Periods, L-functions, and duality of Hamiltonian spaces
MIT number theory seminar

Yiannis Sakellaridis (Johns Hopkins); joint w. David Ben-Zvi (Texas) and Akshay Venkatesh (IAS)

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Abstract

The relationship between periods of automorphic forms and L-functions has been studied since the times of Riemann, but remains mysterious. In this talk, I will explain how periods and L-functions arise as quantizations of certain Hamiltonian spaces, and will propose a conjectural duality between certain Hamiltonian spaces for a group G, and its Langlands dual group $\check{G}$, in the context of the geometric Langlands program, recovering known and conjectural instances of the aforementioned relationship. This is joint work with David Ben-Zvi and Akshay Venkatesh.

1 Introduction

•Riemann: $\int_0^\infty y^{\frac{s}{2}} \sum_{n=1}^\infty e^{-n^2\pi y} dx = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, and proof of functional equation based on symmetry of the theta series:

$$\theta(y) = \sum_{n=1}^{\infty} e^{-\pi n^2 y} = y^{-\frac{1}{2}}\theta(y^{-1}).$$

•Iwasawa–Tate reformulation as ($x \leftrightarrow \sqrt{y}$)

$$\int_{k^\times \backslash \mathbb{A}^\times} \chi(x) \sum_{\gamma \in k^\times} \Phi(\gamma x) d^\times x \text{ ``="} L(\chi, s),$$

where the critical local calculation is that, for $\Phi_v = 1_{\mathfrak{o}_v}$,

$$\int_{k_v^\times} \chi_v(x)\Phi_v(x) d^\times x = L_v(\chi_v, 0).$$

•Generalized by Godement–Jacquet to Mat_n under the action of $G = (\mathrm{GL}_n \times \mathrm{GL}_n)/\mathbb{G}_m$ (set $[G] = G(k)\backslash G(\mathbb{A})$), $\varphi \in \pi \otimes \tilde{\pi}$ a cusp form,

$$\int_{[G]} \varphi(g_1, g_2) \sum_{\gamma \in \mathrm{Mat}_n(k)} \Phi(g_2^{-1}\gamma g_1) d(g_1, g_2) \text{ ``="} L(\pi, \frac{1-n}{2}),$$

where the critical local calculation is that, for $\Phi_v = 1_{\mathrm{Mat}(\mathfrak{o}_v)}$,

$$\int_{\mathrm{Mat}_n(k_v)} \langle \varphi_1(g), \varphi_2 \rangle \Phi_v(g) d^\times g = L_v(\pi_v, \frac{1-n}{2}).$$

Integrals that use other spaces:

•Hecke: $X = \mathrm{PGL}_2$, $G = \mathrm{GL}_1 \times \mathrm{PGL}_2$, $\chi \otimes \pi \ni \chi \otimes \varphi$,

$$\int_{[G]} \chi(a)\varphi(g) \sum_{\gamma \in X(k)} \Phi(a^{-1}\gamma g) d(a,g) \text{ ``$=$'' } \int_{[\mathbb{G}_m]} \chi(a)\varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a \text{ ``$=$'' } L(\chi \otimes \pi, \frac{1}{2}).$$

Local integral:

$$\int_{G(k_v)} \chi_v(a) \otimes W_{\varphi,v}(g)\Phi_v(a^{-1} \cdot g) d(a,g) \text{ ``$=$'' } L(\chi_v \otimes \pi_v, \frac{1}{2}).$$

Note: Over function fields, φ everywhere unramified. There is a canonical choice $\Phi = \prod_v 1_{X(\mathfrak{o}_v)}$. The associated theta series will be denoted by Θ_X,

$$\Theta_X(g) = \sum_{\gamma \in X(k)} \Phi(\gamma g) \in C^\infty([G]),$$

where $[G] = G(k)\backslash G(\mathbb{A})$.

