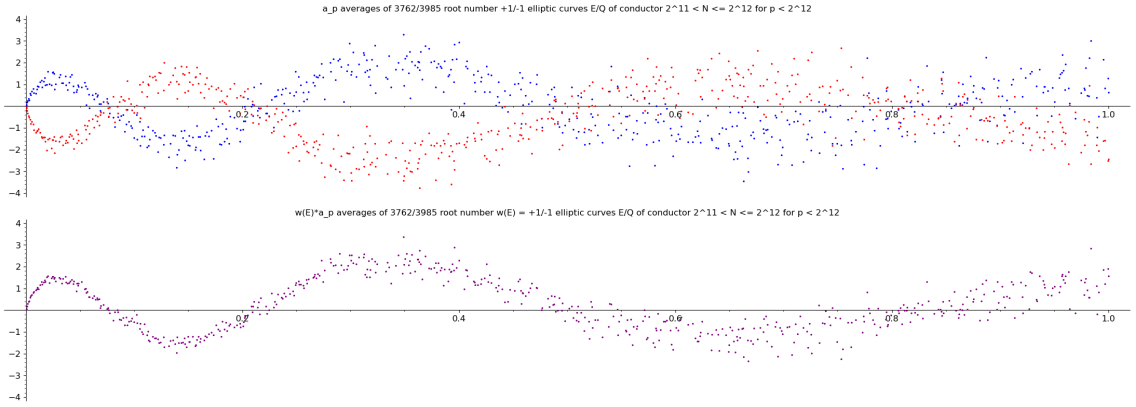
Murmurations of elliptic curves

In their 2022 preprint *Murmurations of elliptic curves* (recently published), He, Lee, Oliver, and Pozdnyakov observed a curious fluctuation in average Frobenius traces of elliptic curves in a fixed conductor interval when separated by rank.



Murmurations of elliptic curves

Elliptic curve L-functions of conductor $N \in (M, 2M]$ for $M = 2^{11}, 2^{12}, \ldots, 2^{17}, 250000$. The x-axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or $m_p := w(E)a_p(E)$ over even/odd or all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$.[3]



[3] Letter to Sarnak, response from Sarnak.

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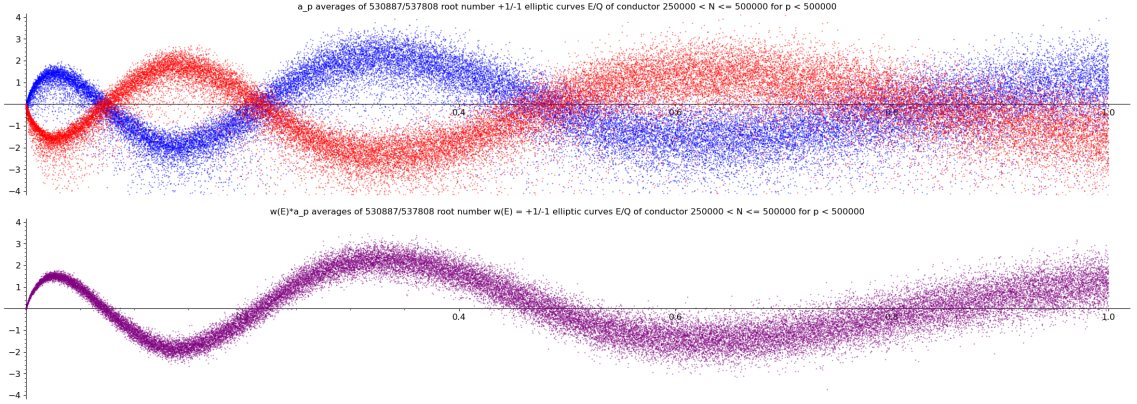

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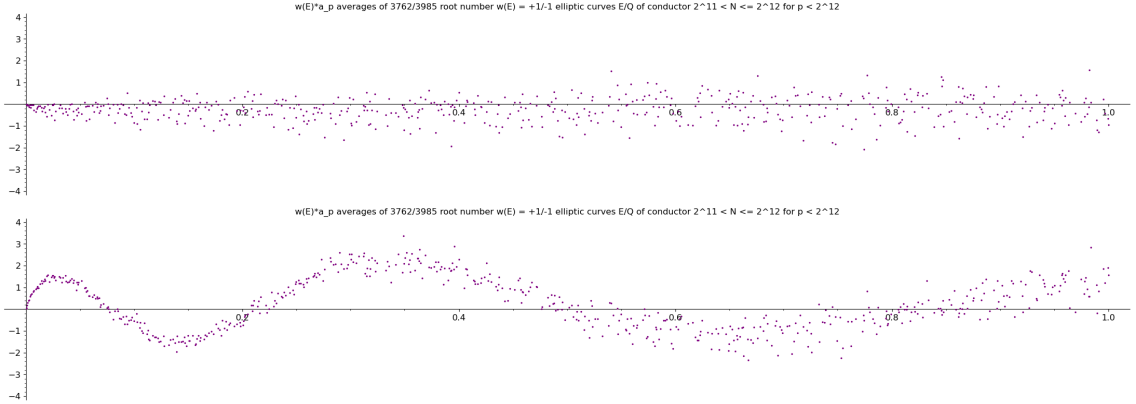
Bias cancellation

There is a negative bias in $\bar{a}_p$ that depends on p but is independent of the root number $w(E)$ and disappears in $\bar{m}_p$.



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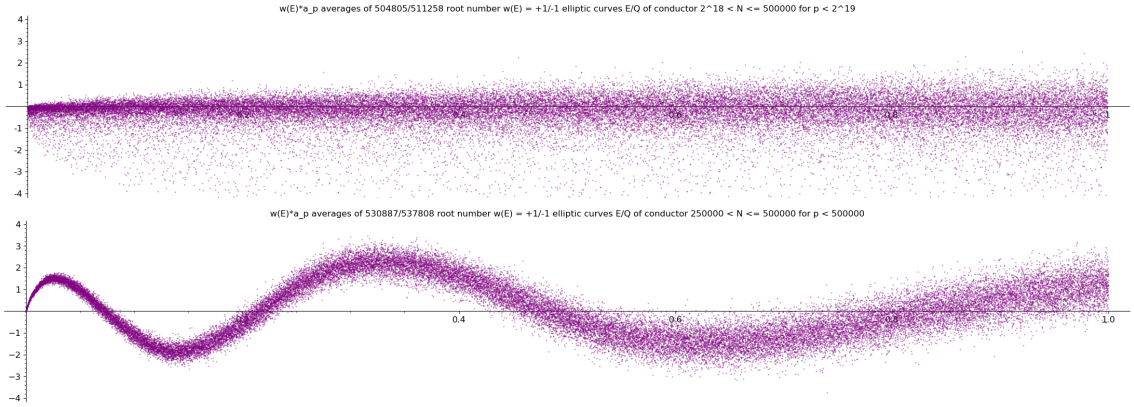
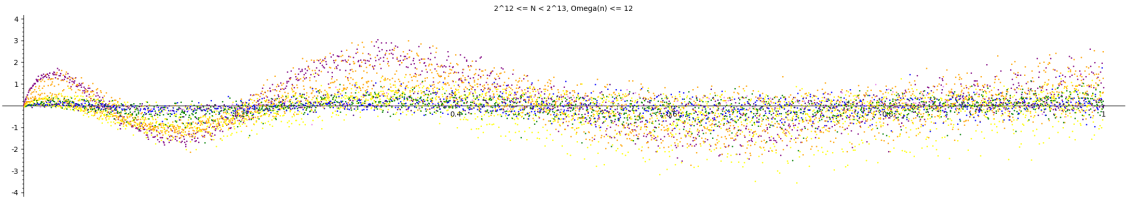


Murmurations of elliptic curves over a_n (not just a_p)

Elliptic curve L-functions of conductor $N \in (M, 2M]$ for $M = 2^{12}, \ldots, 2^{17}, 250000$. The x-axis range is $[0, 2M]$. Dots at $(n, \bar{m}_n)$ show the average of $m_n := w(E)a_n(E)$ over all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$.

The color of the dot indicates the number of prime factors of n (with multiplicity).



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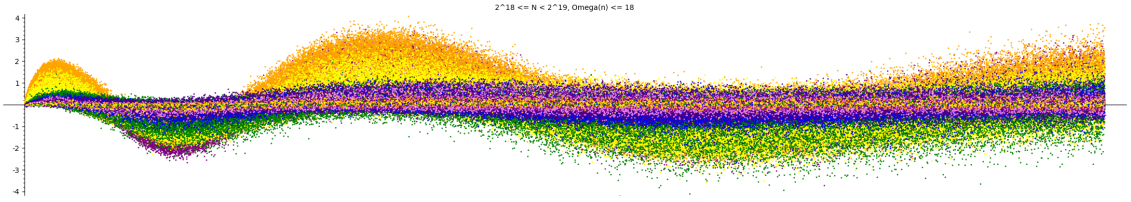
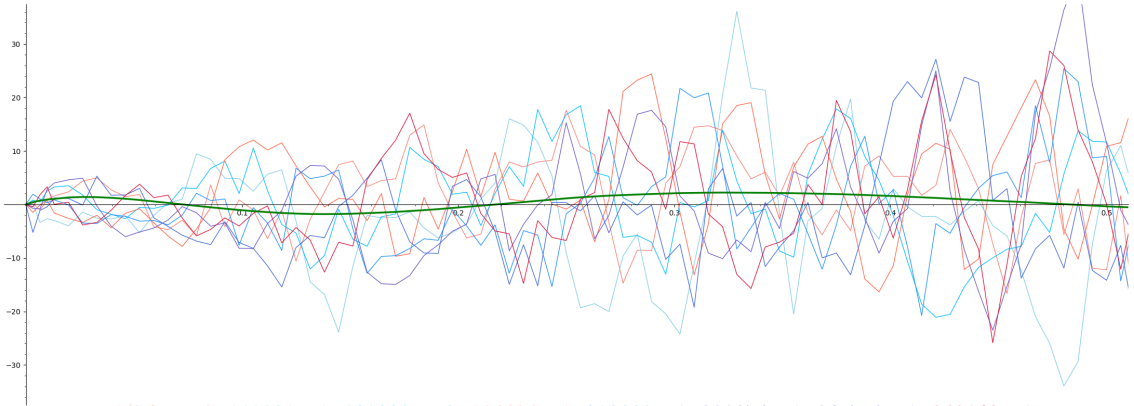


Murmurations are an aggregate phenomenon

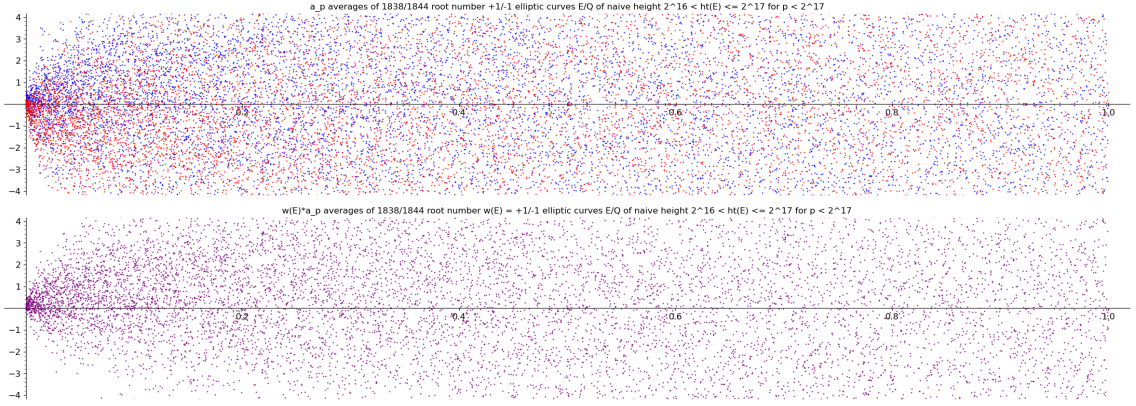
Moving average line plots of $\bar{m}_p$ for 8 individual and all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$, using subintervals of size $\sqrt{M}$ for $p \leq 2M$, with $M = 2^{17}$.



