Murmurations: A computational perspective

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Simons Collaboration in Arithmetic Geometry, Number Theory, and Computation

Elliptic curves and their L-functions

Let $E/\mathbb{Q}$ be an elliptic curve, say $E\colon y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$.
For primes $p \nmid \Delta(E) := -16(4A^3 + 27B^2)$ this equation defines an elliptic curve $E/\mathbb{F}_p$.
For all such primes p we have the trace of Frobenius $a_p(E) := p + 1 - \#E(\mathbb{F}_p) \in \mathbb{Z}$.

One can also define $a_p(E)$ for $p|\Delta(E)$, and then construct the L-function

$$L(E,s) := \prod_p (1 - a_p p^{-s} + \chi(p) p^{1-2s})^{-1} = \sum_{n\ge 1} a_n n^{-s}$$

where $\chi(p) = 0$ for $p|N(E)$ and $\chi(p) = 1$ otherwise and $N(E)|\Delta(E)$ is the conductor.

But in fact the a_p for $p \nmid \Delta(E)$ determine $L(E,s)$ (via strong multiplicity one), and also the conductor and root number $w(E) = \pm 1$, which appear in the functional equation

$$\Lambda(E,s) = w(E) N(E)^{1-s} \Lambda(E, 2-s)$$

where $\Lambda(s) := \Gamma_{\mathbb{C}}(s) L(E,s)$. The L-function $L(E,s)$ determines the isogeny class of E.

Arithmetic statistics of Frobenius traces of elliptic curves $E/\mathbb{Q}$

Three conjectures from the 1960s and 1970s (the first is now a theorem):

1. **Sato–Tate**: The sequence $x_p := a_p(E)/\sqrt{p}$ is equidistributed with respect to the pushforward of the Haar measure of of $\mathrm{ST}(E)$ $(= \mathrm{SU}(2)$ if E does not have CM).

2. **Birch and Swinnerton-Dyer**:

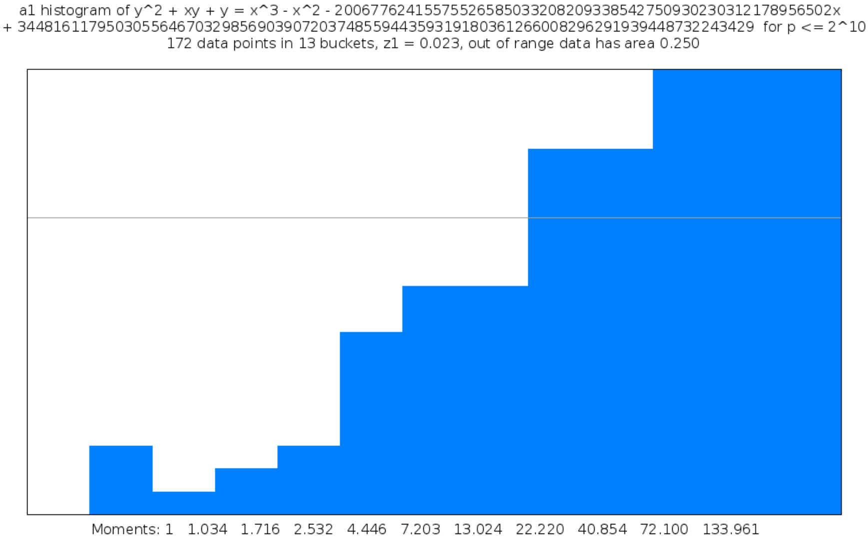
$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{p\le x} \frac{a_p(E)\log p}{p} = \frac{1}{2} - r,$$

3. **Lang–Trotter**: For every nonzero $t \in \mathbb{Z}$ there is a real number $C_{E,t}$ for which

$$\#\{p \le x : a_p(E) = t\} \sim C_{E,t} \frac{\sqrt{x}}{\log x}.$$

These conjectures depend only on $L(E,s)$ and generalize to other L-functions.

Example: Elkies' curve of rank ≥ 28 ($= 28$ under GRH).



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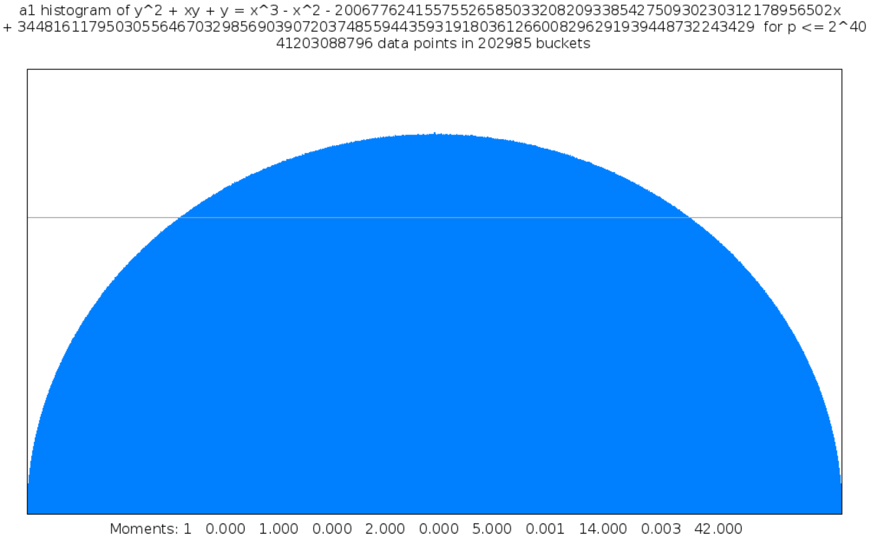

How rank effects trace distributions

An early form of the BSD conjecture implies that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{p\le x} \frac{a_p(E)\log p}{p} = \frac{1}{2} - r, \tag{1}$$

and sums of this form (Mestre-Nagao sums) are often used as a tool when searching for elliptic curves of large rank (which necessarily have large conductor N).[1] [2]

Theorem (Kim-Murty 2023)

If the limit on the LHS of (1) exists then it equals the RHS with r the analytic rank, and the L-function of E satisfies the Riemann hypothesis.

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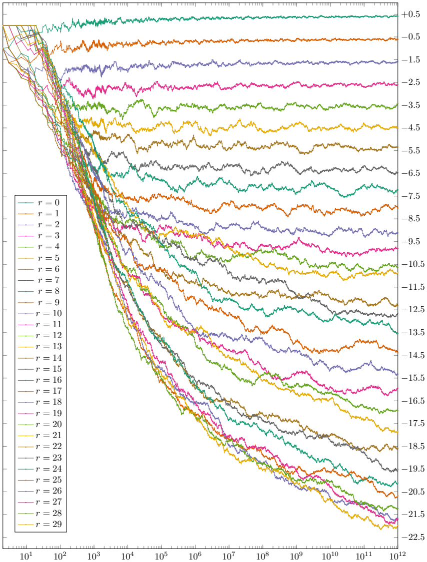
[1]See Sarnak's 2007 letter to Mazur.

[2]See Kazalicki-Vlah for some recent machine-learning work on this topic.