When $X = H\backslash G$, φ unramified, up to measures $\langle \varphi, \Theta_X \rangle_{[G]} = \int_{[H]} \varphi(h) dh$.

•Waldspurger: Same as Hecke, but with $X = T\backslash(T \times \mathrm{PGL}_2)$, $T \hookrightarrow \mathrm{PGL}_2$ a non-split 1-dimensional torus, then

$$\left| \int_{[G]} \chi(a)\varphi(g) \sum_{\gamma \in X(k)} \Phi(a^{-1}\gamma g) d(a,g) \right|^2 \text{ ``=''} \left| \int_{[T]} \chi(a)\varphi(a) d^\times a \right|^2 \text{ ``=''} \ L(\chi \otimes \pi, \otimes, \frac{1}{2}).$$

•(Gan–)Gross–Prasad: $X = \mathrm{SO}_n \backslash (\mathrm{SO}_n \times \mathrm{SO}_{n+1}) = H\backslash G$,

$$\left| \int_{[G]} \varphi(g) \sum_{\gamma \in X(k)} \Phi(\gamma g) dg \right|^2 \text{ ``=''} \left| \int_{[H]} \varphi(h) dh \right|^2 \text{ ``=''} \ L(\pi, \otimes, \frac{1}{2}).$$

Here the relevant local integral is that of Ichino–Ikeda,

$$\int_{H(k_v)} \langle \pi(h)\phi_v, \phi_v \rangle \, dh_v.$$

[Technicalities:

1. Sum over X vs $X^\bullet$ — can regularize.
2. Unitary normalization — produces L-values at $\frac{1}{2}$ in all examples above.
3. The good choice that makes "=" true varies by example: In Godement–Jacquet case need $\int_{[\mathrm{PGL}_n]^{\mathrm{diag}}} \varphi = 1$. In Hecke case need Whittaker normalization $W_\varphi(1) = 1$. The two *differ* by $\sqrt{L(\pi, \mathrm{Ad}, 1)}$, i.e.,

   if $W_\phi(1) = 1$ then $\int_{[G]} |\phi|^2 = L(\pi, \mathrm{Ad}, 1)$,

   at least for unramified data (& suitable choice of Haar measures). Morally, the last two examples should have Whittaker normalization, but this is not always possible (e.g., may need to replace G by a non-quasisplit inner form); so, we use the unitary normalization and correct by the factor $L(\pi, \mathrm{Ad}, 1)$.
4. Volume factors (OK) and small rational factor $\frac{1}{|S_\varphi|}$ missing from the Gross–Prasad example.]

The local factors in general are not known to have a meaningful formula for the period itself; for the (abs. value) square of the period, we have the Ichino–Ikeda conjecture. Venkatesh realized that it is related to the local Plancherel formula, so conjecturally,

$$\sum_{\varphi \in ON(\pi)} \left| \int_{[G]} \varphi(g) \sum_{\gamma \in X(k)} \Phi(\gamma g) dg \right|^2 =? \cdot \prod_v^* \langle \Phi_v, \Phi_v \rangle_{\pi_v}, \tag{1.1}$$

where

$$\langle \Phi_v, \Phi_v \rangle_{L^2(X(k_v))} = \int_{\hat{G}} \langle \Phi_v, \Phi_v \rangle_{\pi_v} \mu(\pi_v)$$

is the local Plancherel formula.

[Actually, over the tempered dual $\widehat{G_X}$ of another group G_X; its L-group ${}^L G_X \subset {}^L G$ is *the L-group of X*.]

The L-function has completely disappeared! Where is it? By a stupefying calculation, when Φ_v is "the basic function of X" (when X is smooth affine: $\Phi_v = 1_{X(\mathfrak{o}_v)}$),

$$\langle \Phi_v, \Phi_v \rangle_{\pi_v} = L_v(\pi_X, \rho_v) \tag{1.2}$$

for a distinguished representation $\rho_X : {}^L G_X \to \mathrm{GL}(V_X)$ of the L-group.