147455.b2, 163839.a1, 180222.be2, 196606.b1, 212990.l1, 229374.a1, 245758.a1, 262143.d1

Ordering by naïve height

Elliptic curves with $\mathrm{ht}(E) := \max(4|A|^3, 27B^2)$ in $(M, 2M]$ for $M = 2^{16}, \ldots, 2^{26}$. The x-axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or m_p over even/odd or all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$.



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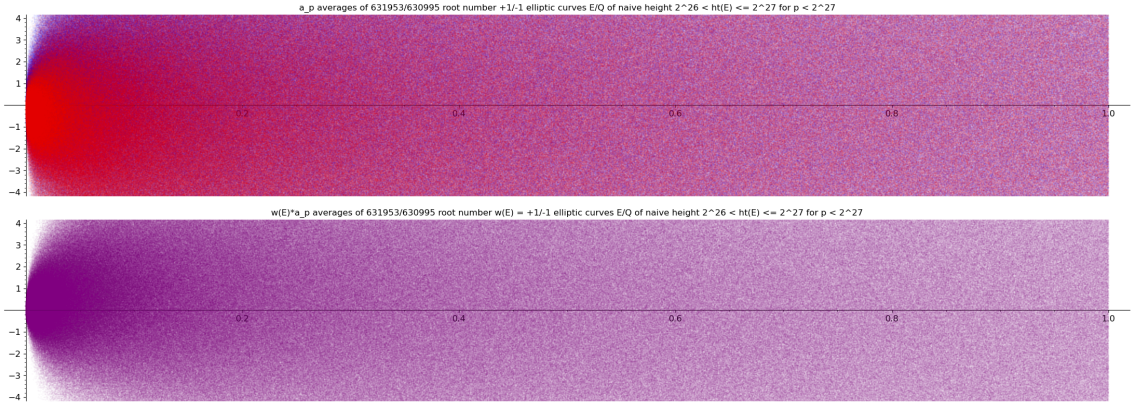
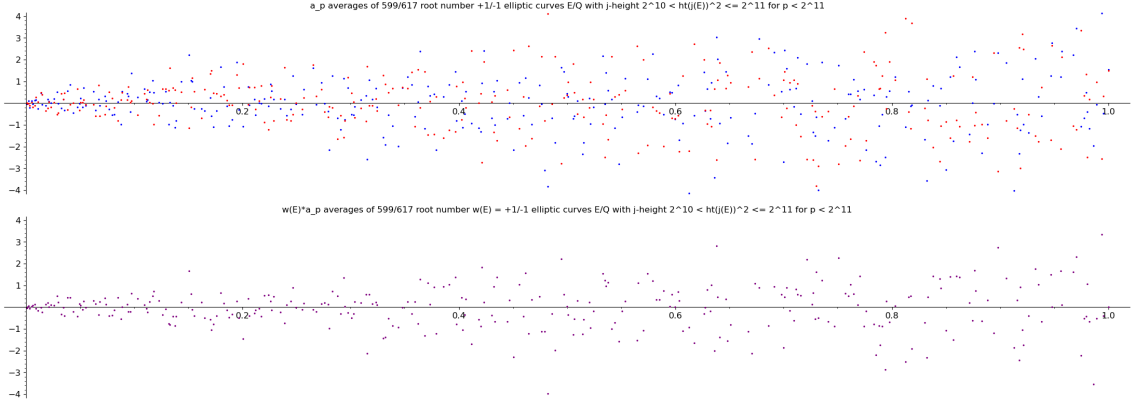


Ordering by j-invariant height

Elliptic curves with $\mathrm{ht}(j(E))^{12/5}$ in $(M, 2M]$ for $M = 2^{11}, \ldots, 2^{19}$.
The x-axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or m_p over even/odd or all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$.



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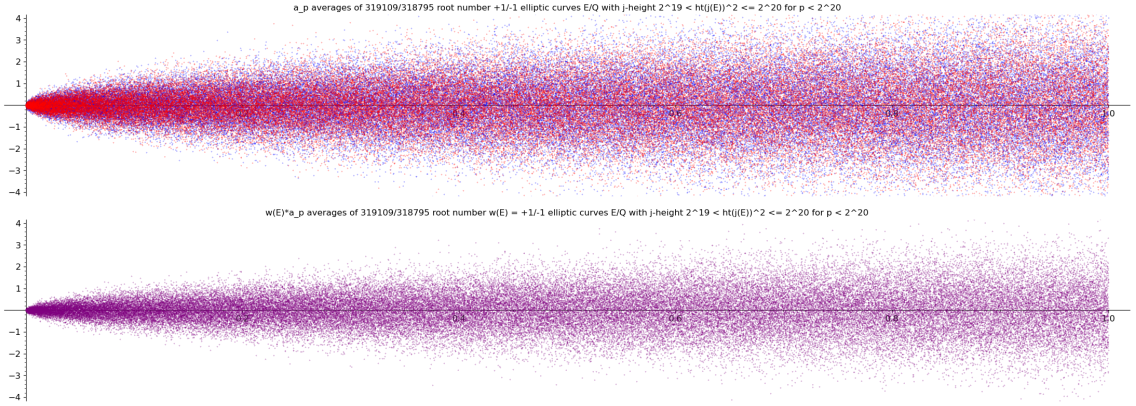
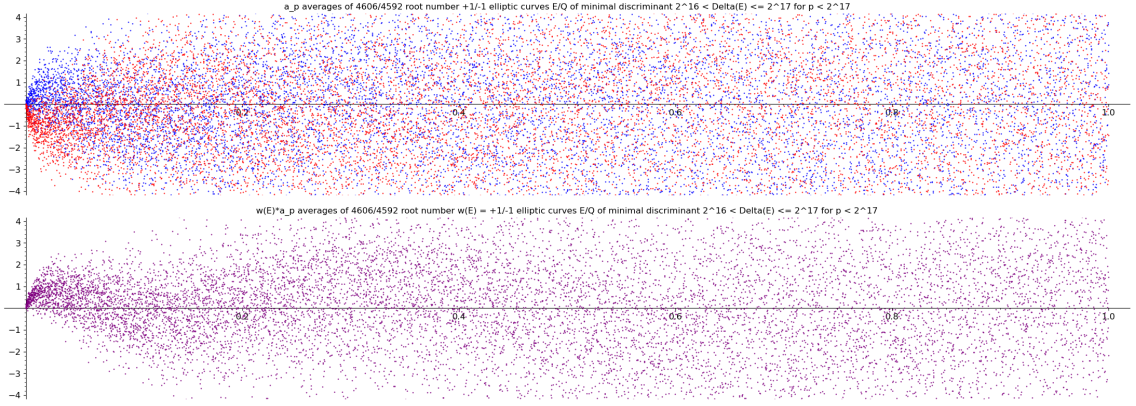


Ordering by absolute minimal discriminant

Elliptic curves with minimal discriminant $\Delta(E)$ in $(M, 2M]$ for $M = 2^{16}, \ldots, 2^{23}$. The x-axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or m_p over even/odd or all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$.



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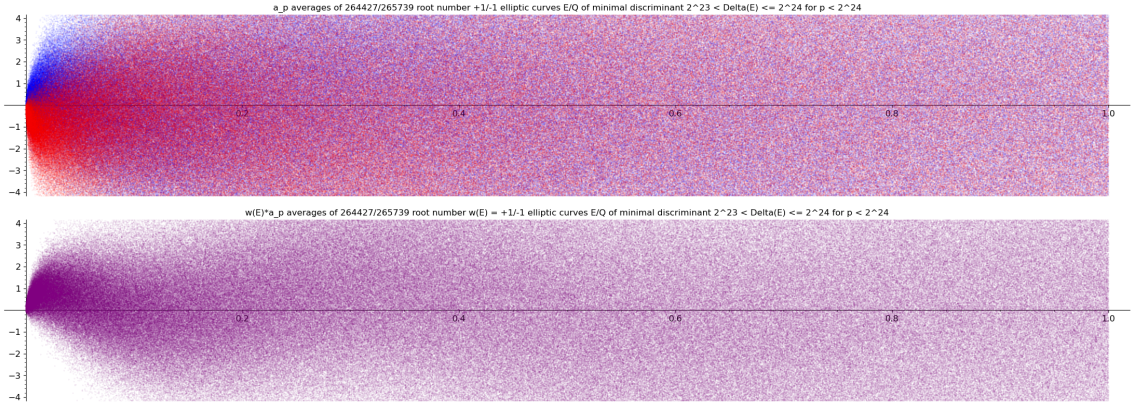

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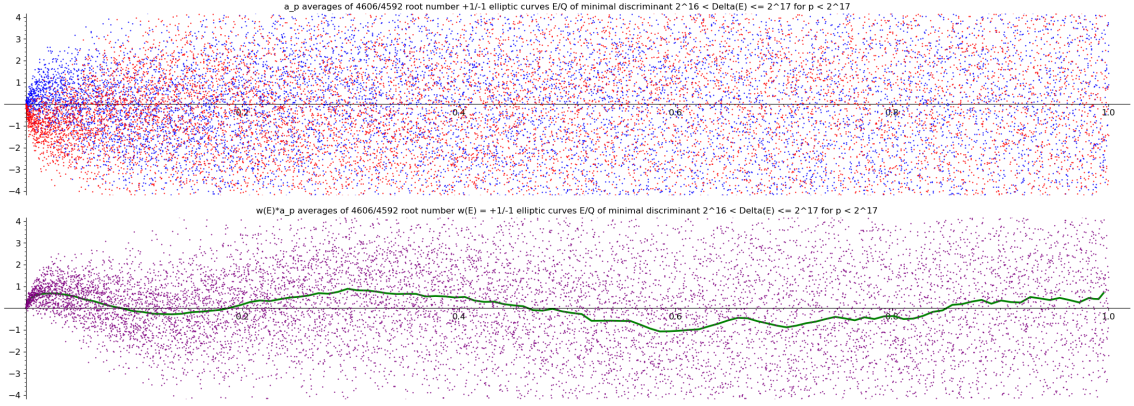

