§1 Elliptic curves

§2 Archimedean theory

§3 Non-Archimedean theory

§4. Global Theory

$$E: y^2 = x^3 + ax + b \qquad \Delta = 4a^3 + 27b^2 \neq 0$$
$$a, b \in \mathbb{Q}$$

$$h_{NT}: \quad \begin{array}{ccc} E(\mathbb{Q}) & \longrightarrow & \mathbb{R} \\ & {\scriptstyle x}\searrow \quad \nearrow {\scriptstyle h_{weil}} & \\ & \mathbb{P}^1 & \end{array}$$

$\log \max(|a|, |b|)$

$(u, v) \in \mathbb{Z}^2$
$\gcd = 1$

$$h_{NT}(P) = \frac{1}{2} \lim_{n \to \infty} \frac{h_{wei}(x(2^n P))}{4^n}$$

Local Theory

$$h_{NT}(P) = -\sum_{p \leq \infty} g_p(P).$$

$$g_p: \quad E(\overline{\mathbb{Q}}_p) \longrightarrow \mathbb{R}$$

green's function: $\quad g_p(P) = -\log|z|(P) + O(1)$

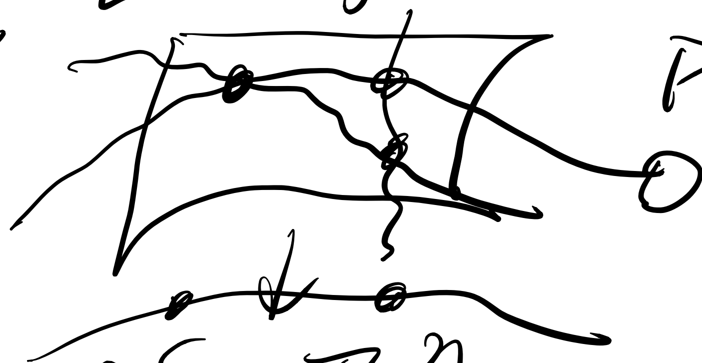
logarithmic singularity at O as $P \to O$

$$z = \frac{x}{y}$$

$$``\Delta" g_p = \delta_O - 1$$

3-case (potential)

1) P = finite, E has good reduction



S Spec $\mathbb{Z}_p$ η

$$g_p(P) = \text{intersection \# of } P, O$$

2) $P = \infty$

$$\Delta\, g_p(P) = \delta_P - 1$$

$$\Delta f = \left(\frac{\partial\bar{\partial}}{\pi i} f\right) / d\mu$$

uniform measure

$$g_p = \sum \frac{\phi_\lambda(p)}{\lambda} \qquad E(\mathbb{C}) = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$$

$$\phi(x+y\tau) = e^{2\pi i (mx+ny)}$$

$$\lambda = 2\pi |n - m\tau|^2$$

p finite bad

$$E^{an} = \mathbb{C}_p^\times / q^{\mathbb{Z}} \xrightarrow[v_q]{\cancel{\mathrm{Val}}} S^1$$

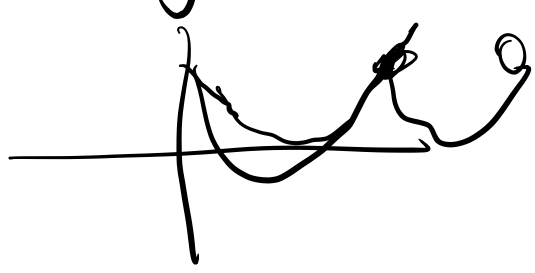
$$S^1 = \mathbb{R}/\mathbb{Z}$$

$q \in \mathbb{C}_p$, $|q| = 1$

$$g_p(p) = \underline{i(p, 0)} + \underline{g_{S^1}(v_q(p))}$$

$$-\frac{d^2}{dt^2} g = \delta_0 - 1$$

$$g = \underline{\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}}$$



$0 \le x < 1$

$$\int g\, dx = 0$$

Problem:

To extend these local height construction to all polarized varieties on local or global fields.

Archimedean case

X compact complex manifold

ω Kähler form $\left\{ \begin{array}{l} (1,1)\text{-form} \\ \bar{\omega} = \omega \\ d\omega = 0 \end{array} \right.$

$\exists$ local coordinate

$$\omega = \sum a_{ij}\, dz^i\, d\bar{z}^j$$

$$a_{ij} = \delta_{ij} + O(|z|^2)$$

1) de Rham theory

$$A^{pq}(X) = \text{span of smooth } (p,q)\text{-form}$$

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X)$$

$$A^*(X) \xrightarrow{d} A^{*+1}(X)$$

$$H^*(X, \mathbb{C}) = \mathbb{H}^*(A^*(X), d)$$

$A^{p,q}$ has Hermitian product.
$\ni$
α, β

$$\langle \alpha, \beta \rangle = \int_X \langle \alpha, \beta \rangle(x) \frac{\omega^n}{n!}$$

$$\underset{L^2}{A^*(X)} \underset{d^*}{\overset{d}{\rightleftarrows}} \underset{L^2}{A^*(X)}$$

