Euler systems & explicit reciprocity laws for GSp(4)

I Introduction

V = p-adic Gal repⁿ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$
geometric

* L-function

$$L(V,s) = \prod_{\ell} P_\ell(\ell^{-s})^{-1}$$

$P_\ell(X)$ = Euler factor at ℓ

* $\mathrm{Sel}(\mathbb{Q},V) \subset H^1(\mathbb{Q},V)$ cut out by local condⁿs

Bloch-Kato conj for V:

$$\mathrm{ord}_{s=0} L(V,s) = \dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V^*(1)) \underbrace{- \dim_{\mathbb{Q}_p} H^0(\mathbb{Q}, V^*(1))}_{\text{often } 0.}$$

Def^n (Euler system for V)
Collection of cohomology classes $(z_m)_{m\geq 1}$
$z_m \in H^1(\mathbb{Q}(\mu_m), V)$
Satisfying ES norm relations
as m varies

ES machine (Kato, Rubin):
if $z_1 \neq 0$ (+ tech. hyp.), then $Sel(\mathbb{Q}, V) = 0$.

II Setup

Π = cuspidal auto rep^n of GSp_4,
unram at p, cohomological,
wts (r_1, r_2), $r_1 \geq r_2 \geq 0$
$\alpha, \beta, \gamma, \delta$ = roots of Hecke poly at p
$0 \leq v_p(\alpha) \leq v_p(\beta) \leq \ldots$
complex abs. value $p^{\frac{r_1+r_2+3}{2}}$

V_π = spin rep^n - 4-dim^l p-adic rep^n of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$

$L(V_\pi, s) = L(\pi, s - \frac{r_1+r_2+3}{2})$

L-factor at p : $(1-\alpha p^{-s})(1-\beta p^{-s})\cdots$

critical range: $2+r_2 \le n \le 2+r_1$ \} disjoint
geometric range: $1 \le n \le r_2+1$

Aim: prove cases of BK conjecture for $V_\pi(n)$, n critical

Assume throughout:

$$v_p(\alpha) = 0, \quad v_p(\beta) = r_2+1$$

III The Euler system

Theorem (Loeffler-Skinner-Z., '17)
For n geometric, $\exists$ ES for $V_\pi^*(1-n)$
Constr^n of bottom class: $z_\pi(n) \in H^1(\mathbb{Q}, V_\pi^*(1-n))$

$H = GL_2 \times_{GL_1} GL_2$, $G = GSp_4$

$\iota : H \hookrightarrow G$ $\quad \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right] \longmapsto \begin{pmatrix} a & & & b \\ & a' & b' & \\ & c' & d' & \\ c & & & d \end{pmatrix}$

Y_H, Y_G Shimura varieties

$\iota : Y_H \hookrightarrow Y_G$

$$\rightsquigarrow \begin{array}{ccccc} H^2_{et}(Y_{H,\bar{\mathbb{Q}}_p}) & \xrightarrow{\iota_*} & H^4_{et}(Y_{G,\bar{\mathbb{Q}}_p}) & \xrightarrow{pr_{\pi,n}} & H^1(\mathbb{Q}, V_\pi^*(1-n)) \\ \cup & & & & \cup \\ \mathrm{Eis}^{\sqcup 2}_{GL_2,et} & & \longmapsto & & z_\pi(n) \\ \| & & & & \\ \mathrm{Eis}_{H,et} & & & & \end{array}$$

Fact: can construct a measure

$z_p^{Mot}(\pi) \in \Lambda(\Gamma)$, $\Gamma \cong \mathbb{Z}_p^*$

measuring the non-triviality of $z_\pi(n)$
n geometric.

ES machine: for n <u>critical</u>,

$\boxed{\text{IF}}$ $z_p^{Mot}(\pi)(n) \neq 0 \Rightarrow \mathrm{Sel}_{BK}(\mathbb{Q}, V_\pi^*(1-n)) = 0$.

Problem: need criterion for
$\mathcal{L}_p^{mot}(\pi)(n) \neq 0$, $\quad$ n critical.

Aim: prove explicit reciprocity law:
if n critical and $\mathcal{L}(V_\pi, n) \neq 0$, then
$\mathcal{L}_p^{M}(\pi)(n) \neq 0$.

Note: geometric input is ES constr^n
for twists in geom. range.

Strategy (following BDP):
use analytic cont^n from geometric
to critical range

Step 1: construct p-adic $\mathcal{L}$-fct.
$\mathcal{L}_p^{an}(\pi) \in \Lambda(T)$ st.

$$\mathcal{L}_p^{an}(\pi)(n) = \frac{\mathcal{L}(V_\pi, n)}{\text{period}}, \quad n \text{ critical}$$

Step 2: show for n geometric,

$$\mathcal{L}_p^{an}(\pi)(n) = \mathcal{L}_p^{Mot}(\pi)(n)$$

Step 3: use analytic cont^n in Hida fam. to deduce that $\mathcal{L}_p^{an}(\pi) = \mathcal{L}_p^{Mot}(\pi)$

IV Const^n of $\mathcal{L}_p^{an}(\pi)$

(joint with Loeffler - Pilloni - Skinner)

Need: const^n of $\mathcal{L}_p^{an}(\pi)$ compatible with ES const^n

Def^n: X_H, X_G toroidal compactifications of Y_H, Y_G

$\rightsquigarrow \iota: X_H \longrightarrow X_G$

$$\pi \rightsquigarrow \eta \in \underbrace{H^2(X_G)}_{}$$

coherent cohom. with suitable coefficient system.