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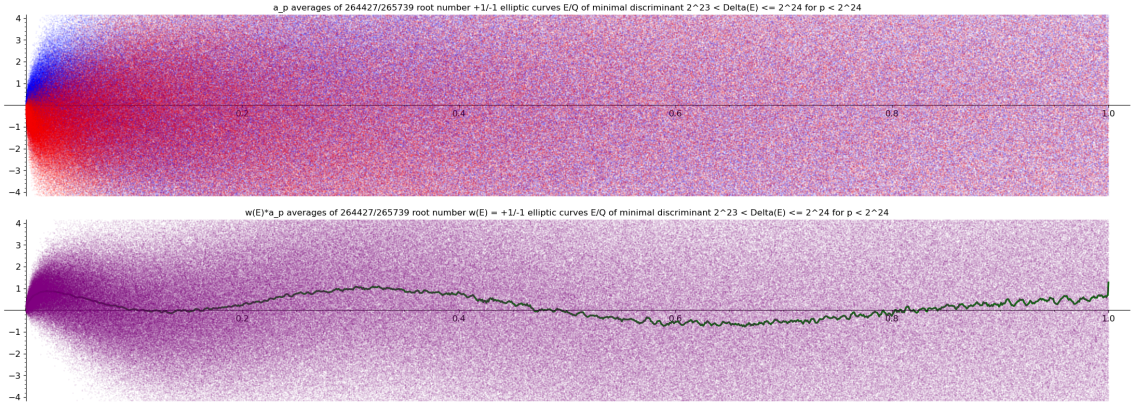

Ordering by naïve height (redux)

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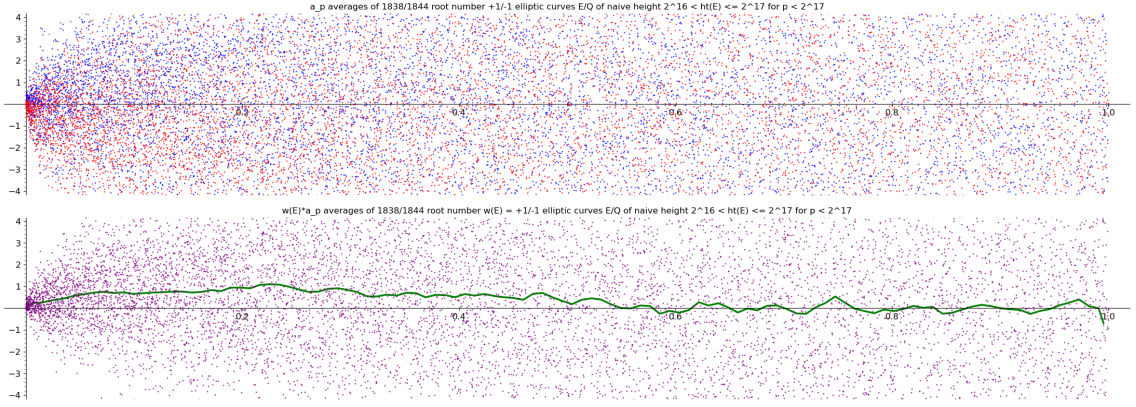

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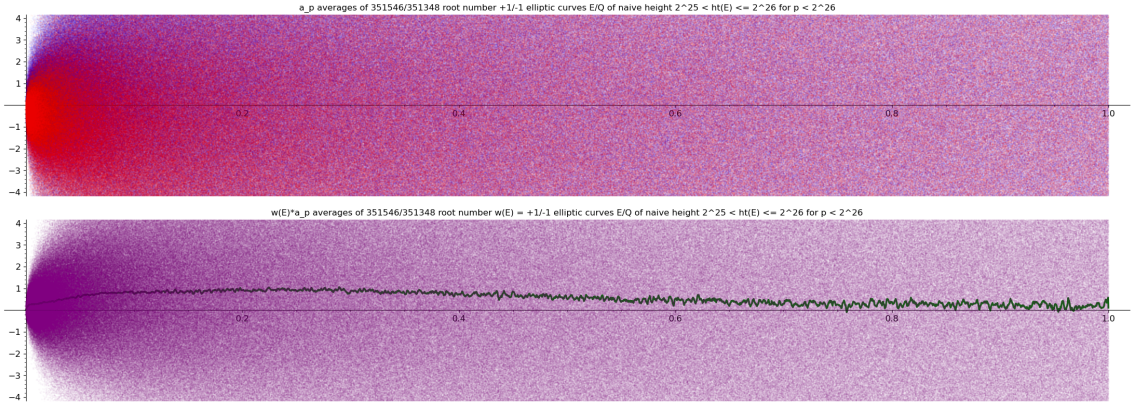


Local averaging

Rather than averaging a_p's for L-functions with conductor in an interval, we may instead compute local averages of a_p for each L-function in our family with p/N varying over some interval, and then average these local averages.

For example, we may divide the interval $[0,1]$ into n intervals $(x, x+\frac{1}{n}]$, with $x = 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$. For each L-function in our family we compute a_p for all primes $p \leq N$, and then for $x = 0, \frac{1}{n}, \ldots, \frac{n-1}{n}$ we compute the average $\alpha_x(E)$ of $a_p(E)$ for

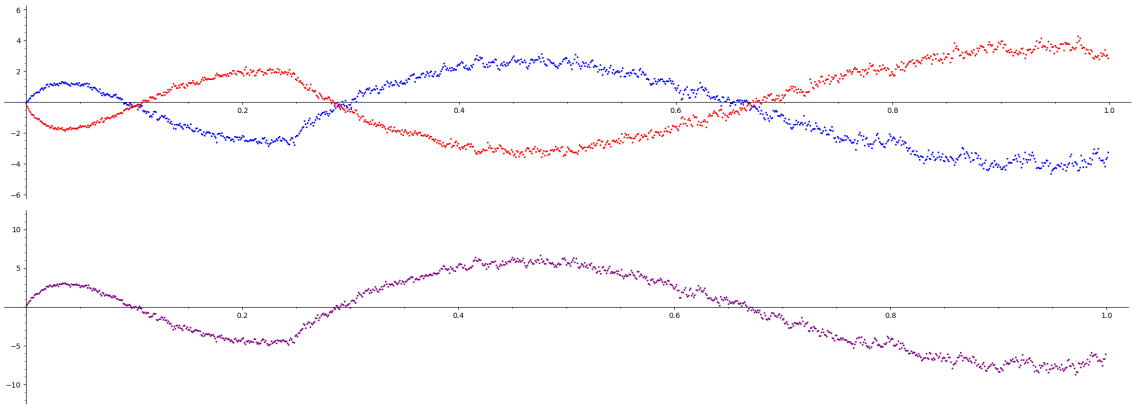
$$\frac{p}{N} \in \Big(x, x+\frac{1}{n}\Big],$$

yielding a vector of n real numbers. We then average these vectors over all L-functions in our family of a given root number or rank, up to an increasing bound $X \to \infty$.

With this setup, we do not need to order by conductor, but the order matters.

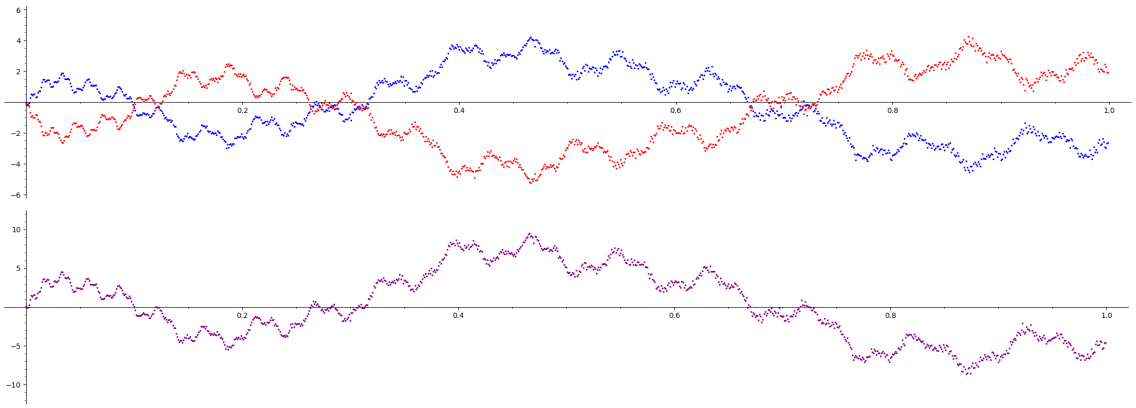
Local averaging: elliptic curves ordered by conductor

Elliptic curve L-functions of conductor $N \le M$ for $M = 2^{12}, 2^{13}, \ldots, 2^{17}, 2^{18}$. The x-axis range is $[0, 1]$. A blue/red (or purple) dot at $(x, \bar{\alpha}_x)$ shows the average $\bar{\alpha}_x$ of $\alpha_x(E)$ (or $w_p(E)\alpha_x(E)$) over even/odd rank (or all) $E/\mathbb{Q}$ with $N_E \le M$.

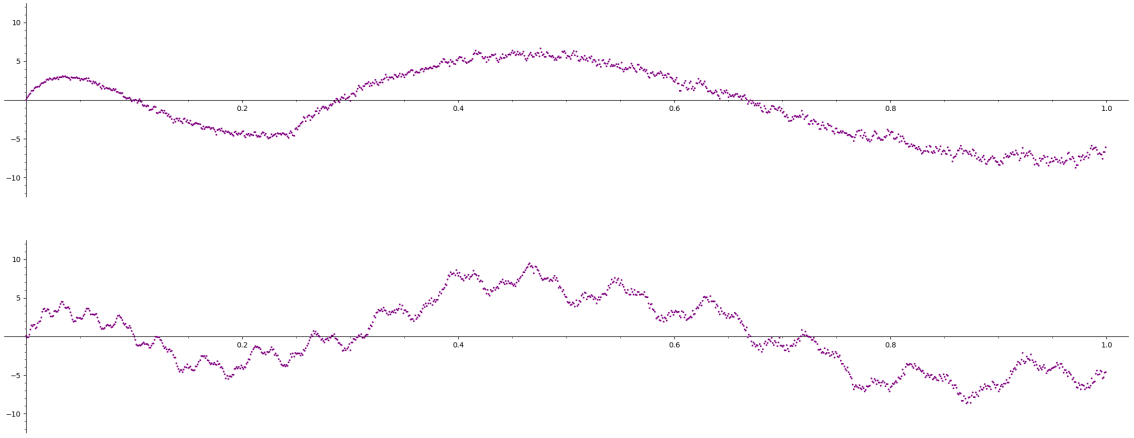


Local averaging: elliptic curves ordered by height

Elliptic curves with $\mathrm{ht}(E) := \max(4|A|^3, 27|B|^2) \le M$ for $M = 2^{18}, \ldots, 2^{27}$.
The x-axis range is $[0, 1]$. A blue/red (or purple) dot at $(x, \bar{\alpha}_x)$ shows the average $\bar{\alpha}_x$ of $\alpha_x(E)$ (or $w_p(E)\alpha_x(E)$) over even/odd rank (or all) $E/\mathbb{Q}$ with $\mathrm{ht}(E) \le M$.