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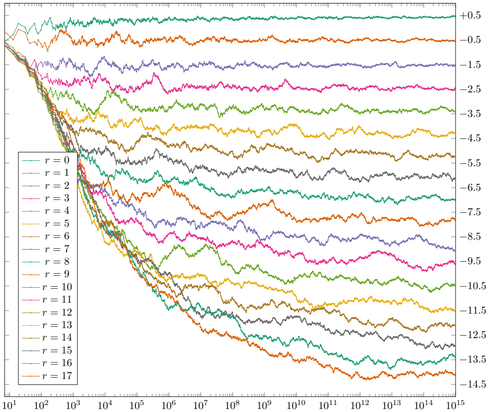

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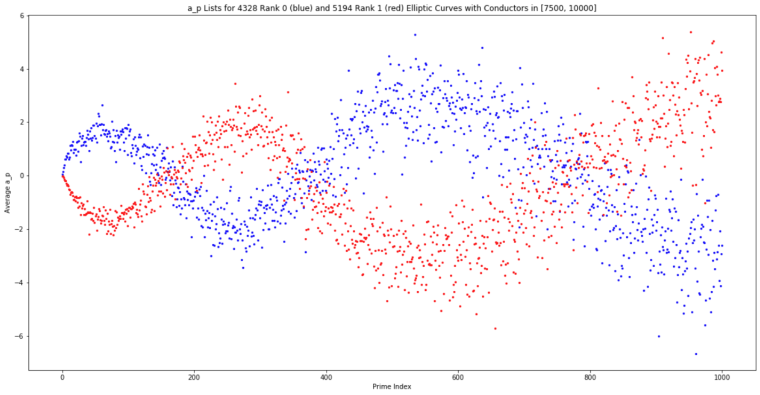
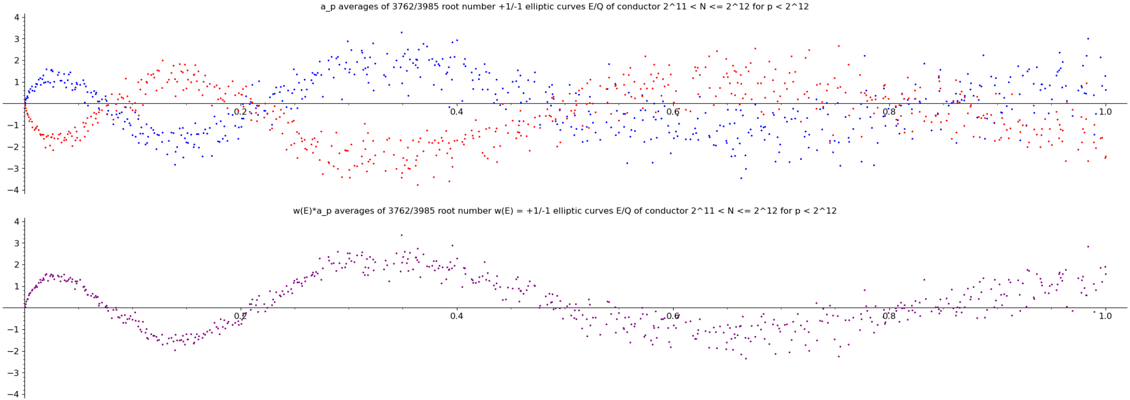
Murmurations of elliptic curves

In their 2022 preprint *Murmurations of elliptic curves*, Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver, and Alexey Pozdnyakov observed a curious fluctuation in average Frobenius traces of elliptic curves in a given conductor interval depending on the rank.



Murmurations of elliptic curves

Elliptic curve L-functions of conductor $N \in (M, 2M]$ for $M = 2^{11}, 2^{12}, \ldots, 2^{17}, 250000$. The x-axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or $m_p := w(E)a_p(E)$ over even/odd or all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$.



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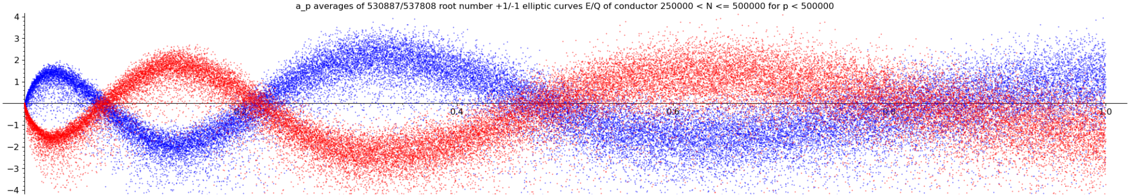

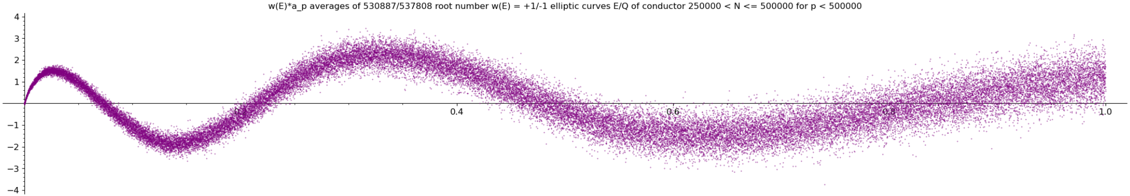

Murmurations of elliptic curves over a_n (not just a_p)

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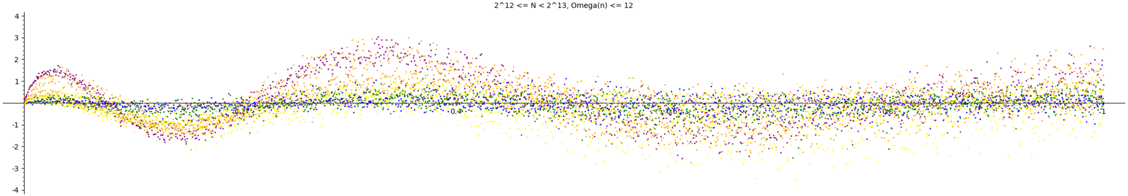
The color of the dot indicates the number of prime factors of n (with multiplicity).



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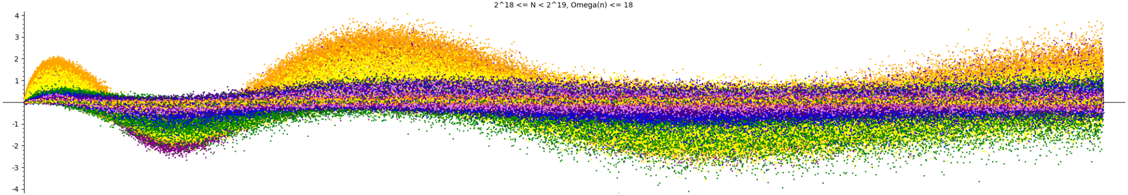
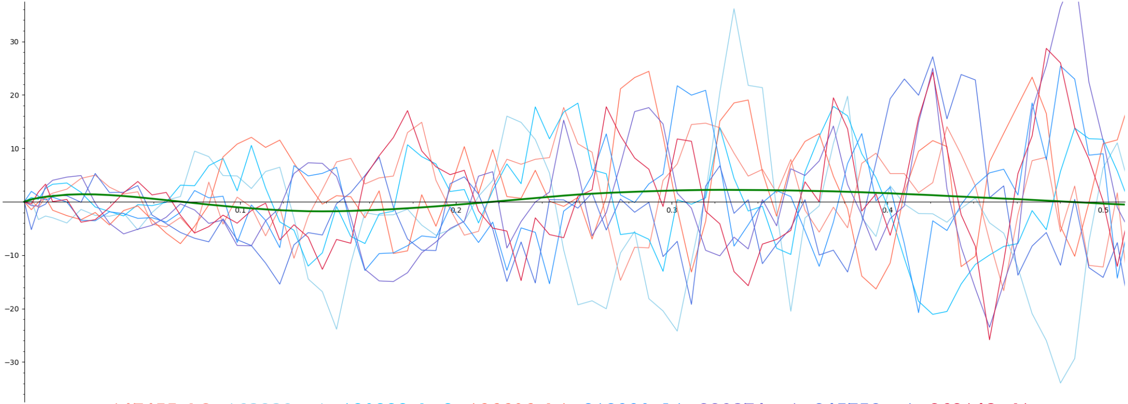


