Talks / Exposés

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Exceptional jumps of Picard rank of K3 surfaces number fields

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1 § Let X be a K3 surface over a number field K:

$(H^1(X, \mathcal{O}_X) = 0, \ \Omega^2_X \simeq \mathcal{O}_X)$ $X \hookrightarrow \mathbb{P}^3$ deg 4.

$\mathcal{X} \xrightarrow{\pi} S \hookrightarrow \operatorname{Spec} \mathcal{O}_K = S$

smooth projective

For $\mathfrak{p} \hookrightarrow S$ ($\mathfrak{p}$ prime of K)
↗ closed point

$$\operatorname{Pic}(X_{\bar{K}}) \hookrightarrow \operatorname{Pic}(\mathcal{X}_{\bar{\mathfrak{p}}})$$

$\Rightarrow \quad \underset{\mathbb{Z}}{\overset{\text{IS}}{\,}} \; \rho(X_{\bar{K}}) \qquad\qquad \underset{\mathbb{Z}}{\overset{\text{IS}}{\,}} \; \rho(\mathcal{X}_{\bar{p}})$

$$\Rightarrow \quad \rho(X_{\bar{K}}) \leq \rho(\mathcal{X}_{\bar{p}})$$

__Charles:__ what can be said about $NL = \{ s \in S, \rho(X_{\bar{K}}) < \rho(\mathcal{X}_{p}) \}$

↑ Noether-Lefchetz locus

__Thm:__ (SSTT 2019)

Assume that X has everywhere good reduction, Then

NL is infinite

<u>Rq</u>: If $\rho(X_{\bar{k}})$ is odd then $NL = S$

<u>Reason</u>: Tate Conjecture

$\Rightarrow \rho(X_{\bar{s}})$ = multiplicity of roots of unity of Fr $\curvearrowright$ $H^2_{et}(X_{\bar{s},et}, \mathbb{Q}_\ell(1))$

= even

$\Rightarrow \rho(X_{\bar{k}}) < \rho(X_{\bar{s}})$

$\Rightarrow \rho(X_{\bar{k}})$ is even and NL can be of density zero (Charles).

Corollary: For X as in the thm, one of the following is true:

(1.) $X_{\bar{K}}$ has infinitely many rational curves

(2) X has infinitely many supersingular reduction
$(\rho(X_p) = 22)$

→ (1) : Chen - Gounelas - Liedtke.

2 {

Thm: (Charles 2014): K number field

E_1, E_2 elliptic curves/K

$$NL = \{ \mathfrak{p} \subseteq O_K, \quad E_{1,\mathfrak{p}},\ E_{2,\mathfrak{p}} \text{ smooth and geometrically isogenous} \}$$

Then NL is infinite

Thm (Ananthshankar–Tang):

$A \to \operatorname{Spec} K$ abelian surface with real multiplication:

F quadratic real field $F \hookrightarrow \operatorname{End}_{\mathbb{Q}}(A)$

Then $\{ \mathfrak{p},\ A_{\bar{\mathfrak{p}}} \simeq E \times E \}$ is infinite.

Our method yields also

Thm (SSTT 19): $A \to K$ abelian surface then.

$\{ \mathfrak{p}, \quad A_{\bar{\mathfrak{p}}}$ is not simple $\}$

is infinite

3 § Complex picture:

Let $\pi : \mathcal{X} \longrightarrow S$ family of K3 surfaces

S complex quasi. proj curve / $\mathbb{C}$

$s \in S$, $\quad \rho(s) = \operatorname{rk} \operatorname{Pic}(\mathcal{X}_s)$

$$M = \min_{s \in S} \rho(s)$$

$$NL = \{ s \in S,\ \rho(s) > M \}$$

- Borcherds - Katzarkov - Pantev - Shepherd Barron ('98)

Assume | π is not isotrivial
| S projective

$\Rightarrow$ $NL \neq \emptyset$

$\rightarrow$ Green - Ogviso ('03)

π not - isotrivial

$\Rightarrow$ NL is dense for the analytic topology.