Local averaging: elliptic curves ordered by conductor vs height



Arithmetic L-functions

We call an L-function analytic if it has the properties every good L-function should: analytic continuation, functional equation, Euler product, temperedness, central character; see FPRS18; it is analytically normalized if its central value is at $s = 1/2$.

An analytically normalized L-function $L_{\mathrm{an}}(s) = \sum a_n n^{-s}$ is arithmetic if $a_n n^{\omega/2} \in \mathcal{O}_K$ for some number field K and $\omega \in \mathbb{Z}_{\geq 0}$. The least such ω is the motivic weight.
Its arithmetic normalization $L(s) := L_{\mathrm{an}}(s + \omega/2)$ has coefficients in $\mathcal{O}_K$ and satisfies

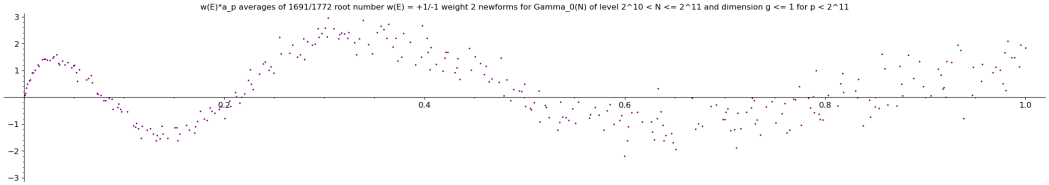
$$\Lambda(s) = N^{1-s} w \bar{\Lambda}(1 + \omega - s).$$

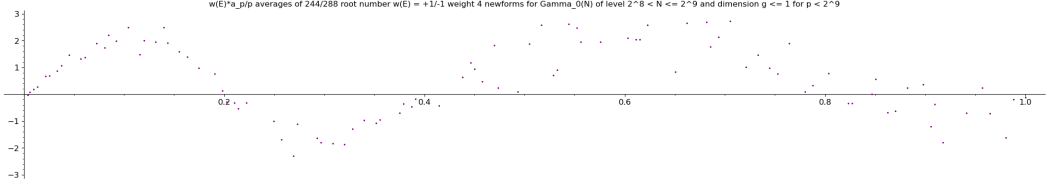
L-functions of abelian varieties have motivic weight $\omega = 1$.
L-functions of weight-k holomorphic cuspforms have motivic weight $\omega = k - 1$.

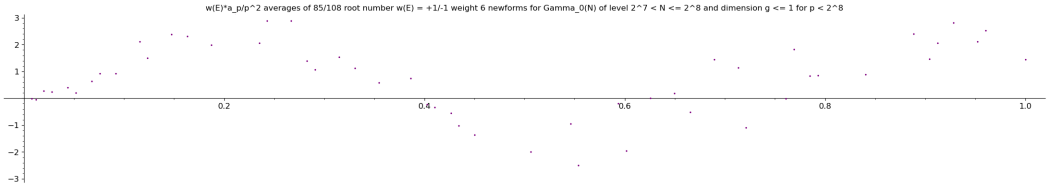
We consider Galois-closed families of self-dual arithmetically normalized L-functions.
In any such family the values of a_p and m_p are integers and $w = \pm 1$.

When averaging a_p's in motivic weight $\omega > 1$ we normalize them via $a_p \mapsto a_p / p^{(\omega-1)/2}$.
This ensures that we always have $|a_p| = O(\sqrt{p})$, as with elliptic curves.

Newforms for $\Gamma_0(N)$ of weight $k = 2, 4, 6$ with rational coefficients.







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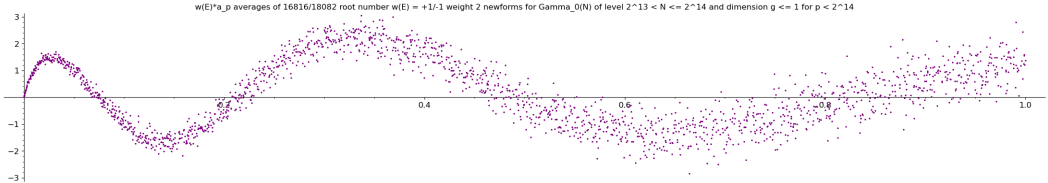
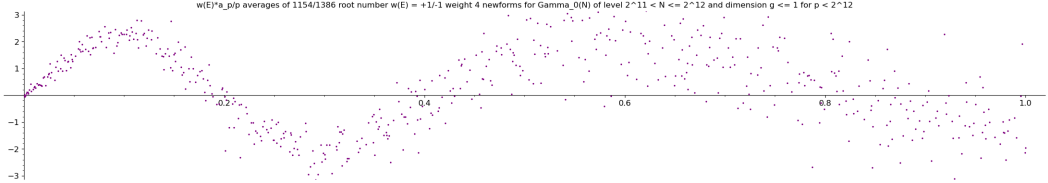
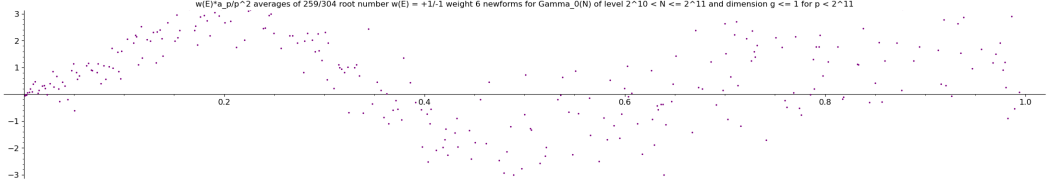
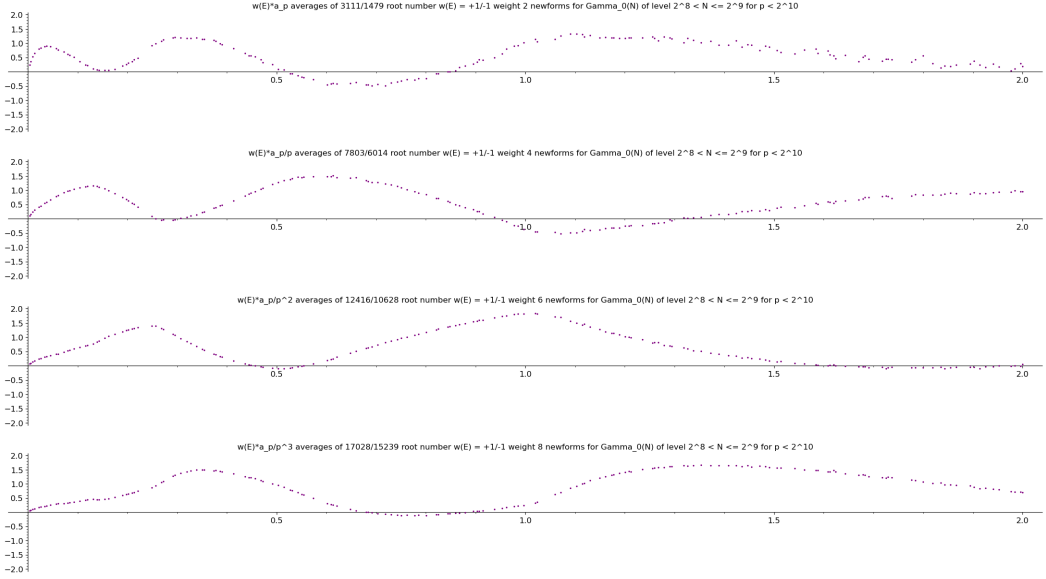






Newforms for $\Gamma_0(N)$ of weight $k = 2, 4, 6, 8$.



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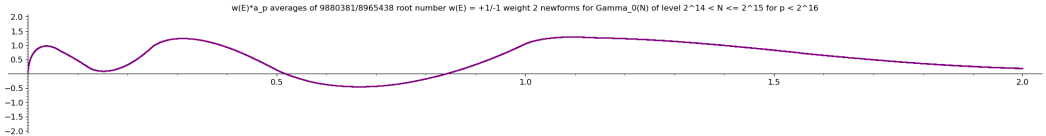
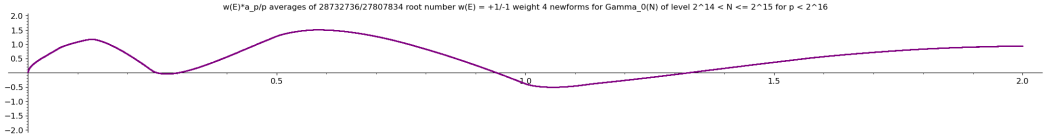
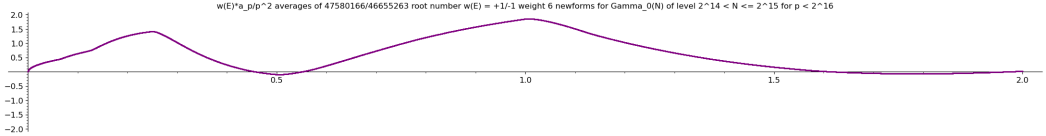
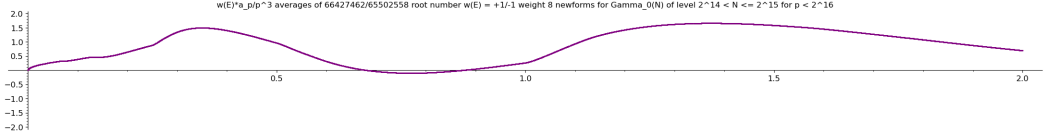








Zubrilina's theorem

Definition. Let $U_n \in \mathbb{Z}[x]$ denote the Chebyshev polynomial defined by $U_n(\cos\vartheta)\sin\vartheta = \sin((n+1)\vartheta)$. The murmuration density function is

$$M_k(y) := D_k\Big(Ay \ - (-1)^{k/2} B \sum_{1\le r\le 2y} c(r)\sqrt{4y^2-r^2}\ U_{k-2}\big(\tfrac{r}{2y}\big) - \pi y^2 \delta_{k=2}\Big),$$

$A := \prod_p \big(1+\frac{p}{(p+1)^2(p-1)}\big)$, $B := \prod_p \frac{p^4-2p^2-p+1}{(p^2-1)^2}$, $c(r) := \prod_{p|r}\big(1+\frac{p^2}{p^4-2p^2-p+1}\big)$, $D_k := \frac{12}{(k-1)\pi\prod_p(1-\frac{1}{p^2+p})}$.