Murmurations are an aggregate phenomenon

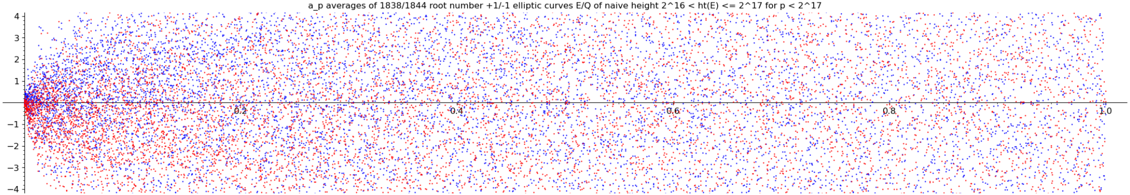
Moving average line plots of $\bar{m}_p$ for 8 individual and all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$, using subintervals of size $\sqrt{M}$ for $p \leq 2M$, with $M = 2^{17}$.

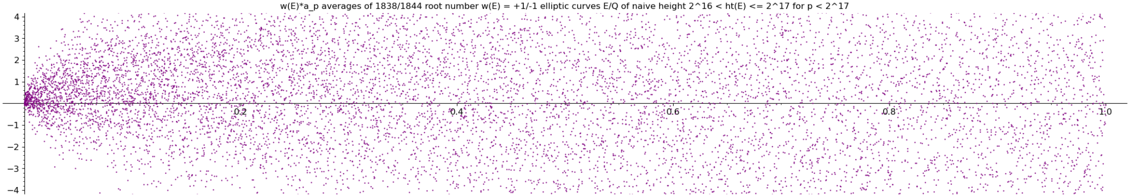


`147455.b2`, `163839.a1`, `180222.be2`, `196606.b1`, `212990.l1`, `229374.a1`, `245758.a1`, `262143.d1`

Murmurations depend critically on the conductor

Elliptic curves with $\mathrm{ht}(E) := \max(4|A|^3, 27B^2)$ in $(M, 2M]$ for $M = 2^{16}, \ldots, 2^{25}$. The x-axis range is $[0, 2M]$. A blue/red or purple dot at $(p, \bar{a}_p \text{ or } \bar{m}_p)$ shows the average of a_p or m_p over even/odd or all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$.





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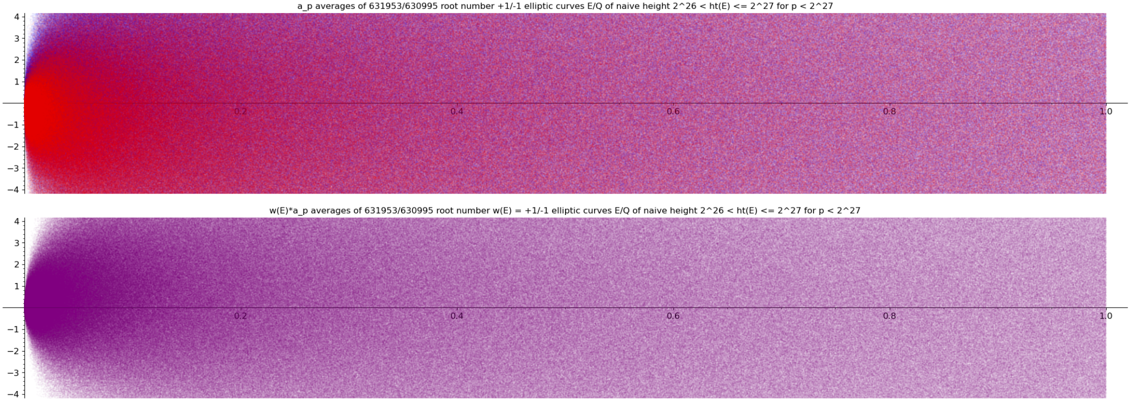

Murmurations scale

Elliptic curves in the SWDB of conductor $N \in (M, 2M]$ for $M = 2^{12}, \ldots, 2^{25}$.
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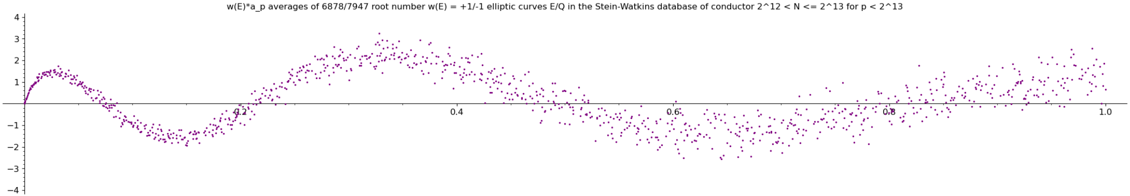

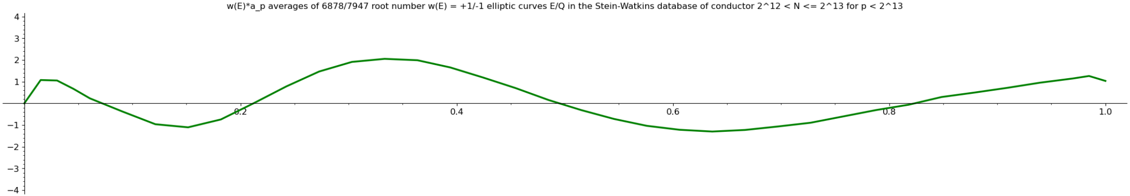

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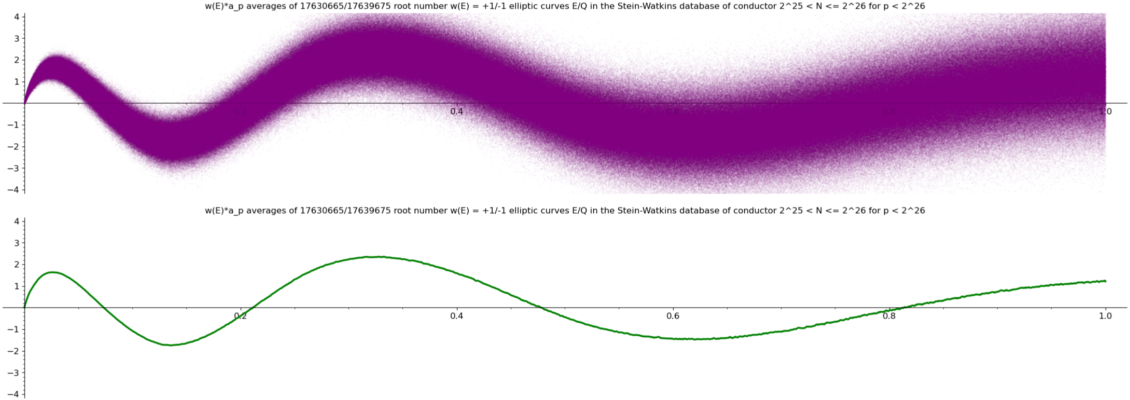


Arithmetic L-functions

We call an L-function is analytic if it has the properties every good L-function should: analytic continuation, functional equation, Euler product, temperedness, central character; see FPRS18; it is analytically normalized if its central value is at $s = 1/2$.