Two directions one can go towards:

1. Generalize the above to non-smooth affine varieties X. Then $1_{X(\mathfrak{o})}$ should be replaced by $IC_{X(\mathfrak{o})}$, and we have generalizations of (1.2) w. Jonathan Wang.

2. (Today:) Stick to $X =$smooth, and seek a deeper explanation for (1.1), (1.2).

It should be mentioned that there are other examples that display the behavior of (1.1), (1.2), without coming from homogeneous G-spaces, i.e.:

Howe duality (theta correspondence): $G = G_1 \times G_2 \hookrightarrow \mathrm{Sp}(M)$ a dual pair (ignore metaplectic covers). Weil representation $\omega = \otimes_v' \omega_v$ of $\mathrm{Sp}(M)(\mathbb{A})$, theta series $\Theta : \omega \to C^\infty([G])$ — in Schrödinger model: $\Phi \in \mathcal{S}(X(\mathbb{A}))$, where X: Lagrangian (*not G-stable*!),

$$\Theta_\Phi(g) = \sum_{\gamma \in X(k)} (\omega(g)\Phi)(\gamma).$$

Rallis inner product (RIP) formula, say for $G_1 = \mathrm{SO}_{2n}$, $G_2 = \mathrm{Sp}_{2n}$: $\pi = \tau \otimes \theta(\tau)$, $\Phi \in \omega$, then

$$\langle \Theta_\Phi, \Theta_\Phi \rangle_\pi = \prod_v \langle \Phi_v, \Phi_v \rangle_{\pi_v},$$

where

$$\langle \Phi_v, \Phi_v \rangle_{\omega_v} = \int \langle \Phi_v, \Phi_v \rangle_{\pi_v} \mu(\pi_v);$$

and moreover

$$\langle \Phi_v, \Phi_v \rangle_{\pi_v} = L_v(\tau_v, \frac{1}{2})$$

at almost every place. (Omitting factors that don't depend on the representation!)

2 Derived endomorphisms and the Plancherel formula

Now let $\mathbb{F} = \overline{\mathbb{F}_q}$, $F = \mathbb{F}((t)) \supset \mathfrak{o} = \mathbb{F}[[t]]$, and for an affine variety X think of $X(F) = LX(\mathbb{F})$, $X(\mathfrak{o}) = L^+(\mathbb{F})$, points of the loop and the arc space.

We'll take X = a smooth, affine, spherical G-variety.

Set $\mathrm{Shv}(LX/L^+G)$ denote an appropriate — bounded DG – category of L^+G-equivariant "sheaves" on LX — should be l-adic for translation to functions, but once we abstract from functions one can also take $\mathbb{F} = \mathbb{C}$ and work with D-modules. Let k: coefficient field, characteristic 0.

[Technicalities:

1. $X \hookrightarrow V$ (a vector space),

$$X(F) = \varinjlim_r X^r, \text{ where } X^r = t^{-r}V(\mathfrak{o}) \cap X(F),$$

and

$$t^{-r}V(\mathfrak{o}) = \varprojlim_s t^{-r}V(\mathfrak{o}/t^s), \text{ so } X^r = \varprojlim_s X^r_s$$

and similarly for $L^+G = \varprojlim_s G_s$, so we have maps $X^r_s/G_{s+k+1} \to X^r_s/G_{s+k}$. Sheaves are defined by pullback and pushforward along such maps of *bounded, constructible, equivariant* complexes of sheaves.

No good theory of t-structures (to the best of my knowledge), but at least for $r = 0$, because X is smooth, the transition maps are smooth, so we have the "basic object" (constant sheaf) k_{L^+X}.