$\Rightarrow$ can consider

$\langle z^*(\eta), f \sqcup g \rangle$ — f, g (nearly) holom. MFs, wts k, l, $k + l = r_1 - r_2 + 2$

cup prod in coherent cohom.

Theorem (Piatetski-Shapiro, Harris):

$\langle z^*(\eta), \underbrace{\mathcal{E}is_k \sqcup \mathcal{E}is_l}_{\text{suitable Eisenstein series}} \rangle = (*) L(V_\pi, n)$ n critical

Idea: interpolate $\overbrace{\langle z^*(\eta), \mathcal{E}is_k \sqcup \mathcal{E}is_l \rangle}^{(*)}$ p-adically

need: lift of $z^*(\eta)$ to $H^2_c(X_H^{ord})$

Pilloni: if $v_p(\alpha) = 0$, then η lifts to $\eta^{\geq 1} \in H^2_c(X_a^{\geq 1})$

$X_a^{\geq 1} = (p\text{-rank} = 1)\text{-locus of } X_a$

check: $\iota^*(\eta^{\geq 1}) \in H^2_c(X_H^{ord})$

$\Rightarrow$ $(*) = \langle \iota^*(\eta^{\geq 1}), (\mathrm{Eis}_k \sqcup \mathrm{Eis}_\ell)|_{X_H^{ord}} \rangle$

define

$$\mathcal{L}_p^{an}(\pi) = \langle \iota^*(\eta^{\geq 1}), \underbrace{\mathcal{E}_1 \sqcup \mathcal{E}_2}_{\text{suitable } p\text{-adic families of Eis series}} \rangle$$

$\Rightarrow$ by const^n, $\mathcal{L}_p^{an}(\pi)$ interpolates $L(V_\pi, n)$, n critical

Remark: (L-Z, 21): vary r_1, r_2 (uses Boxer-Pilloni).

V Relating $\mathcal{L}_p^{an}(\pi)$, $\mathcal{L}_p^{Mot}(\pi)$ in geometric range

By const^n,

$$\mathcal{L}_p^{Mot}(\pi)(n) = \langle \log_{BK}(\underbrace{z_\pi(n)}_{\text{étale cohom.}}), \eta_{dR} \rangle$$

where $\eta_{dR} \in \mathrm{Fil}^n \mathbb{D}_{dR}(V_\pi)$ arising from η via Hodge-to-dR spectral seq.

étale cohom. $\xleftrightarrow[\text{Ertl-Yamada}]{\text{Nekovar-Niziol}}$ syntomic cohom.
(hybrid between dR and rigid cohom. for smooth/ss schemes using φ)

$\Rightarrow$ express $\mathcal{L}_p^{Mot}(\pi)(n)$, n geom., via syntomic cohom.

$$\mathcal{L}_p^{Mot}(\pi)(n) = \langle \mathcal{E}is_{H,syn}, \iota^*(\tilde{\eta}) \rangle \qquad (\star)$$

where $\tilde{\eta}$ = lift of η_{dR} to syntomic cohom.

$\tilde{\eta} \in H^3_{syn}(X_a)$

to compare $(\star)$ to $\mathcal{L}_p^{an}(\pi)(n)$, need to restrict to X_H^{ord}.

Pilloni/Lan-Skinner: $\tilde{\eta}$ lifts to $\tilde{\eta}^{\geq 1} \in H^3_{syn,c}(X_a^{\geq 1})$

$$\Rightarrow (\star) = \langle \mathcal{E}is_{H,syn}|_{X_H^{ord}}, \iota^*(\tilde{\eta}^{\geq 1}) \rangle$$
$$= \langle (\mathcal{E}is_{H,syn}|_{X_H^{ord}}), \iota^*(\tilde{\eta}^{\geq 1}|_{X_{G}^{ord}}) \rangle \quad (\star\star)$$

Need: relate this pairing to coherent cohom.

NOTE: over X_H^{ord}, X_G^{ord}, φ lifts to char. 0.

$\Rightarrow$ can describe $\mathrm{ris}_{H,\mathrm{syn}}|_{X_H^{\mathrm{ord}}}$, $(\tilde{\eta}^{\geq 1})|_{X_G^{\mathrm{ord}}}$ in terms of coherent cohom.

(tool: Poznan spectral sequence, relating coherent and syntomic cohom.
$\leftrightarrow$ syntomic analogue of H-to-dR s.s.)

$\Rightarrow$ express ($\star\star$) as pairing in coherent cohom + recognize it as $\mathcal{L}_p^{an}(\pi)(n)$.

$\square$

Remarks: generalisations to other ES:

$GSp_4 \times GL_2$ (Hsu–Jin–Sakamoto, LZ).

$\mathrm{Res}^K_{\mathbb{Q}} GL_2$, K real quad (Lei-L-Z, Grossi) $\}$ in progress

$U(2n-1,1)$ (Graham–Shah)

[illegible])