An analytically normalized L-function $L_{\rm an}(s) = \sum a_n n^{-s}$ is arithmetic if $a_n n^{\omega/2} \in \mathcal{O}_K$ for some number field K and $\omega \in \mathbb{Z}_{\geq 0}$. The least such ω is the motivic weight. Its arithmetic normalization $L(s) := L_{\rm an}(s + \omega/2)$ has coefficients in $\mathcal{O}_K$ and satisfies

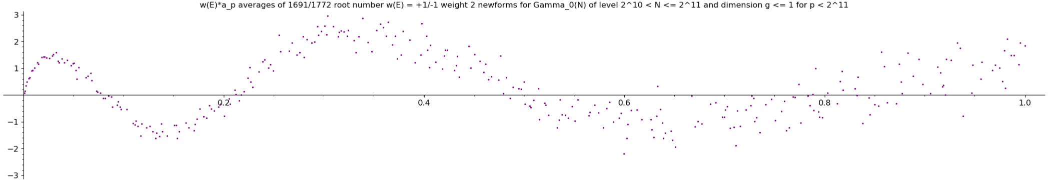
$$\Lambda(s) = N^{1-s} w \bar{\Lambda}(1 + \omega - s).$$

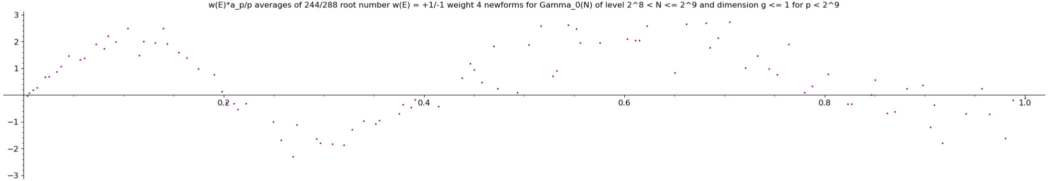
L-functions of abelian varieties have motivic weight $\omega = 1$.
L-functions of weight-k holomorphic cuspforms have motivic weight $\omega = k - 1$.

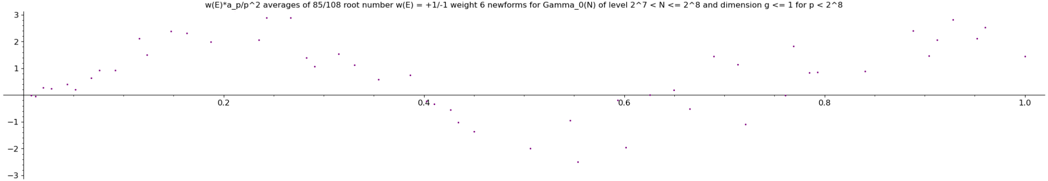
We consider Galois-closed families of self-dual arithmetically normalized L-functions. In any such family the values of a_p and m_p are integers and $w = \pm 1$.

When averaging a_p's in motivic weight $\omega > 1$ we normalize them via $a_p \mapsto a_p/p^{(\omega-1)/2}$. This ensures that we always have $|a_p| = O(\sqrt{p})$, as with elliptic curves.

Newforms for $\Gamma_0(N)$ of weight $k = 2, 4, 6$ with rational coefficients.







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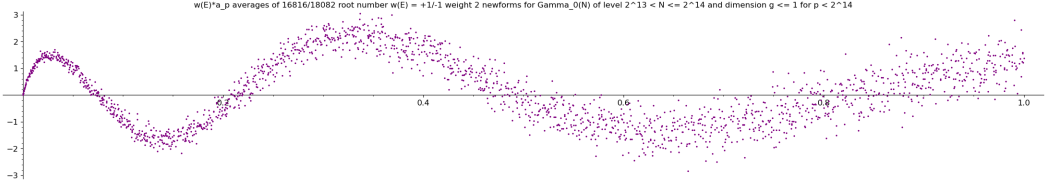

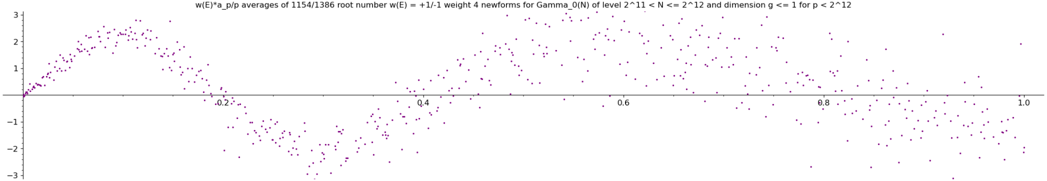

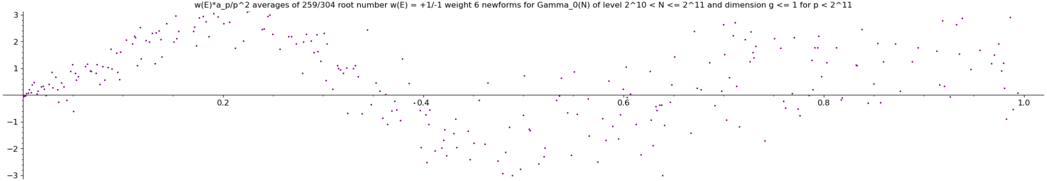

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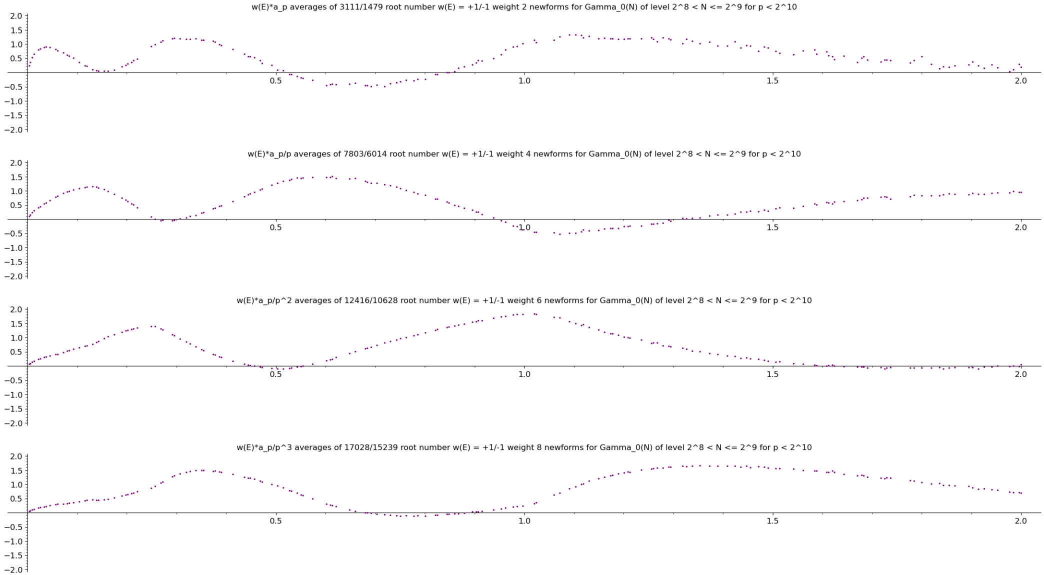

