Minimal model program for semi-stable threefolds in mixed characteristic.

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§1.

MMP

... higher dim^{l} analogue of classification of surfaces.

Castelnuovo : X : sm. proj. surf / $\mathbb{C}$.

then $\exists$ sequence

$$X = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{r-1}} X_r$$

: composition of contractions of (−1) - curves.

X_r : "simplest birat^l model" of X ($\nexists$ (−1) curve)

MMP

input : X : proj var / k : fld. w/ "mild" singularity.

(−1)-curve contr. φ_i (e.g. K_X : $\mathbb{Q}$-Cart.)

⇝ generalized to

K_{X_i}-neg. ext. ray contraction $\psi_i : X_i \to Z_i$ (proper birat^l)

ψ_i : ∃ 2 cases. { (a) divisorial contr. (contracting a divisor)

(b) flipping contr. (o/w)

(a) ⇝ $X_{i+1} := Z_i$. $\varphi_i := \psi_i$.

(b) ⇝ singularity of Z_i : very bad.

need to construct a flip

<u>Def</u>

$$\begin{array}{ccc} X_i & & X_i^+ \\ & {\scriptstyle \psi_i}\searrow \quad \swarrow{\scriptstyle \psi_i^+} & \\ & Z_i & \end{array}$$

mor. of norm. vars.

suppose: (ψ_i : isom in codim 1 ($\underset{df}{\Leftarrow}$ small)

K_{X_i} : $\mathbb{Q}$-Cartier.

$-K_{X_i}$: ψ_i-ample.

ψ_i^+ : proper birat^l mor is a flip of ψ_i

$\underset{df}{\Longleftrightarrow}$ ψ_i^+ : small. $K_{X_i^+}$: $\mathbb{Q}$-Cartier.

$K_{X_i^+}$: ψ_i^+-ample.

Rem If flip exists,

$$X_i^+ = \mathrm{Proj}_{Z_i} \bigoplus_m \psi_{i*}\mathcal{O}_{X_i}(mK_{X_i})$$

∴ existence of flip
⟺ finite generation of ~~~

⤳ $X_{i+1} := X_i^+$, $\varphi_i := \psi_i^{+\,-1} \circ \psi_i$ "flip"

Goal of MMP

find a sequence

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots\cdots \dashrightarrow X_r$$

: comp. of div. contr. & flip.

s.t. K_{X_r} : nef
or $\exists\, X_r \to Z$: "Mori fiber space"

((non minimal $\times \mathbb{P}^1$ ∈ [illegible] --))

MMP: very strong tool in alg. geom.!

Rem
- $\exists$ log version of MMP.

  input $\rightsquigarrow (X, \Delta)$

  (var. w/ eff. $\mathbb{Q}$-div)

  $K_X \rightsquigarrow K_X + \Delta$.

  : important for induction argument on dimension.
- $\exists$ relative version of MMP.

  V : scheme.

  input $\rightsquigarrow X$: proj / V.

  $K_X \rightsquigarrow K_{X/V}$.
- In particular. we can consider MMP in mixed char.

… Today's topic.

§2

Setting

V: spec. of Dedekind domain of mixed char.

$\pi: X \to V$ proj. X: integral.

$\rightsquigarrow$ can define relative canonical div.

$K_X := K_{X/V}$. (weil)

Consider MMP/ V ⁉

Known results

can run MMP/V if

- $\dim X = 2$ & X has "mild" sing. (Tanaka)
- $\dim X = 3$ & X: str. semist. ($V = \operatorname{Spec} \mathbb{Z}_p \rightsquigarrow X$: loc. ét over $\mathbb{Z}_p[x_1 \cdots x_3]/(x_1 \cdots x_i - p)$)

  where $\forall$ res char of $V \neq 2, 3$

  (Kawamata)
- 〃 & 〃

  where V: cf. char. of any p

  (Hacon-Witaszek)

Thm (T-Yoshikawa)

Can run a MMP / V

if $\dim X = 3$ &

X: str semist / V.

(any char)

Rmk • Recent paper by

Bhatt-Ma-Patakfalvi-Schwede-Tucker-Waldron-Witaszek

: can run a MMP / V

in more gen^{l} setting

if $^{\text{v}}$char of $V \neq 2, 3, 5$.