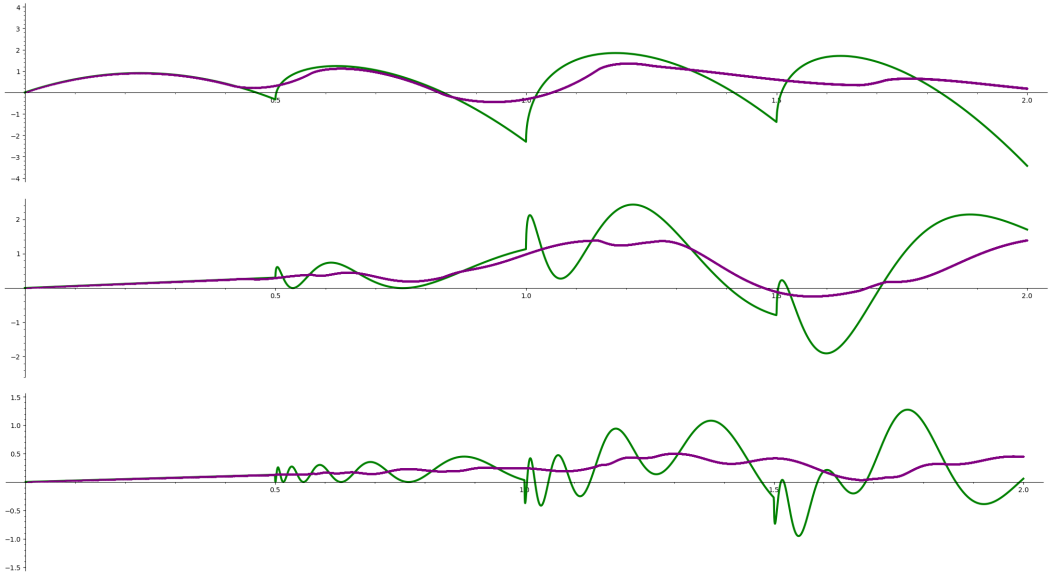
Theorem [Zubrilina 2023]. Let $\sum a_n(f)q^n$ denote a weight-k newform for $\Gamma_0(N)$ with root number $w(f)$. Let $X, Y, P \to \infty$ with P prime, $Y \sim X^{1-\delta}$, $P \ll X^{1+\delta_1}$, $\delta, \delta_1 > 0$ and $2\delta_1 < \delta < 1$, and put $y := \sqrt{P/X}$. Then for every $\varepsilon > 0$ we have

$$\frac{\sum_{N\in[X,X+Y]}^{\square\text{-free}} \sum_f w(f)a_P(f)P^{(1-k/2)}}{\sum_{N\in[X,X+Y]}^{\square\text{-free}} \sum_f 1} = M_k(y) + O_\varepsilon(X^{-\delta'+\varepsilon} + P^{-1})$$

where $\delta' := \max(\delta/2-\delta_1, (\delta+1)/9-\delta_1)$; for $\delta_1 < 2/9$ we can choose δ so $\delta' > 0$.

Zubrilina's theorem for $k = 2, 14, 32$

(click here for other k)



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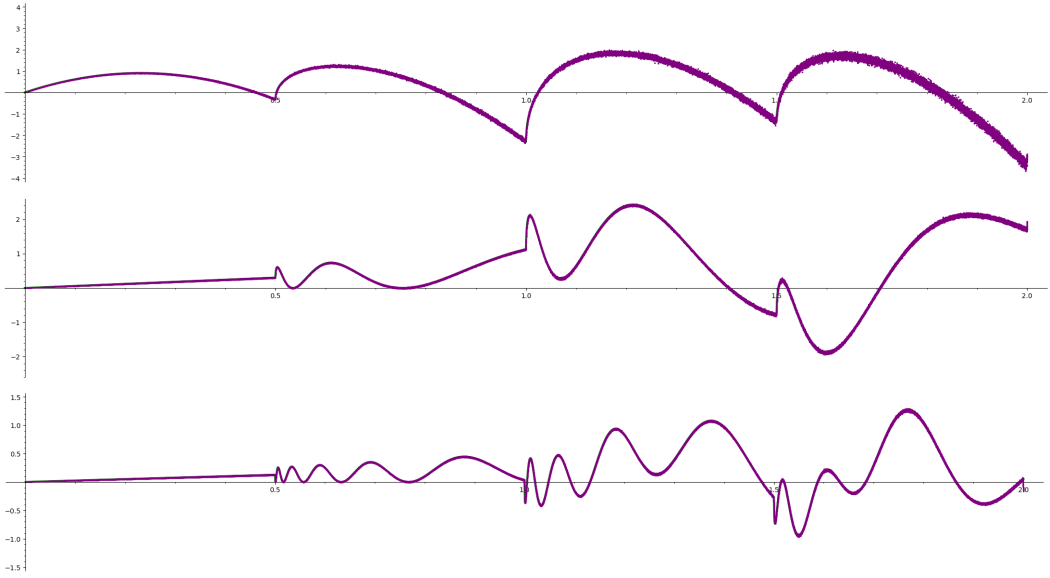
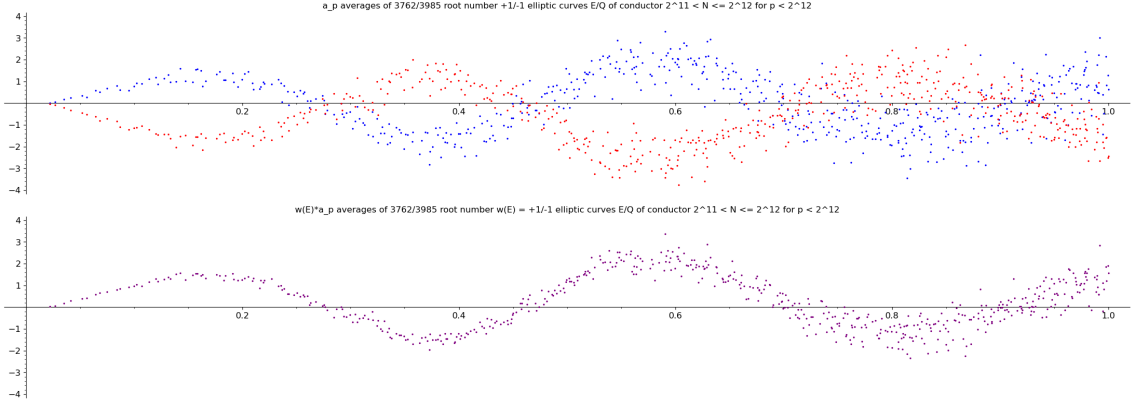


Murmurations of elliptic curves with squareroot normalization

Elliptic curve L-functions of conductor $N \in (M, 2M]$ for $M = 2^{11}, 2^{12}, \ldots, 2^{17}, 250000$. The x-axis range is $[0, 2M]$. A blue/red or purple dot at $(\sqrt{p}, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or $m_p := w(E)a_p(E)$ over even/odd or all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$.



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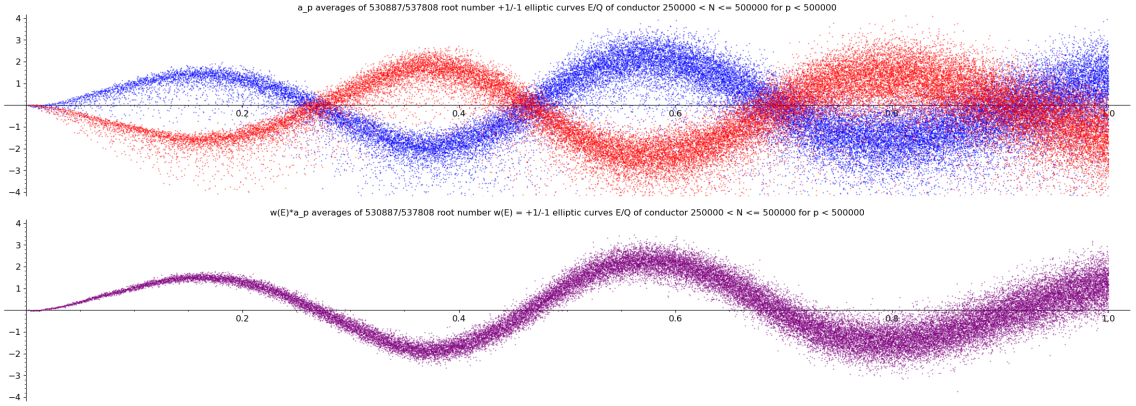


A murmuration theorem for elliptic curves

Let $\mathcal{E}(X) := \left\{ y^2 = x^3 + Ax + B : A, B \in \mathbb{Z}, p^4 | A \Rightarrow p^6 \nmid B, \max(4|a|^3, 27b^2) \leq X \right\}$ be the set of isomorphism classes of elliptic curves over $\mathbb{Q}$ of naive height at most X.

Theorem (S–Sawin 2025)

For any smooth $W : \mathbb{R}_{>0} \to \mathbb{R}$ with compact support, the limit

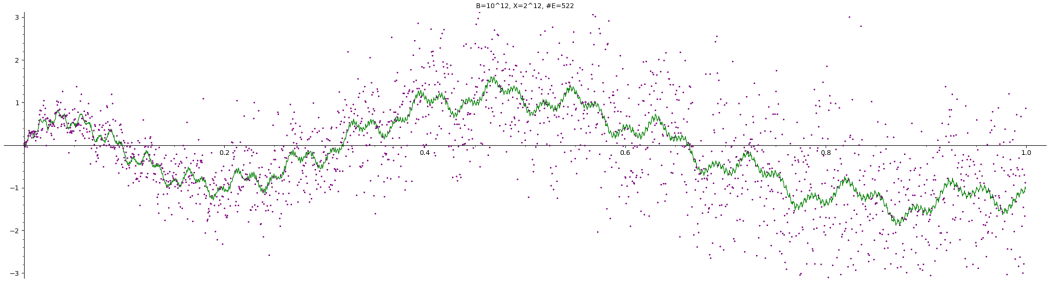
$$\lim_{X \to \infty} \frac{1}{\#\mathcal{E}(X)} \sum_{E \in \mathcal{E}(X)} \frac{w(E)}{N_E} \sum_{n \geq 1} W(n/N_E) a_n(E)$$

exists and is equal to

$$\int_0^\infty 2\pi W(u) \sum_{n=1}^\infty \frac{\prod_{p|n} \ell_{p^{\nu_p(n)}}}{\sqrt{n}} \sqrt{u} J_1(4\pi\sqrt{un}) du,$$

with $\ell_{2^\nu} = \frac{t_2(\nu+2)}{1023}, \ell_{3^\nu} = \cdots, \ell_{p^\nu} = \frac{p^9 - p^8}{p^{10}-1} t_p(\nu+2)$, *where* $t_p(k) = \mathrm{tr}(T_p)$ *on* $S_k(1)$.