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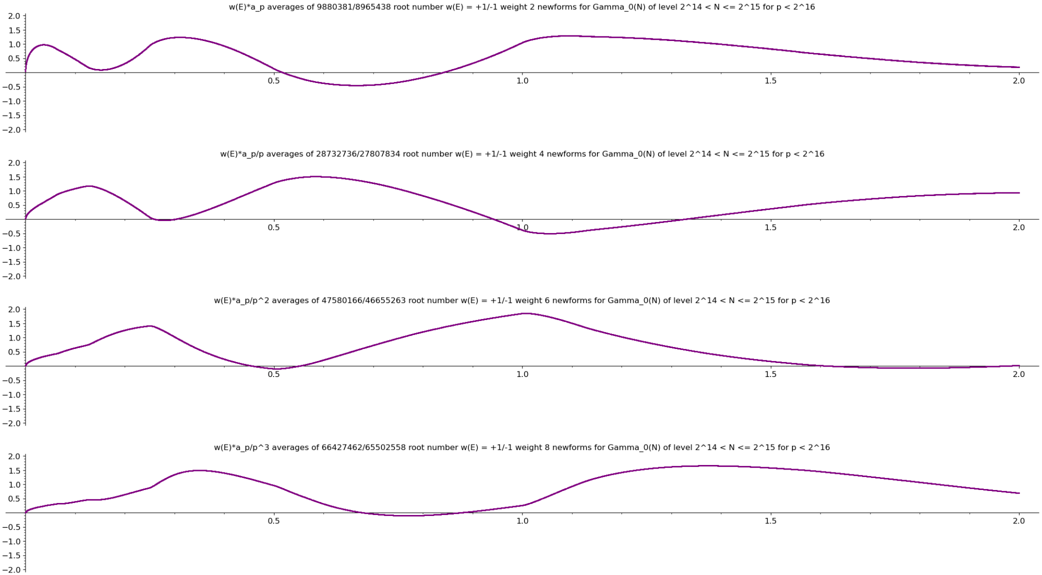


Zubrilina's theorem (click me!)

Definition. Let $U_n \in \mathbb{Z}[x]$ denote the Chebyshev polynomial defined by $U_n(\cos\vartheta)\sin\vartheta = \sin((n+1)\vartheta)$. The murmuration density function is

$$M_k(y) := D_k\Big(Ay \ - (-1)^{k/2} B \sum_{1\le r\le 2y} c(r)\sqrt{4y^2-r^2}\ U_{k-2}\big(\tfrac{r}{2y}\big) - \pi y^2 \delta_{k=2}\Big),$$

$A := \prod_p \big(1 + \frac{p}{(p+1)^2(p-1)}\big)$, $B := \prod_p \frac{p^4-2p^2-p+1}{(p^2-1)^2}$, $c(r) := \prod_{p|r}\big(1+\frac{p^2}{p^4-2p^2-p+1}\big)$, $D_k := \frac{12}{(k-1)\pi\prod_p(1-\frac{1}{p^2+p})}$.

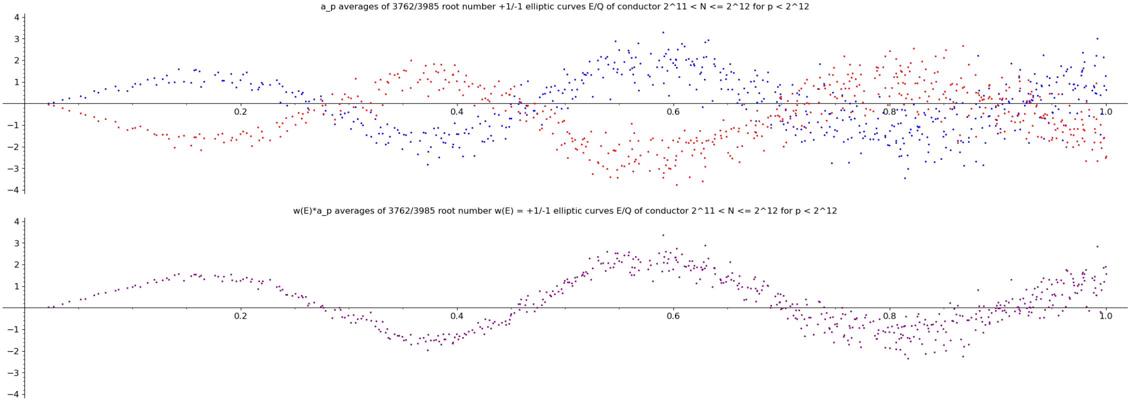
Theorem [Zubrilina 2023]. Let $\sum a_n(f)q^n$ denote a weight-k newform for $\Gamma_0(N)$ with root number $w(f)$. Let $X, Y, P \to \infty$ with P prime, $Y \sim X^{1-\delta}$, $P \ll X^{1+\delta_1}$, $\delta, \delta_1 > 0$ and $2\delta_1 < \delta < 1$, and put $y := \sqrt{P/X}$. Then for every $\varepsilon > 0$ we have

$$\frac{\sum_{N\in[X,X+Y]}^{\square\text{-free}} \sum_f w(f) a_P(f) P^{(1-k/2)}}{\sum_{N\in[X,X+Y]}^{\square\text{-free}} \sum_f 1} = M_k(y) + O_\varepsilon(X^{-\delta'+\varepsilon} + P^{-1})$$

where $\delta' := \max(\delta/2 - \delta_1, (\delta+1)/9 - \delta_1)$; for $\delta_1 < 2/9$ we can choose δ so $\delta' > 0$.

Murmurations of elliptic curves with squareroot normalization

Elliptic curve L-functions of conductor $N \in (M, 2M]$ for $M = 2^{11}, 2^{12}, \ldots, 2^{17}, 250000$. The x-axis range is $[0, 2M]$. A blue/red or purple dot at $(\sqrt{p}, \bar{a}_p$ or $\bar{m}_p)$ shows the average of a_p or $m_p := w(E)a_p(E)$ over even/odd or all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$.