2. Crash course on equivariant derived category, and the case of $B\mathbb{G}_m$. Let X be a G-space. If (say, in the topological setting) we were able to find a contractible cover $\tilde{X} \xrightarrow{f} X$, with a free G-action, we would have

$$\operatorname{Shv}(X/G) = \{(\mathcal{F}, \mathcal{G}, \alpha) | \mathcal{F} \in \operatorname{Shv}(X), \mathcal{G} \in \operatorname{Shv}(\tilde{X}/G), \alpha : f^*\mathcal{F} \simeq \pi^*\mathcal{G}\}.$$

In the algebraic setting, we approximate this by "n-acyclic covers", e.g., for $B\mathbb{G}_m = \mathrm{pt}/\mathbb{G}_\mathrm{m}$:

$$\tilde{X} = \mathbb{A}^\infty \setminus \{0\} = \varinjlim \mathbb{A}^n \setminus \{0\},$$

$$\tilde{X}/G = \mathbb{P}^\infty = \varinjlim \mathbb{P}^n,$$

and $\operatorname{Shv}(X/G) = \varprojlim H^\bullet(\mathbb{P}^n) = k[\eta]$, $\deg(\eta) = 2$.

3. To “center” the L-functions, we need “metaplectic correction”:

   Fact: $X \to H\backslash G$ a vector bundle with some fiber S_+, with H reductive. For simplicity, assume that $\det S_+$ extends uniquely to a character of G, gives $G(F) \to \mathbb{G}_m(F) \xrightarrow{\text{val}} \mathbb{Z}$, and twist the action of $G(F)_n$ by $\langle n \rangle = [n] \left(\frac{n}{2}\right)$. This allows for an extension of this definition to include the Weil representation, i.e., when X is not a G-space, but the Lagrangian fiber of a G-equivariant symplectic induction

$$M = S \times^H_{\mathfrak{h}^*} T^*G,$$

   S a symplectic H-vector space, $S_+ \subset S$ a Lagrangian subspace.

4. Whittaker-type induction: M could be a "twisted cotangent space", e.g.,

$$M = \{d\psi\} \times_{\mathfrak{n}^*} \mathfrak{g}^* \times^N G = T^*((N,\psi)\backslash G),$$

the cotangent space of the Whittaker model. More generally, ψ could be a central character of a Heisenberg subgroup quotient, and we could induce the associated irreducible representation, e.g.,

$$\mathrm{Ind}_{\widetilde{\mathrm{Sp}}(W)\ltimes U}^{\widetilde{\mathrm{Sp}}(W')} \omega_\psi,$$

where W is a symplectic space, $W' = W \oplus l \oplus l'$ is its sum with a 2-dimensional symplectic space, $U =$ the unipotent radical of the parabolic stabilizing the isotropic subspace (line) l, and ω_ψ the oscillator representation associated to $l^\vee \otimes W$. These are the Fourier–Jacobi models.

To see how these fit into the same framework, the relevant space is not the spherical variety X but M, a *coisotropic Hamiltonian G-space*, satisfying certain conditions. Coisotropic: $\mathbb{F}[M]^G$ is Poisson–commutative — the analog of the spherical condition.

Under certain conditions, there is a unique closed orbit $M_0 \subset M$ with nilpotent image under the moment map $M \xrightarrow{\mu} \mathfrak{g}^*$. An $\mathfrak{sl}_2$-triple (h, e, f) with $f \in \mu(M_0)$ gives rise to a Heisenberg group such as N, U above, with a central additive character.

]

Sheaf–function dictionary: In an l-adic setting, the inner product of functions should be obtained as Frobenius trace of derived homomorphisms (Ext) of sheaves:

Given two l-adic sheaves $\mathcal{F}, \mathcal{G}$ on an $\mathbb{F}_q$-variety Y, let f and $g^\vee$ be the trace functions associated to respectively $\mathcal{F}$ and $D\mathcal{G}$, with D the Verdier dual. Then

$$\sum_{Y(\mathbb{F}_q)} f(y)g^\vee(y) = \mathrm{tr}(\mathrm{Fr}, \mathrm{Hom}(\mathcal{F}, \mathcal{G})^\vee).$$

Conjecture 1: There is a representation (ρ_X, V_X) of $\check{G}_X$ and an equivalence of k-linear triangulated categories:

$$\mathrm{Shv}(LX/L^+G) = \mathrm{QC}^{/\!\!/}_{\mathrm{perf}}(V_X/\check{G}_X),$$

sending the basic object k_{L^+X} to the structure sheaf. [More structures to be added.]