A murmuration theorem for elliptic curves



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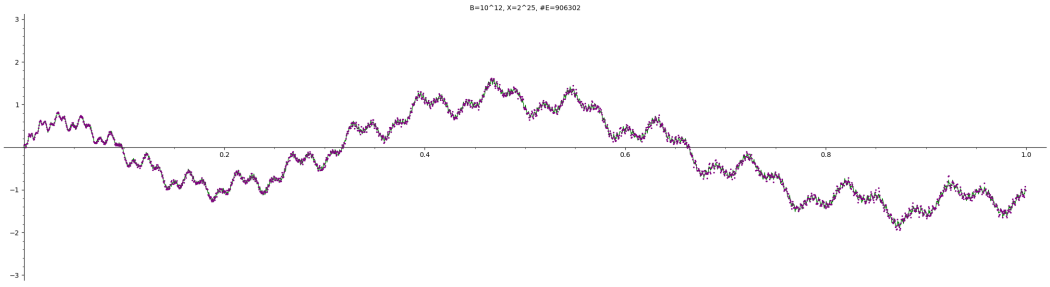


The Voronoi summation formula

Lemma (Kowalski-Michel-VanderKam 2002,
Blomer-Fouvry-Kowalski-Michel-Milićević-Sawin 2023, Sawin-S 2025)

Let E be an elliptic curve over $\mathbb{Q}$, $N(E)$ the conductor of E, and $\varepsilon(E)$ the epsilon factor of E. Let q be a positive integer, a an integer coprime to q, and $W\colon (0,\infty) \to \mathbb{R}$ a smooth compactly supported function. Let $\overline{aN(E)}$ denote the modular inverse of $aN(E)$ modulo q. We have

$$w(E)\sum_{n=1}^{\infty}\frac{a_n(E)}{\sqrt{n}}W\left(\frac{n}{N(E)}\right)e\left(\frac{an}{q}\right)$$
$$=\frac{\sqrt{N(E)}}{q}\sum_{n=1}^{\infty}\frac{a_n(E)}{\sqrt{n}}e\left(\frac{\overline{aN(E)}n}{q}\right)\int_0^{\infty}2\pi W(u)J_1\left(4\pi\frac{\sqrt{un}}{q}\right)du.$$

Proof strategy

The Voronoi summation formula can be shown to imply

$$\frac{w(E)}{N(E)} \sum_{n=1}^{\infty} W\Big(n/N(E)\Big) a_n(E) = \int_0^{\infty} 2\pi W(u) \sum_{n=1}^{\infty} \frac{a_n(E)}{\sqrt{n}} \sqrt{u} J_1(4\pi\sqrt{un}) du. \qquad (2)$$

Noting that $a_n(E) = \prod_{p|n} a_{p^{v_p(n)}}(E_{A,B})$ and each $a_{p^{v_p(n)}}(E_{A,B})$ is a p-adically continuous function of A, B, one can show

$$\lim_{X\to\infty} \mathbb{E}_{\{E\in\mathcal{E}(X)\}} [a_n(E)] = \int_{\prod_p \mathbb{Z}_p^2 \setminus (p^4\mathbb{Z}_p \times p^6\mathbb{Z}_p)} \prod_{p|n} a_{p^{v_p(n)}}(E) = \prod_{p|n} \ell_{p,v_p(n)}, \qquad (3)$$

Taking limits on both sides of (2), proving absolute convergence, moving the limit on the RHS inside the sum, and applying (3) yields the theorem.

Refinements

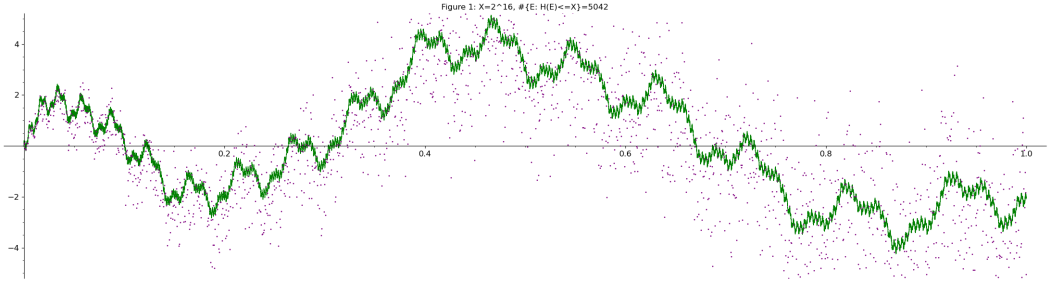
Conjecture

For real numbers $0 < C_1 < C_2$ *we have*

$$\lim_{X\to\infty} \mathbb{E}_{\{E\in\mathcal{E}(X)\}}\Big[\frac{\log(N(E)^{\frac{C_1+C_2}{2}})}{N(E)} \sum_{\substack{p\in(C_1N(E),C_2N(E)]\\ p \text{ prime}}} \varepsilon(E)a_p(E)\Big]$$

$$= \int_{C_1}^{C_2} 2\pi\sqrt{u} \sum_{\substack{q\in\mathbb{N}\\ \text{squarefree}}} \sum_{m\in\mathbb{N}} \frac{\mu(\gcd(m,q))}{qm\varphi\left(\frac{q}{\gcd(m,q)}\right)} J_1\left(4\pi\frac{\sqrt{u}m}{q}\right) \prod_{p|q} \hat{\ell}_{p,2v_p(m)} \prod_{p|m,p\nmid q} \ell_{p,2v_p(m)}\, du.$$

Refinements (testing the conjecture)



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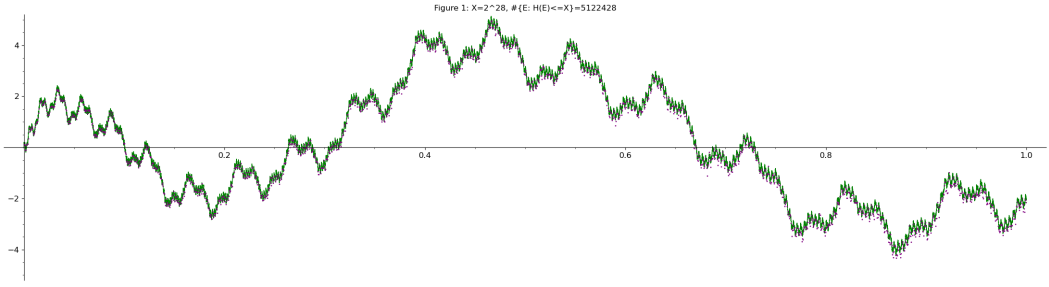


Refinements

Theorem (Sawin-S 2025)

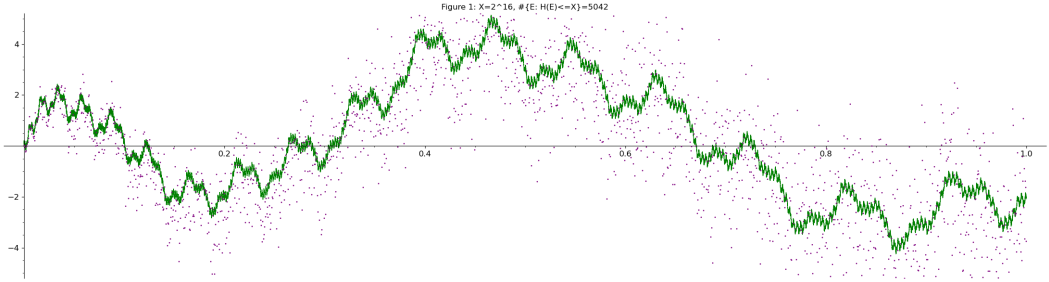
Let W be a smooth, compactly-supported function on $(0,\infty)$. The limit

$$\lim_{P\to\infty}\lim_{X\to\infty}\mathbb{E}_{\{E:E\in\mathcal{E}(X)\}}\Big[\frac{\prod_{p\le P}(1-1/p)^{-1}}{N(E)}\sum_{\substack{n\in\mathbb{N}\\ p\nmid n \text{for} p\le P}}W\left(\frac{n}{N(E)}\right)\varepsilon(E)a_n(E)\Big]$$

exists and is equal to

$$\int_0^\infty W(u)\sqrt{u}\Bigg(2\pi\sum_{\substack{q\in\mathbb{N}\\ \text{squarefree}}}\sum_{m\in\mathbb{N}}\frac{\mu(\gcd(m,q))}{qm\varphi\left(\frac{q}{\gcd(m,q)}\right)}J_1\left(4\pi\frac{\sqrt{u}m}{q}\right)\prod_{p|q}\hat{\ell}_{p,2v_p(m)}\prod_{p|m,p\nmid q}\ell_{p,2v_p(m)}\Bigg)du.$$

Refinements (illustrating the theorem)



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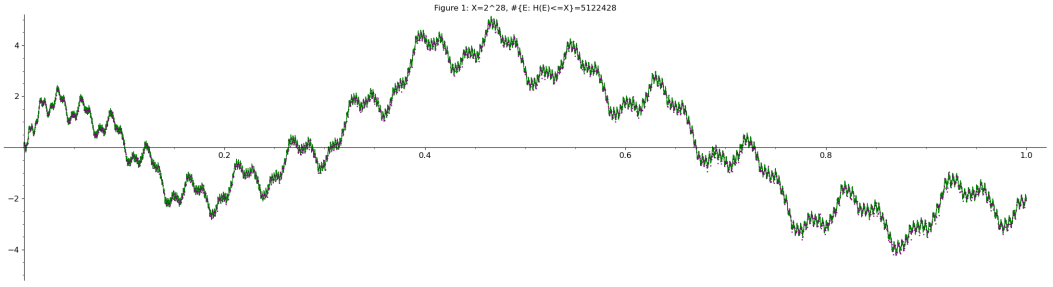

Ordering by (prime) conductor

Let $\mathcal{P}(X)$ be the set of isomorphism classes of $E/\mathbb{Q}$ of prime conductor $N(E) \leq X$.