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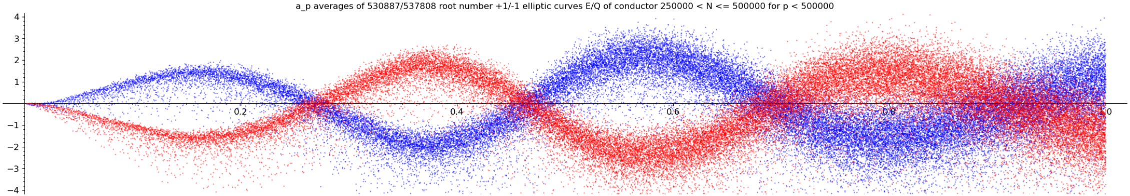
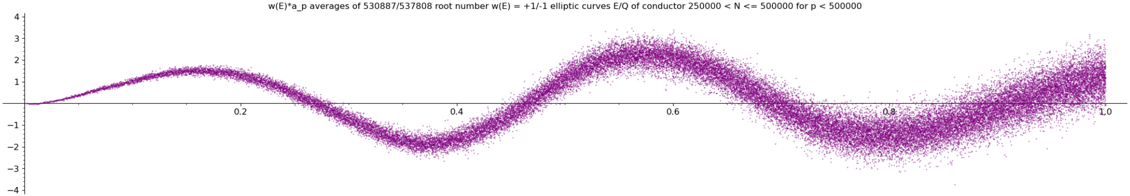
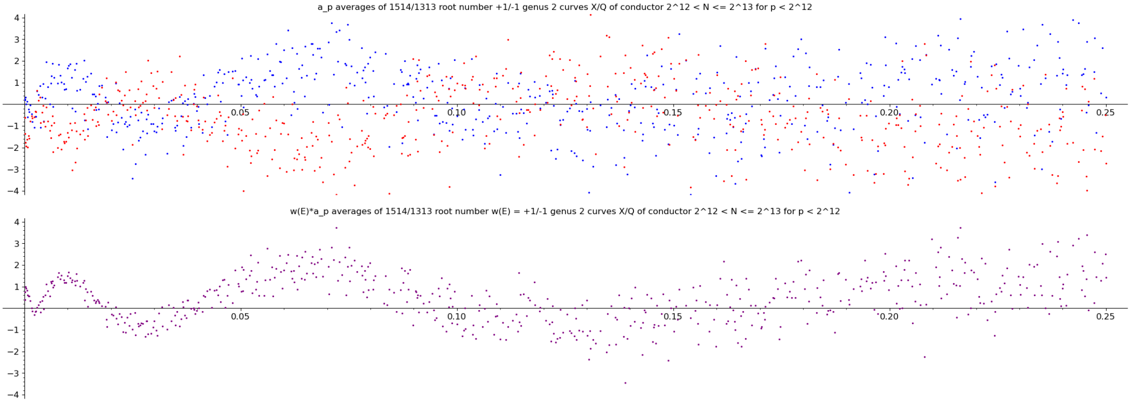




L-functions of genus 2 curves over $\mathbb{Q}$ with Sato-Tate group USp(4).

Recently constructed database of more than 5 million genus 2 curves $X/\mathbb{Q}$ of conductor at most 2^{20} includes 1,440,894 isogeny classes with Sato–Tate group USp(4).
Conductor of $L(X,s)$ in $(M,2M]$ for $M = 2^{12},\ldots,2^{19}$ with x-axis range $[0, M/2]$.



Coming soon to the LMFDB.

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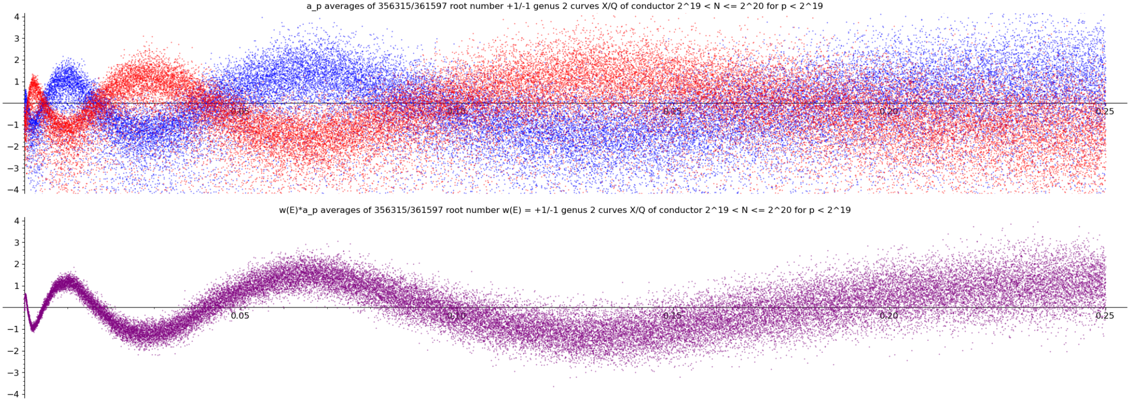

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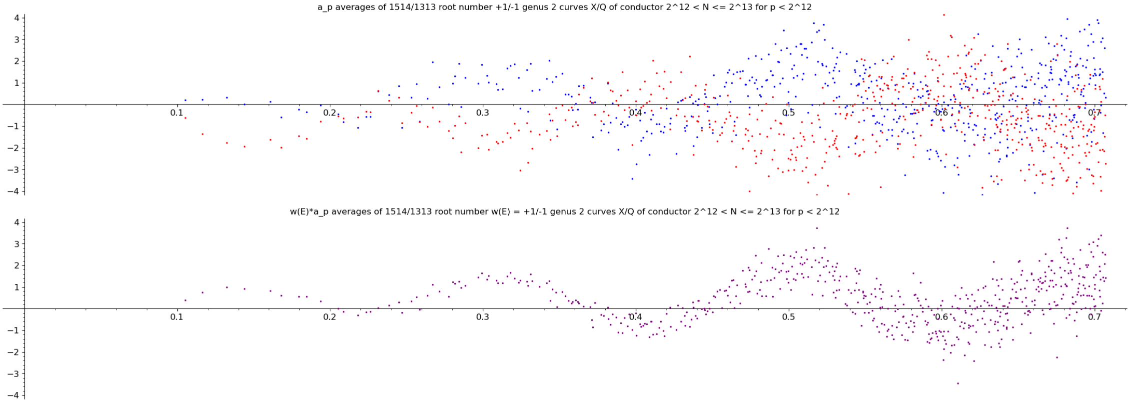

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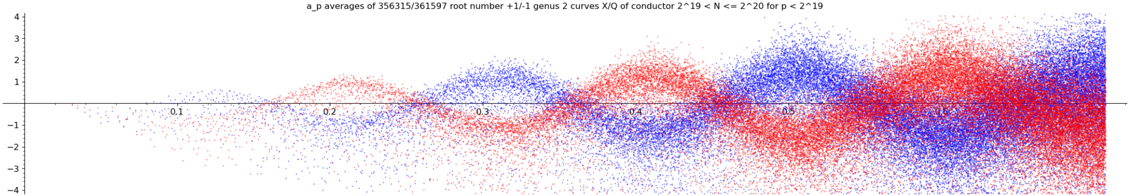