$\partial, \bar\partial$

$$\Delta_d = dd^* + d^*d$$
$$= 2\Delta_\partial = 2\Delta_{\bar\partial}$$

Hodge Thm:

$$H^*(X, \mathbb{C}) = (\ker \Delta)$$
$$= \text{Harmonic forms}$$

Kea App

Construct green's current:

$$Y \hookrightarrow X \quad \text{subvariety} \quad \text{cod} = p$$

$$[Y] \in H^{2p}(X, \mathbb{C})$$

$$\left\{ \begin{array}{l} \underbrace{\frac{\partial\bar\partial}{\pi i} g_Y = \delta_Y - h_Y}_{} = \text{Harmonic-form} \\ g_Y \quad (p-1, p-1)\text{-form on } X \setminus Y \\ \quad \text{logarithmic singularity along } Y. \end{array} \right.$$

$$\int g_Y \cdot h = 0$$

h = Harmonic $(n-p+1, n-p+1)$.

Key: For certain g_Y

$$\boxed{A^{p-1,p-1}(X) \xrightarrow{\partial\bar\partial} A^{p,p}_{\text{clsd}}(X)}$$

$\cancel{\operatorname{Im}\partial + \operatorname{Im}\bar\partial}$

Defn

$$\operatorname{Im}(\partial\bar\partial) = A^p_\varphi(X)$$

$$\text{“} A^p(X) = A^{p,p}_{\text{closed}}(X) \text{”}$$

$$A^p_\varphi(X) = \operatorname{coker}(\partial\bar\partial)$$

Fact $$\boxed{A^p_\varphi(X) = H^{p,p}_{-}(X, \mathbb{C})}$$

Splitting

$$A^{p,p}_{\text{closed}}(X) = \operatorname{Im}(\partial\bar\partial) \oplus \left(\ker\Delta\big|_{A^{p,p}_{\text{clsd}}}\right)$$

Lefschetz Thm

$$A^*(X) \xrightarrow[\wedge\omega]{\mathbb{L}} A^{*+2}(X)$$

$$0 \to A^*_\varphi(X) \to \underset{\text{clsd}}{A^*(X)} \to A^*_\varphi(X) \to 0$$

$\mathrm{Im}(\partial\bar\partial)$ $\qquad\qquad\qquad$ $H^*(X)$.

Exact sequ of $\mathbb{L}$-modules

$A^*(X)$ Lefschetz str.

$$A^i(X) \xrightarrow[\sim]{\mathbb{L}^{n-2i}} A^{n-i}(X)$$

$i \le n/2$ $\qquad\qquad$ $\wedge\,\omega^{n-2i})$

$$(A^{p,p})^* = \mathcal{D}^{n-p,\, n-p}$$

$$\delta_Y, \quad g_Y \in \mathcal{D}^{p-1,\, p-1}$$

$$\in \mathcal{D}^{p,\, p}$$

For any α smooth $(n-p,\, n-p)$

$$\int_X g_Y \frac{\partial\bar{\partial}}{\pi i} \alpha = \int_Y \alpha - \int_X \alpha h_Y$$

$h_Y = \text{Harm}$

Proof: $U_0^i = \ker(L' |_{U^i})$ $i \le n/2$

$$\begin{cases} U^* \simeq \bigoplus \mathbb{L}\, U_0^* \\ U^i \simeq \bigoplus\limits_{j \le n/2} L^{i-j} U_0^j \end{cases}$$

primitive elements $\le i$

$$\begin{tikzcd}
 & & V_0^i \arrow[r, "\simeq"] \arrow[d] & W_0^i \arrow[d] & \\
0 \arrow[r] & U^i \arrow[r] \arrow[d, "L^{n+1-i}"', "\wr"] & V^i \arrow[r] \arrow[d] & W^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U^{n+1-i} \arrow[r] & V^i \arrow[r] & W^{n+1-i} &
\end{tikzcd}$$

$$A^i_\varphi(X) \xrightarrow[\sim]{\mathbb{L}^{n-i}} A^{n-i}_\varphi(X)$$

<u>Künneman</u>

$$A^i_\varphi(X) \xrightarrow[\sim]{\mathbb{L}^{n+1-2i}} A^{n+1-i}_\varphi(X)$$

$\Longrightarrow$ unique splitting

<u>Lemma</u> E field of char $= 0$

(*) $0 \to U^* \to V^* \to W^* \to 0$

$E[\mathbb{L}]$-module graded

n

$$U^i \xrightarrow[\sim]{\mathbb{L}^{n+1-2i}} U^{n+1-i}$$

$$W^i \xrightarrow[\sim]{\mathbb{L}^{n-2i}} W^{n-i}$$

Then (*) is uniquely split

$$X_s \xrightarrow{i} X \quad \text{-dim } n+1$$

$$\begin{array}{ccc} A_*(X_s) & \xrightarrow{i_*} & A_*(X) = A^{n+1-*}(X) \\ & & \downarrow i^* \\ & & A^{n+1-*}(X_s) \end{array}$$