Conjecture

For any smooth $W\colon \mathbb{R}_{>0} \to \mathbb{R}$ with compact support, the limit

$$\lim_{X\to\infty} \frac{1}{\#\mathcal{P}(X)} \sum_{E\in\mathcal{E}(X)} \frac{w(E)}{N_E} \sum_{p\geq 1} W(p/N_E) a_p(E)$$

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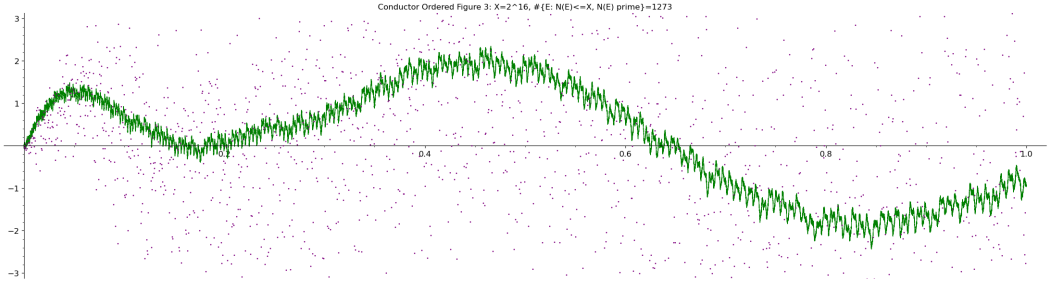
$$\int_0^\infty W(u)\sqrt{u}\left(2\pi \sum_{\substack{q\in\mathbb{N}\\ \text{squarefree}}} \sum_{m=1}^{\infty} \frac{\mu(\gcd(m,q))}{q\varphi\left(\frac{q}{\gcd(m,q)}\right)} \frac{\prod_{p\mid m} \ell'_{p^{2v_p(m)}}}{m} J_1\left(4\pi\frac{\sqrt{u}m}{q}\right)\right) du.$$

with $\ell'_{p^\nu} = -p^{-1}(1+t_p(k))$, where $t_p(k) = \mathrm{tr}(T_p)$ on $S_k(1)$.

Ordering by (prime) conductor (testing the conjecture)



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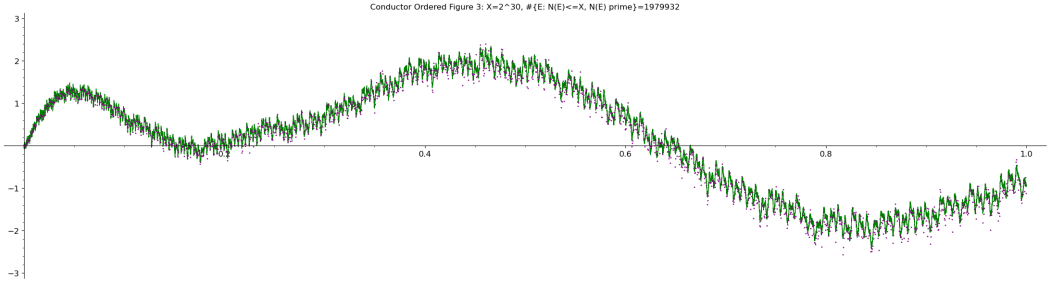
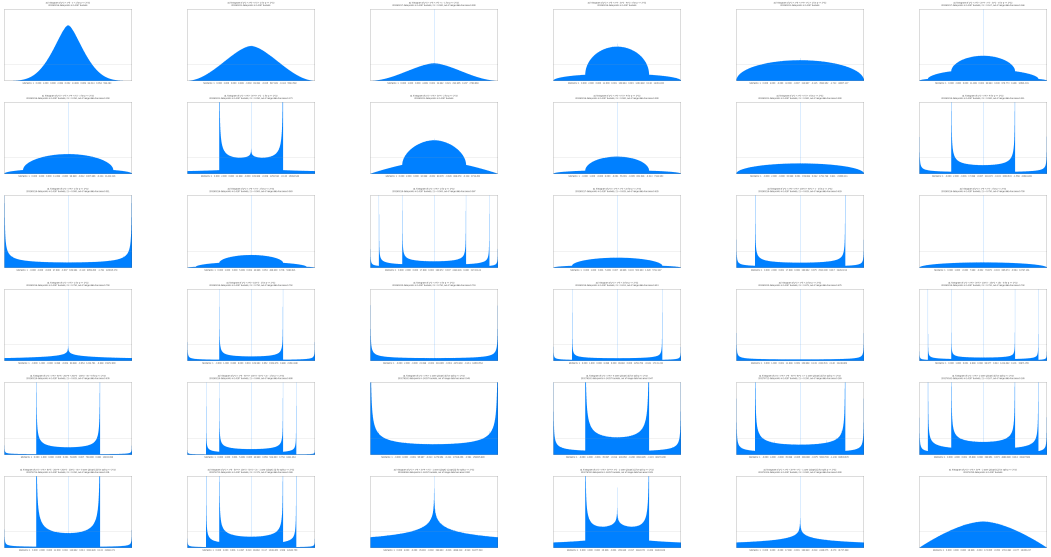
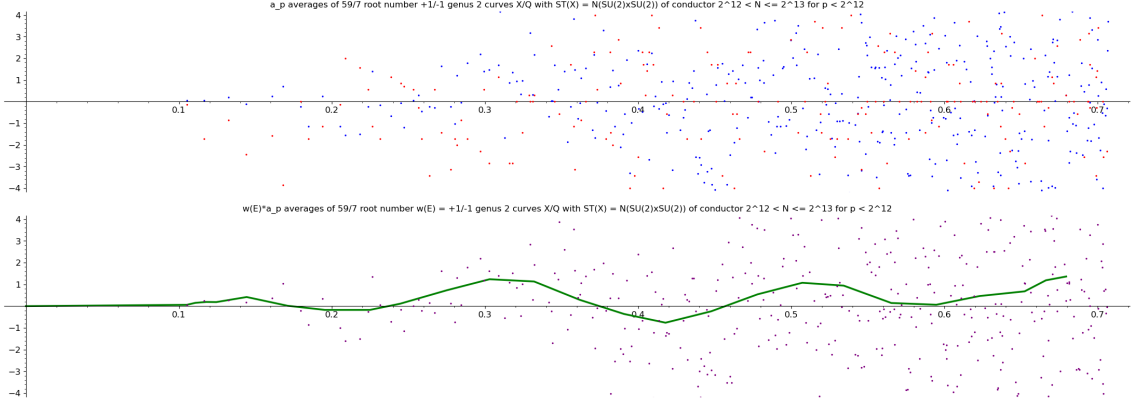


Trace distributions of genus 2 curves



L-functions of genus 2 curves over $\mathbb{Q}$, Sato-Tate group $N(\mathrm{SU}(2) \times \mathrm{SU}(2))$.

These are primitive L-functions arising from Hilbert or Bianchi modular forms.
Conductor of $L(X, s)$ in $(M, 2M]$ for $M = 2^{12}, \ldots, 2^{19}$ with x-axis range $[0, M/2]$.



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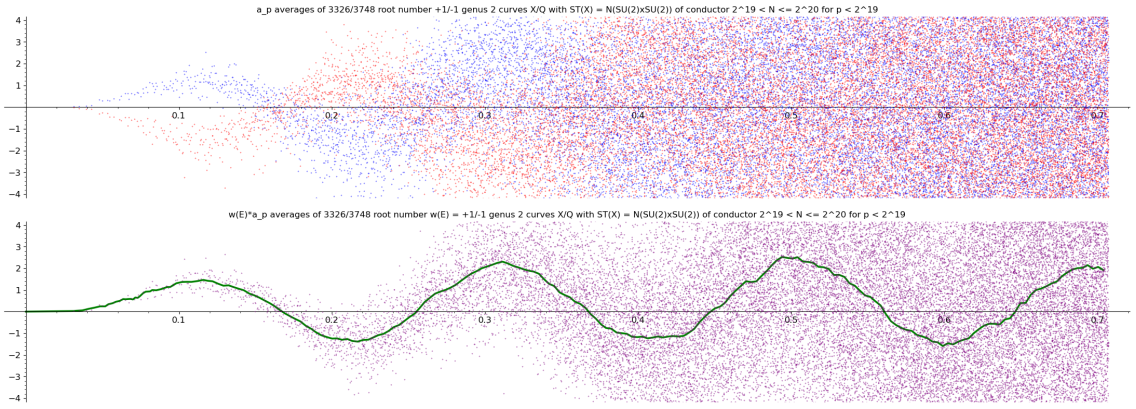

L-functions of products of $E/\mathbb{Q}$, Sato-Tate group $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

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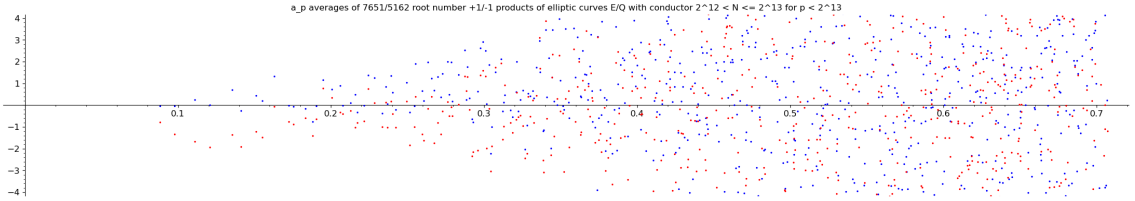