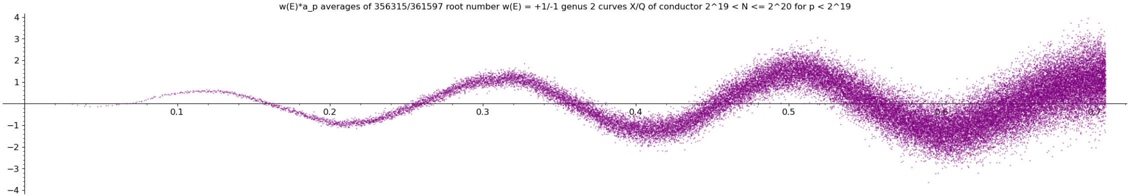

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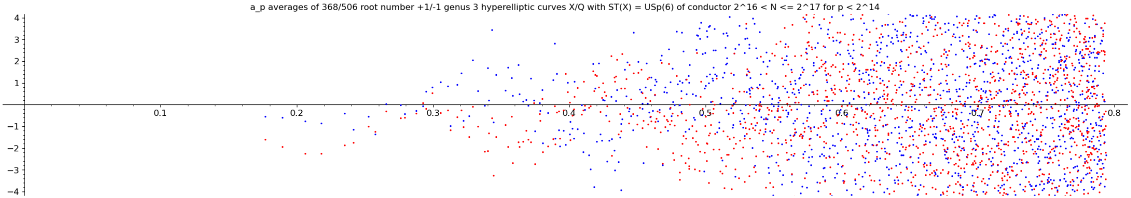
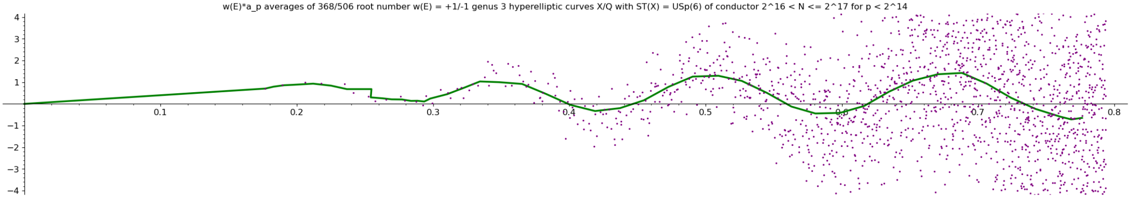
L-functions of genus 3 curves over $\mathbb{Q}$ with Sato-Tate group USp(6).

Recently constructed database of genus 3 curves $X/\mathbb{Q}$ of conductor at most 10^7 includes 59,214 isogeny classes of hyperelliptic curves with ST group USp(6). Conductor of $L(X,s)$ in $(M, 2M]$ for $M = 2^{16}, \ldots, 2^{22}$ with x-axis range $[0, M/2]$.





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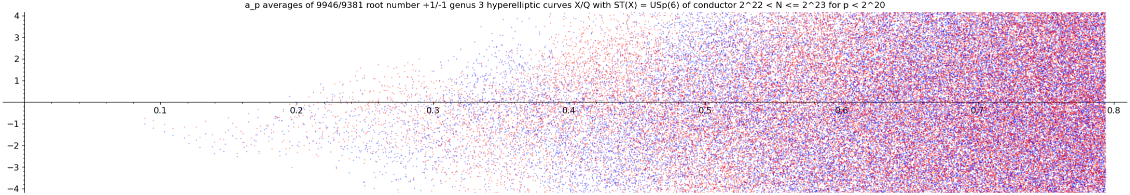

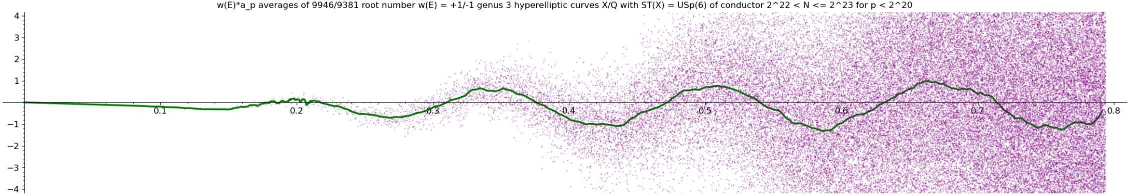

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Computing trace averages of many $E/\mathbb{Q}$

When computing $a_p(E)$ for many elliptic curves $E/\mathbb{Q}$ we construct a lookup table $T[j] = a_p(E)$ for $E\colon y^2 = x^3 + Ax + B$ with $j(E) = j \neq 0, 1728$ and $B = \square$.

- Naive: $O(p)$ per curve.
- Mestre BSGS: $O(p^{1/4} \log p)$ per curve.
- Schoof: $O(\log^5 p)$ per curve.
- SEA: $O(\log^4 p)$ per curve.
- CM torsor (isogenies): $O(\log^3 p)$ per curve (GRH).
- CM torsor (isogenies): $O(\log^2 p)$ per curve (heuristic).
- CM torsor (GCDs): $O(\log p)$ per curve (heuristic).

Complexity estimates ignore $\log\log p$ factors.

Realizing the CM torsor via isogenies

Having computed $a_p(E)$ for one $E/\mathbb{F}_p$, we can (typically) easily compute $\mathrm{End}(E) = \mathcal{O}$, since $\operatorname{disc}\mathcal{O}$ divides $a_p(E)^2 - 4p$. Let $\mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_p) := \{j(E) : E/\mathbb{F}_p \text{ has } \mathrm{End}(E) = \mathcal{O}\}$.

$\mathrm{Gal}(K_{\mathcal{O}}/K) \simeq \mathrm{cl}(\mathcal{O})$ acts on $\mathrm{Ell}(\mathcal{O})$ via (horizontal) isogenies.
If $[\mathfrak{l}] \in \mathrm{cl}(\mathcal{O})$ has norm ℓ and $j_1 \in \mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_p)$ then

$$\Phi_\ell(j_1, [\mathfrak{l}]j_1) = 0,$$

where $\Phi_\ell(X, Y)$ is the classical modular polynomial. Typically $[\mathfrak{l}]j_1$ and $[\bar{\mathfrak{l}}]j_1$ are the only roots of $\Phi_\ell(j_1, X)$ in $\mathbb{F}_p$, and we can choose them to ensure this.

A polycyclic presentation for $\mathrm{cl}(\mathcal{O})$ is a sequence of ideals $\mathfrak{l}_1, \ldots, \mathfrak{l}_k$ such that every $[\mathfrak{a}] \in \mathrm{cl}(\mathcal{O})$ may be written uniquely as

$$[\mathfrak{a}] = [\mathfrak{l}_1^{e_1}] \cdots [\mathfrak{l}_k^{e_k}] \qquad (0 \le e_i < r_i),$$

where $r_i = \min\{r \colon [\mathfrak{l}_i^r] \in \langle [\mathfrak{l}_1], \ldots, [\mathfrak{l}_{i-1}] \rangle\}$ is $[\mathfrak{l}_i]$'s relative order.

Using GCDs

We can replace most root-finding steps with GCDs.

Suppose we have computed a cycle of ℓ-isogenies. After computing a single ℓ'-isogeny, we can compute the next cycle of ℓ-isogenies using GCDs.

$$\begin{tikzcd} j_1 \arrow[r, "\mathfrak{l}"] \arrow[d, "\mathfrak{l}'"'] & j_2 \arrow[r, "\mathfrak{l}"] \arrow[d, dashed, "\mathfrak{l}'"'] & j_3 \arrow[r, "\mathfrak{l}"] & \cdots \arrow[r, "\mathfrak{l}"] & j_r \\ j_1' \arrow[r, dashed, "\mathfrak{l}"] & j_2' & & & \end{tikzcd}$$

Provided $4\ell^2\ell'^2 < |D|$, the monic polynomial

$$\varphi(X) = \gcd(\Phi_\ell(j_1', X), \Phi_{\ell'}(j_2, X)) \in \mathbb{F}_p[X]$$

will have degree 1 and we can compute $j_2' = -\varphi(0)$ as its unique root.

Even when our polycyclic presentation with one generator, we can choose an auxiliary prime ℓ' so $[\mathfrak{l}'] = [\mathfrak{l}]^n$ and use GCDs to a single line of j-invariants after n steps.

Computing murmurations using the trace formula

The sum of $w(f)a_p(f)$ over $f \in S_k^{\mathrm{new}}(N)$ is equal to the trace of $T_n \circ W$ acting on $S_k^{\mathrm{new}}(N)$, where the Fricke involution W is defined by $W(f) := f \mid \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$.