$$A_{n+1-*}(X_s) \xrightarrow{i^* i_*} A^*(X_s)$$

Analys of $\partial\bar\partial$

Define $\quad A_\varphi^*(X_s) = \mathrm{Im}\,(i^* i_*)$

$\quad A_\psi^*(X_s) = \mathrm{coker}\,(i^* i_*)$

$$0 \to A_\varphi^*(X_s) \to A^*(X_s) \to A_\psi^*(X_s) \to 0 \qquad (*)$$

Conclusion We can define harmonic fun by Lefschetz operators.

Non-Archimedean Theory

X = regular

$$\begin{tikzcd} X_s \arrow[d, dash] & X \arrow[d, "\text{flat}"] & X_\eta \arrow[d, dash] \\ s & S = \operatorname{Spec} R & \eta \end{tikzcd}$$

R = Complete DVR

$A_*(X_s)$ = cycles on X_s modulo homology equivalent.

$A^*(X_s)$ = cohomology cl cycles

Assume X_s is strictly normal crossing

$$X_s = \bigcup_{i \in I}^{r} X_i$$

$$I \subset \{1 \cdots r\} \quad X_I = \bigcap_{i \in I} X_i \text{ smooth of cod } |I|.$$

$$A^*(X_s) = A^*_\varphi(X_s) \oplus \mathcal{A}^*_\psi(X_s)$$

$\mathcal{A}^*_\varphi(X_s)$ is called space of

Harmonic forms

<u>Examples when the conjecture holds</u>

1) Curves, surface, abelian varieties

2) The case when X_s is SNC. s.t each strata X_I satisfies Grothendieck standard conjecture

<u>Example</u>: Varieties with the p-adic uniformization by Drinfeld

$$A_*(X_s) = \mathrm{coker}\left(\bigoplus_{i<j} A_*(X_i \cap X_j) \xrightarrow{(\beta_*,\, \gamma_*)} \bigoplus A_*(X_i)\right)$$

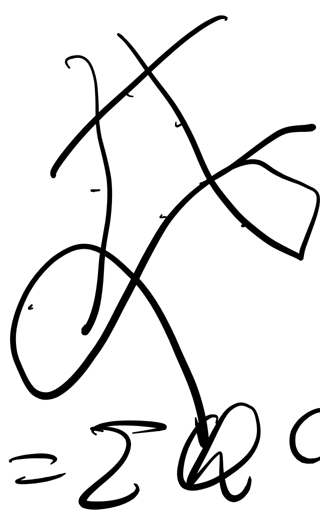
$$\coprod_{i<j} X_i \cap X_j \overset{\beta}{\underset{\gamma}{\rightrightarrows}} \coprod X_i \xrightarrow{\alpha} X_s$$

$$A^*(X_s) = \ker\left(\bigoplus A^*(X_i) \xrightarrow{\beta^* - \gamma^*} \bigoplus_{i<j} A^*(X_i \cap X_j)\right)$$

On smooth varieties X_i, X_{ij}

$$A^*(X_i) = A_{n-*}(X_i)$$

<u>Example</u>



X_s

$$A_1(X_s) = \sum \mathbb{Q}\, c_i, \qquad A^1(X_s) = \text{Function on}$$

set $\{c_1 \cdots c_r\}$.

Conjecture: Let $\mathcal{L}$ be an ample line bundle on X which induces an Lefschetz operator $\mathbb{L}$.

Then 1) $\forall i \leq \frac{n+1}{2}$

$$\mathbb{L}^{n+1-2i} : A^i_\varphi(X_S) \xrightarrow{\sim} A^{n+1-i}_\varphi(X_S)$$

2) $\forall i \leq \frac{n}{2}$

$$\mathbb{L}^{n-2i} \quad A^i_\psi(X_S) \xrightarrow{\sim} A^{n-i}_\psi(X_S)$$

Theorem / Definition

Assume Lefschetz conjecture. Then The short exact sequence $(*)$ is uniquely split

Application: Define intersection pairing for $Y, Z \subset X$

$\dim Y + \dim Z = \dim X - 1.$

Global Theory: $X/\mathcal{O}_K$ arithmetic variety, $\mathcal{L}$

$$\widehat{Ch}^*(X) = \widehat{Ch}^*_0(X) \oplus A^*(X)$$

$A^*(X) \cong$ cohomology

$\langle\ \rangle_{BB}$

$$\langle [X], (pt) \rangle_{AA} = \frac{\langle c_1(\mathcal{L})^{n+1} \rangle_{AA}}{(n+1)\deg_{\mathcal{L}}(X)}$$

$$\left(A^*(X), \langle\ \rangle_{AA} \right)$$

of Drinfeld upper-half space