Thm has applications to studies of reduction of varieties

$V = \operatorname{spec} K$ K: hens dvf w/ perf residue.

- (Matsumoto, Liedtke–Matsumoto)

X: K3 surf. $/K$ w/ pot. semist. red.

If $G_K \curvearrowright H^2_\ell(X_{\bar{K}})$: unram. then X admits good red. after a fin unr. ext. of K.

- (Chiarellotto–Lazda)

X: ab surf $/K$ admits str. semist. minimal (alg. sp) model after a fin ext of K

§3 Sketch of Pf

MMP consists of 3 theorems.

(1) existence of K_X-neg. ext. ray cont.

(2) — " — of flips.

(3) termination of flips.

(1) ⟸ Kawamata's argument

(3) ⟸ classical method.

Today, we sketch (2) part.

(2)

(⟸) (2)' existence of special kind of flip
(Fujino) ("pl-flip w/ ample div. in boundary".)

($\Leftarrow$) Tanaka's result for surfaces

(Shokurov's reduction to pl-flip.) + "lifting of sections"

$\circledast$

Bhattの結果がでる。か

Toy model of $\circledast$

$X \to Z$: proj / V, Z : affine.

normal CM, int.

$S \subseteq X$: prime Cartier div.

L : $\mathbb{Q}$-Cartier div.

$A := L - (K_X + S)$: ample / Z

$\leadsto H^0(\mathcal{O}_X(L)) \to H^0(\mathcal{O}_S(L_S))$: surj ??

cf.

equal char case (Hacon-Witaszek)

want : $H^1(\mathcal{O}_X(L-S))$
(ideally) $= H^1(\omega_X(A)) = 0$

If X: "klt" of char 0,
this holds by Kodaira vanish.
char $p>0$ case ... $\nexists$ such vanish.
Instead, we use

① : $H^1(\omega_X(p^e A)) = 0$ for $e \gg 0$.

$$\omega_X(p^e A) \cong F^{e*}A \qquad F: \text{abs. Frob}$$

$\exists$ comm diagram

$$\begin{array}{ccccccccc} 0 \to & F_*^e\omega_X(p^e A) & \to & F_*^e\omega_X(p^e A + S) & \to & F_*^e\omega_S(p^e A_S) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \omega_X(A) & \to & \omega_X(A+S) & \to & \omega_S(A_S) & \to 0 \end{array}$$

(vertical ... induced by the trace map $F_*^e\omega_X \to \omega_X$)

$$
\begin{array}{ccccc}
\to H^0(\omega_X(p^eA+S)) & \to & H^0(\omega_S(p^eA_S)) & \to & H^1(\omega_X(p^eA)) \overset{①}{=} 0 \\
\downarrow & & \downarrow (*) & & \downarrow \\
H^0(\omega_X(A+S)) & \to & H^0(\omega_S(A_S)) & \to & H^1(\omega_X(A)) \\
\| & & \| & & \\
H^0(\mathcal{O}_X(L)) & & H^0(\mathcal{O}_S(L_S)) & &
\end{array}
$$

$\therefore$ suffices to show $(*)$: surj.

... holds true if S: globally F-regular ②

H-W proves ② for certain surfaces to (2′)

mixed case

Instead of ① & ②, we use

Fact ①′ (Bhatt)

X: C.M. A: ample Cartier,

then $\exists f: Y \to X$ alteration

s.t. (gen. fin. proj surj)

$$H^i(\omega_Y(f^*A)) \to H^i(\omega_X(A)) : 0\text{-map}.$$

for $(\forall i > 0)$

Def ②′

X: normal int. G-proj / V

Δ: eff. $\mathbb{Q}$-Weil. $K_X + \Delta$: $\mathbb{Q}$-Cart.

(X, Δ): globally T-regular

$\underset{\text{def}}{\iff}$ $^{\forall} f: Y \to X$: alt. w/ Y: normal.

$$H^0(\omega_Y(\lceil -f^*(K_X + \Delta) \rceil)) \to H^0(\mathcal{O}_X)$$

: surj. (trace map)

(2') reduced to ②' for certain surfaces.

Key: inversion of adjunction for ②'

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