By massaging a theorem of Popa, one finds that

$$\mathrm{tr}(T_n \circ W, S_k(N)) = -\frac{1}{2} \sum_{\substack{t^2 N < 4n \\ D := t^2N^2 - 4nN}} g_k(t^2N, n) h^*(D, N) - \frac{1}{2} s_k(N, n) + \delta_{k=2} \sigma_N(n) - \delta_{\substack{N=1 \\ n=\square}} \frac{k-1}{12} n^{k/2-1},$$

$$h^*(D, N) := \sum_{\substack{u|N \\ u^2|D}} \mu(u) H'(\tfrac{D}{u^2}), \quad s_k(N, n) := \frac{\varphi(N)}{N^{k/2}} \sum_{\substack{uv=Nn \\ N|(u+v)}} \min(u, v)^{k-1}, \quad \sigma_N(n) := \sum_{\substack{m|n \\ m \perp N}} \frac{n}{m},$$

where $g_k := g_k(b, c)$ is defined by $g_2 := 1,\ g_4 := b - c,\ g_{k+4} := (b - 2c)g_{k+2} - c^2 g_k$.

Assaf's recent paper gives formulas for $\mathrm{tr}(T_n \circ W, S_k^{\mathrm{new}}(N))$ via $tr(T_n \circ W, S_k(N))$.

Key point: For $n = O(N)$ the sum contains $O(1)$ terms!

Computing murmurations using the trace formula

We compute $h^*(D,N)$ as the product of a multiplicative function and a class number

$$h^*(D,N) = \sum_{u|N, u^2|D} \mu(u) H'(\tfrac{D}{u^2}) = \sum_{u|\frac{c}{w}} \mu(u) H'(\tfrac{D}{u^2}) = \varphi_1^{D/w^2}(w) h'(\tfrac{D}{w^2}),$$

where $\varphi_1^D(n)$ is the multiplicative function defined on prime powers as

$$\varphi_1^D(p^e) = 1 + \frac{p^e - 1}{p-1}\Big(p - (\tfrac{D}{p})\Big).$$

The class numbers for $|D| \leq 2^{40}$ have been computed by Jacobson and Mosunov and can be downloaded from the LMFDB, and can be crammed into a 1.125TB lookup table. Using a memory mapped file on fast SSD it takes 40s to load.

It then takes less than a minute to compute $\mathrm{tr}(T_p \circ W, S_k^{\mathrm{new}}(N))$ for $2^{18} \leq N < 2^{19}$ and $p \leq 2^{19}$ for any reasonably small k (on 256 cores).

Computing murmurations of genus 2 and genus 3 curves

The average polynomial time algorithms described in [Harvey-S 2016] and [Costa-Harvey-S 2022] can readily compute the desired trace sums.

The main challenge is finding curves (and abelian varieties) of small conductor.

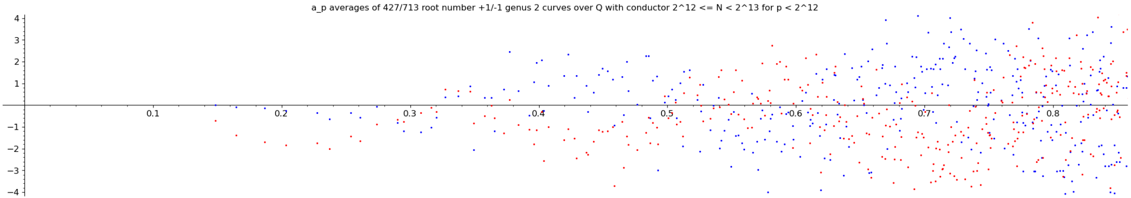
The algorithms described in [BSSVY 2016] and [S 2018] enumerate curves by discriminant, but curves with very large discriminants can have very small conductors.

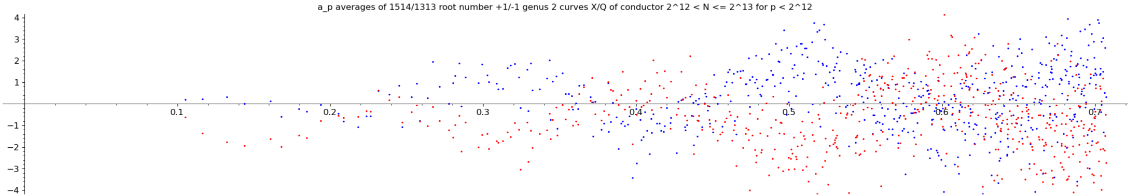
This is already an issue in genus 1 with the Stein-Watkins database: it misses about $1/4$ of the isogeny classes of conductor up to $5 \cdot 10^5$, despite ranging up to 10^8, but the situation is much worse in higher genus.

Curves may have bad reduction at primes of good reduction for the Jacobian (this happens a lot!). The genus 2 murmurations here use a new dataset of some 5 million curves with conductor below 10^6 (98% of these are not in the LMFDB yet).

L-functions of genus 2 curves over $\mathbb{Q}$ with Sato-Tate group USp(4).

Before and after genus 2 murmuration plots (top LMFDB, bottom new dataset).





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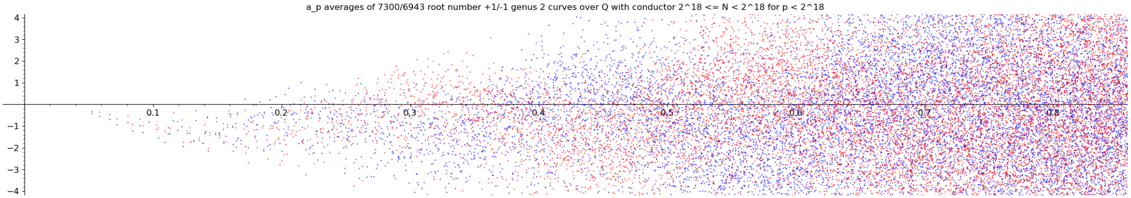
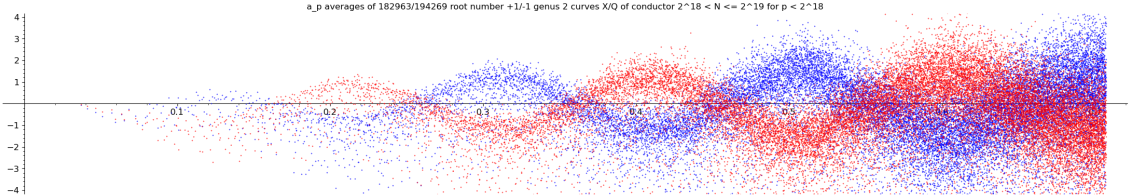
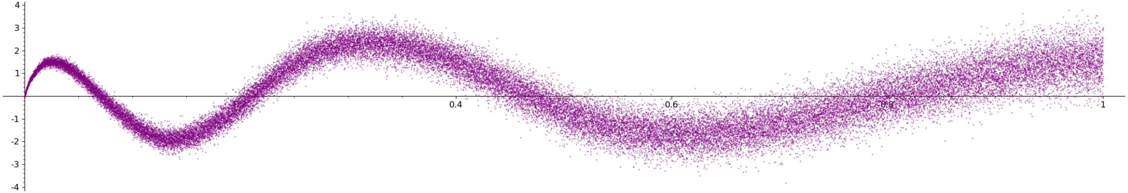




Thank you!



Animations available at https://math.mit.edu/~drew/murmurations.html.