Here, $\mathrm{QC}^{/\!\!/}_{\mathrm{perf}}(V_X/\check{G}_X)$ denotes (a "shearing" — will discuss below) of the triangulated category of (generated by) perfect complexes of $\check{G}_X$-equivariant $k[V_X]$-modules.

The data (ρ_X, V_X) come with an ${}^LG_X$-action when (G, X) are defined over a finite field, and are those of the L-value associated to the *square* of the corresponding period.

Generalizes:

- Bezrukavnikov–Finkelberg derived Satake: $G = H \times H$, $X = H$, $\check{G}_X = \check{H}$, $V_X = \check{\mathfrak{h}}^*$.

$$\mathrm{Shv}(L^+H \backslash LH / L^+H) = \mathrm{QC}_{\text{perf}}(\check{\mathfrak{h}}^*[2]/\check{H}).$$

- Braverman–Finkelberg–Ginzburg–Travkin:
  $G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$, $X = \mathrm{GL}_{n+1}$, $\check{G}_X = \check{G}$, $V_X = \mathrm{Hom}(k^n, k^{n+1}) \oplus \mathrm{Hom}(k^n, k^{n+1})^\vee$.

- Trivial period, $X = \mathrm{pt}$, $\mathrm{Shv}(LX/L^+G) = \mathrm{Shv}(\mathrm{pt}/\mathrm{L}^+\mathrm{G}) = \mathrm{Shv}(\mathrm{BG}) = \mathrm{k}[\mathfrak{t} /\!\!/ \mathrm{W}] = \mathrm{k}[\check{\mathfrak{t}}^* /\!\!/ \mathrm{W}]$.

“Corollary” of the conjecture, under a purity assumption: The unramified Plancherel formula, $k = \mathbb{C}$,

$$\left\langle 1_{X(\mathfrak{o})}, 1_{X(\mathfrak{o})} \right\rangle = \mathrm{tr}(\mathrm{Fr}, \mathbb{C}[V_X]^{\check{G}_X}) = \int_{\check{G}_X^{\mathrm{compact}}} L(\pi, V_X) d\pi,$$

Here and later, $L(\pi, V_X)$ denotes the special value of an L-function (or product thereof) at points depending on a grading — V_X is really a $\check{G} \times \mathbb{G}_m$-space, with $\mathbb{G}_m$-action depending on the cohomological grading.

E.g., in the group case, $X = H$, $\check{G}_X /\!\!/ \check{G} = \check{T}_H /\!\!/ W_H \ni \chi$, $d\pi = L(\chi, (\check{\mathfrak{h}}/\check{\mathfrak{t}}_H)^*, 0)^{-1}$, and we get the unramified Plancherel measure (up to zeta factors depending on our choice of measures)

$$\frac{L(\chi, \check{\mathfrak{h}}^*, 1)}{L(\chi, (\check{\mathfrak{h}}/\check{\mathfrak{t}}_H)^*, 0)}.$$

In the trivial case, $X = \mathrm{pt}$, we obtain

$$\langle 1_{\mathrm{pt}}, 1_{\mathrm{pt}} \rangle = \frac{1}{\mathrm{Vol}\, G(\mathfrak{o})} = \frac{1}{\# G(\mathbb{F}_q)} = L(\check{\mathfrak{t}}^* /\!\!/ W), \text{ the “motive of } G\text{”.}$$

More structures:

$$\mathrm{Shv}(LX/L^+G) = \mathrm{QC}^{/\!/}_{\text{perf}}(V_X/\check{G}_X),$$

Note that the right hand side can also be written as $\check{M}/\check{G}$, where $\check{M} = V_X \times^{\check{G}_X} \check{G}$.