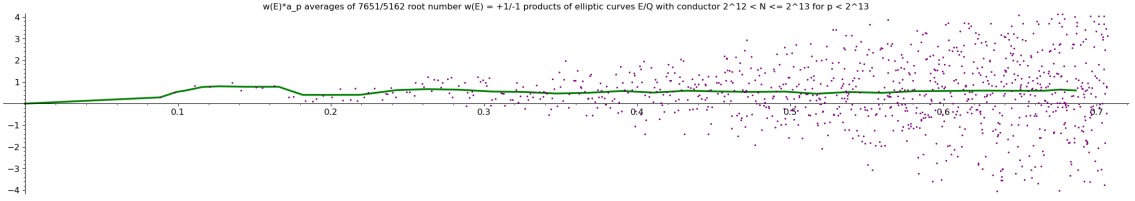

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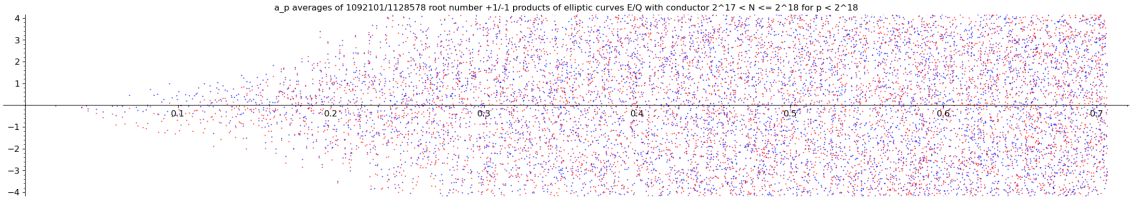
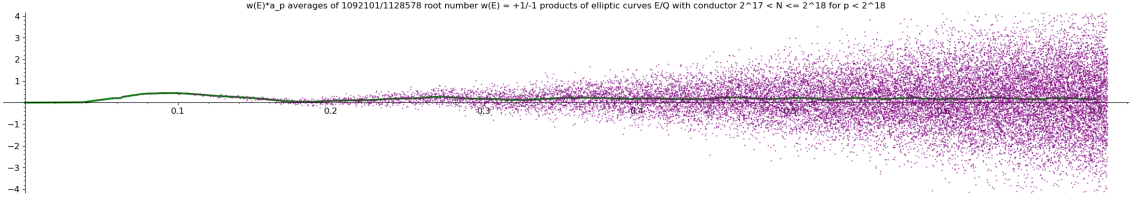
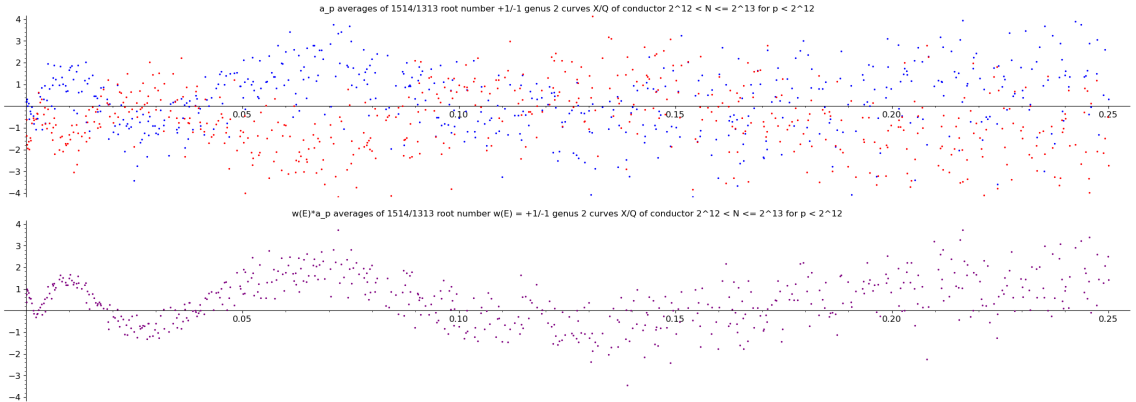




L-functions of genus 2 curves over $\mathbb{Q}$ with Sato-Tate group USp(4).

Recently constructed database of more than 6 million genus 2 curves $X/\mathbb{Q}$ of conductor at most 2^{20} includes about 1.7 million isogeny classes with Sato–Tate group USp(4). Conductor of $L(X,s)$ in $(M, 2M]$ for $M = 2^{12}, \ldots, 2^{19}$ with x-axis range $[0, M/2]$.



Coming soon to the LMFDB.

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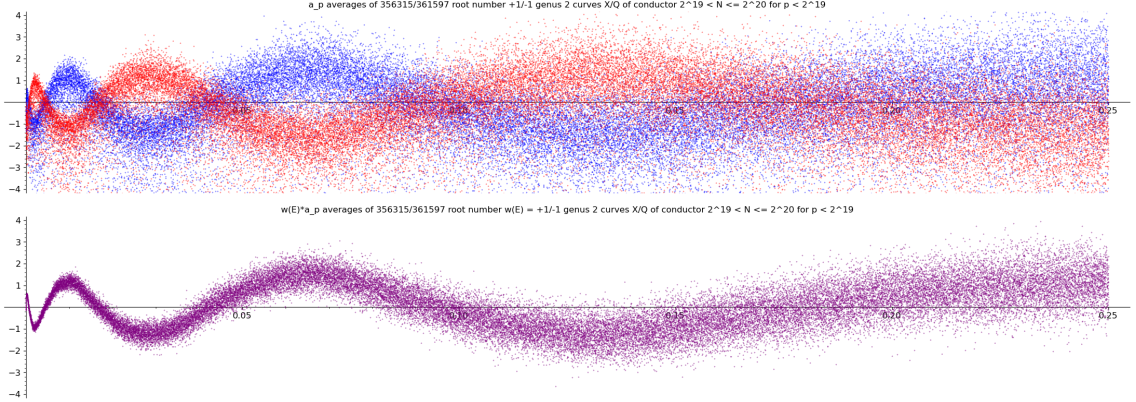

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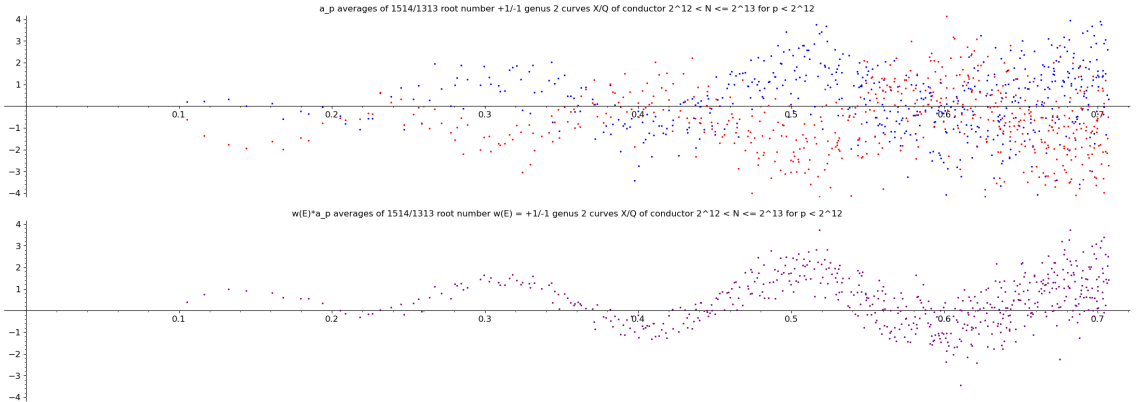
Recently constructed database of more than 6 million genus 2 curves $X/\mathbb{Q}$ of conductor at most 2^{20} includes about 1.7 million isogeny classes with Sato–Tate group USp(4). Conductor of $L(X,s)$ in $(M, 2M]$ for $M = 2^{12}, \ldots, 2^{19}$ with x-axis range $[0, M/2]$.



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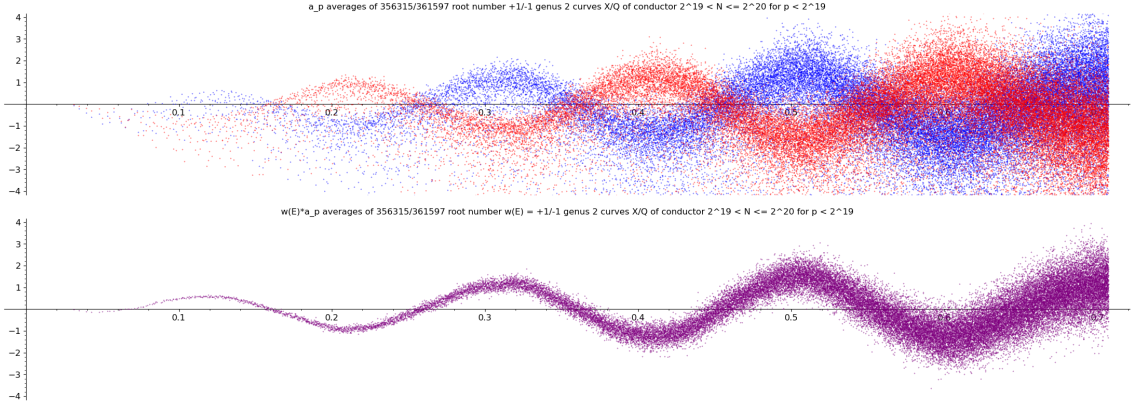
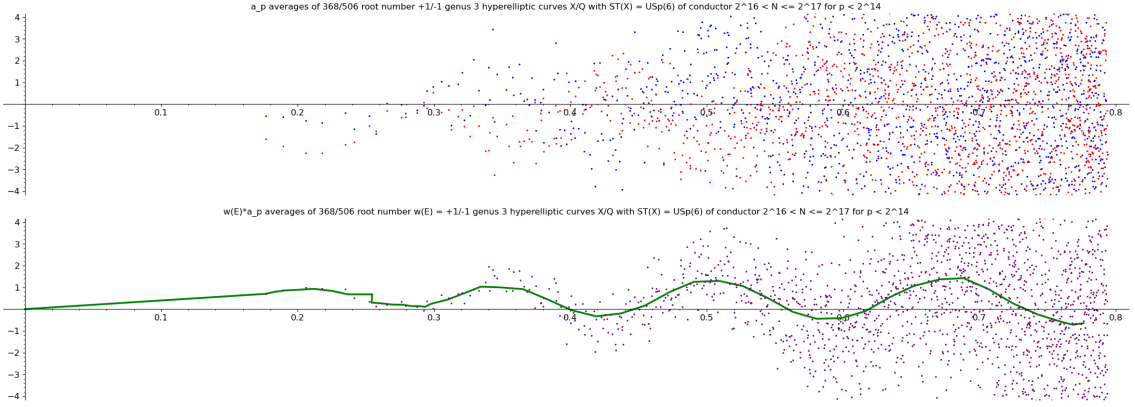


Coming soon to the LMFDB.

L-functions of genus 3 curves over $\mathbb{Q}$ with Sato-Tate group USp(6).

Recently constructed database of genus 3 curves $X/\mathbb{Q}$ of conductor at most 10^7 includes 59,214 isogeny classes of hyperelliptic curves with ST group USp(6). Conductor of $L(X,s)$ in $(M, 2M]$ for $M = 2^{16}, \ldots, 2^{22}$ with x-axis range $[0, M/2]$.



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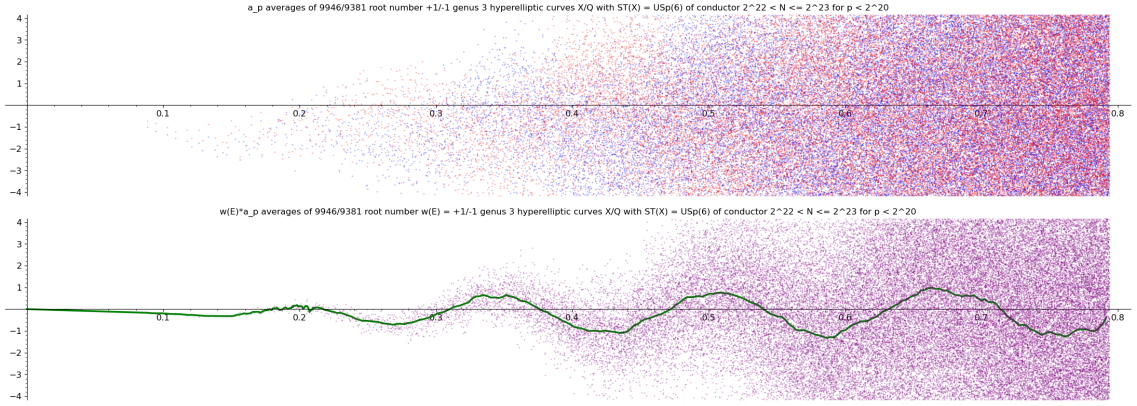

Thank you!

Animations available at https://math.mit.edu/~drew/murmurations.html.