$\rightarrow$ (T2018): $\lambda \in \mathrm{Pic}(\mathcal{X}_s) \rightarrow (\lambda.\lambda) \in \mathbb{Z}$

$\Rightarrow$ NL is equidistributed in S w.r.t $\{ s \in S, \ \lambda \in Pic(\mathcal{X}_s) \ (\lambda.\lambda) = n \}$

to the measure given by the curvature of the Hodge bundle as $n \longrightarrow +\infty$

4 § Ideas of the proof

Let $\pi : \mathcal{X} \longrightarrow S$ family of K3 surfaces / $\mathbb{C}$

(S: curve)

For simplicity, assume $\underline{M = 1}$

$\Longleftrightarrow$ $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ ample line bundle

+ π non-isotrivial.

$\rightarrow$ Variation of Hodge structures:

$$\mathbb{V}_{\mathbb{Z}} \underset{= c_1(\mathcal{L})^{\perp}}{\subseteq} R^2\pi_* \underline{\mathbb{Z}}_{\mathcal{X}} \qquad (H^2(\mathcal{X}_s, \mathbb{Z}))$$

+ holomorphic filtration:

$$F^2 V \subseteq F^1 V \subseteq F^0 V = \mathbb{V}_{\mathbb{Z}} \otimes \mathcal{O}_S$$

+ quadratic form: $\mathbb{V}_{\mathbb{Z}} \times \mathbb{V}_{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}_S$

$\to (V_{\mathbb{Z}}, F^{\cdot}V, Q)$ Polarized Variation of Hodge structure

$\to$ Let $(L, Q) \simeq (V_{\mathbb{Z}, s}, Q)$

$\uparrow$ quadratic lattice $\qquad$ $\uparrow$ fiber

has signature $(2, ~~19~~)$

$b \geq 1$

Period domain:

$$D = \{ \omega \in \mathbb{P}(L_{\mathbb{C}}),\ (\omega, \omega) = 0,\ (\omega, \bar{\omega}) > 0 \}$$

$\shortparallel$

weight 2 polarized HS on (L, Q)

For $\Gamma \subseteq O(2,19)$ arithmetic subgroup

we have a period map:

$$\rho: S \longrightarrow {}_{\Gamma}\backslash D$$
$$s \longmapsto F^2 V_s \in D$$

F^2V: Hodge line bundle.

$\rightarrow$ ${}_{\Gamma}\backslash D$: orthogonal Shimura variety

it has an algebraic structure (Baily-Borel)

+ Borel: $\rho: S \longrightarrow {}_{\Gamma}\backslash D$ is algebraic

Special divisors:

$m \in \mathbb{Z}_{<0}$,

$\lambda \in L$, $Q(\lambda) = m$, $\lambda^{\perp} = \{x \in D,\ (x,\lambda) = 0\}$

$$\rightarrow \underline{\underline{Z(m)}} = \Gamma \backslash \Big(\bigcup_{\substack{\lambda \in L \\ Q(\lambda) = m}} \lambda^{\perp} \Big) \hookrightarrow \Gamma \backslash D$$

↑ Special divisor.

and $NL(S) = \bigcup_m Sn\, \varphi^{-1}(Z(m))$

$$\rightarrow \quad \begin{matrix} \mathcal{O}(-1)/_D \\ \downarrow \\ D \end{matrix} \quad \Rightarrow \quad \begin{matrix} \mathcal{L} = F^2 \mathcal{V} \\ \downarrow \\ \Gamma \backslash D \end{matrix}$$

↑ line bundle.

$[Z(0)] = \mathcal{L}$

Borcherds

$$\Phi = \sum_{m \geqslant 0} [Z(-m)]\, q^{m} \in \operatorname{Pic}(\mathcal{S}_\Gamma)[[q]]$$

$\in \operatorname{Pic}(\mathcal{S}_\Gamma) \otimes M_{1+\frac{19}{2}}$

(Component of a vector valued modular form)

$\Rightarrow$ If S projective then

$$(Z(-m).S) \sim m^{\frac{19}{2}} \cdot (S.\mathcal{L})$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Volume(S)

Mixed char situation:

Canonical models, $M(\mathbb{C}) = \mathcal{S}_\Gamma$

(Deligne - Milne - André)

$\Rightarrow \exists$ DM stack $/\mathbb{Q}$:

$$\mathcal{M} \longrightarrow \operatorname{Spec}(\mathbb{Q}), \quad \mathcal{M}(\mathbb{C}) = \Gamma \backslash D$$

+ (Vasiu, Kisin, Madapusi-Pera, Kim, AGHMP):

$$\longrightarrow \exists \quad \mathcal{M}_{\mathbb{Z}} \longrightarrow \operatorname{Spec}\mathbb{Z}$$

flat normal DM stack.

+ Special divisors:

$$\mathcal{Z}(m)_{\mathbb{Z}} \hookrightarrow \mathcal{M} \quad \text{relative Cartier divisors}$$

$$\mathcal{Z}(m)_{\mathbb{Z}} \xrightarrow{\text{flat}} \operatorname{Spec}\mathbb{Z}, \qquad \mathcal{M} \to \operatorname{Spec}\mathbb{Z}$$

$\rightarrow \quad \mathcal{X} \longrightarrow \operatorname{Spec} O_K = S$

$\rightarrow \quad \psi : S \longrightarrow \mathcal{M}$ (Rizov + Madapusi Pera)

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{M} \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} \mathbb{Z} & \end{array}$$

Arakelov intersection theory:

We define:

$$\widehat{CH}^1(\mathcal{M}) =: \left(\bigoplus (A, g) \right) \Big/ \left(\operatorname{div}(f), -\log|f|^2 \right)_{f \in \mathbb{Q}(\mathcal{M})}$$

A: Cartier divisor

g: Green function

$C^\infty(\mathcal{M}(\mathbb{C}) \setminus A(\mathbb{C}))$

+ logarithmic singularity along $A(\mathbb{C})$.

Borcherds - Bruinier:

$$\widehat{Z}(m) = (Z(m), g_m) \in \widehat{CH}^1(\mathcal{M})$$

Thm (Howard - MP):

$$\sum_{m \geq 0} \widehat{Z}(m)\, q^m \in \widehat{CH}^1(\mathcal{M}) \otimes M_{\frac{1+19}{2}} \text{—}$$

$$\widehat{CH}^1(\mathcal{M}) \longrightarrow \widehat{CH}^1(S) \xrightarrow{\widehat{\deg}} \mathbb{R}$$

$S = \operatorname{Spec} \mathcal{O}_K$

$$\widehat{\mathcal{Z}}(m) \longmapsto (\widehat{\mathcal{Z}}(m).S)$$

$$(\widehat{\mathcal{Z}}(m).S) = \sum_{\sigma : K \hookrightarrow \mathbb{C}} g_m(S^\sigma) + \sum_{p} (S.\mathcal{Z}(m))_p \cdot \log |p|.$$

$\neq 0$

If. $(S.\mathcal{Z}(m))_p \neq 0$ then $\mathcal{X}_{\bar{p}}$ has an extra line bundle

Assume that $NL < +\infty$

then: We prove:

→ $(\hat{Z}(m) . S) \ll |m|^{\frac{b}{2}}$

← modularity

→ for every $\sigma : K \hookrightarrow \mathbb{C}$:

$g_m(S^{\sigma}) \simeq - |m|^{\frac{b}{2}} \log |m| +$ other contribution

we show $= o(|m|^{\frac{b}{2}} \log |m|)$

→ At a finite place $\mathfrak{P}$:

$(S . Z(m))_{\mathfrak{P}} = o(|m|^{\frac{b}{2}} \log |m|)$

→ gives a contradiction.

⇒ NL is infinite

$(1,a)\ (2,b)$ unimodular

$$\to \sum_{m\geq 0} \hat{Z}(-m)\, q^m$$

$$\to \sum_{m\geq 0} (S.\hat{Z}(-m))\, q^m \in M_{1+\frac{b}{2}}(SL_2(\mathbb{Z}))$$

$$\parallel$$

$$(S.\hat{Z}(0)).E_{1+\frac{b}{2}} + \text{cuspidal}$$

Fourier coefficients

$$(S.\hat{Z}(-m)) = (S.\hat{Z}(0))\, E_{1+\frac{b}{2}}(m) + a(-m)$$

$$= O(m^{b/2})$$