Conjecture 2: $\check{M}$ has a natural symplectic structure, with moment map $\check{M} \to \check{\mathfrak{g}}^*$, and the equivalence of Conjecture 1 is equivariant with respect to the Satake category, i.e., the action of $\mathrm{Shv}(L^+G\backslash LG/L^+G)$ on the LHS corresponds to the action of $QC_{\text{perf}}(\check{\mathfrak{g}}^*[2]/\check{G})$ on the RHS.

The Poisson structure should follow by considering loop rotations coming from the action of $\mathbb{G}_m$ on $\mathfrak{o}$. This gives a deformation $\mathrm{Shv}_{\mathbb{G}_m}(L^+G\backslash LG/L^+G)$ of $\mathbb{G}_m$-equivariant sheaves, living over $\mathrm{Shv}(B\mathbb{G}_m) = k[\hbar]$, whose specialization at $\hbar = 0$ is the original category. Alternatively: factorization algebras, E_2-algebras, ...

Example: Tate's thesis. $X = \mathbb{A}^1$, $G = \mathbb{G}_m$, first without the "metaplectic correction". L^+G-strata on LX parametrized by integers, $X^n := t^n X(\mathfrak{o})$. The basic object $\mathcal{F}_0 = k_{X^0} = k_{L^+X}$, acting by the perverse sheaf $k_{t^n\mathbb{G}_m(\mathfrak{o})}$, we get $\mathcal{F}_n = k_{X^n}$. Let $\mathrm{Shv}(L^+G\backslash LG/L^+G) = \mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}(B\mathbb{G}_m) = \mathrm{QC}_{\text{perf}}(B\check{G}) \otimes k[\eta]$, with η in degree 2. Ext-groups:

$$\mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_j) = \begin{cases} k[\eta], \text{ if } i = j, \\ k[0] \otimes k[\eta], \text{ if } i < j, \\ k[-2(i-j)] \otimes k[\eta], \text{ if } i > j, \end{cases}$$

with generators $x \in \mathrm{Hom}^0(\mathcal{F}_i, \mathcal{F}_{i+1})$, $y \in \mathrm{Hom}^2(\mathcal{F}_{i+1}, \mathcal{F}_i)$ with $xy = \eta$.

The category $\mathrm{Shv}(LX/L^+G)$ is generated by those objects, and that means that we can identify

$$\mathrm{Shv}(LX/L^+G) \xrightarrow{\sim} QC_{\text{perf}}(\mathrm{Spec} k[x,y]/\check{G})$$

with x in degree 0, y in degree 2, and $\check{G} = \mathbb{G}_m$-action $z \cdot (x,y) = (zx, z^{-1}y)$, by sending $\mathcal{F}_i$ to $z^i \otimes k[x,y]$. Indeed,

$$\mathrm{Hom}^{\check{G}}_{k[x,y]\text{-mod}}(z^i \otimes k[x,y], z^j \otimes k[x,y]) = \begin{cases} k[x,y]^{\check{G}} = k[xy], \text{ if } i = j, \\ k[0] \otimes k[xy], \text{ if } i < j, \\ k[-2(i-j)] \otimes k[xy], \text{ if } i > j, \end{cases}$$

Moreover, $\mathrm{Spec} k[x,y] = T^*[2]\mathbb{A}^1$ is Hamiltonian with moment map $T^*[2]\mathbb{A}^1 \to \check{\mathfrak{g}}^* = \mathrm{Spec} k[\eta]$ given by $\eta = xy$, under which the above isomorphism becomes $\mathrm{Shv}(BG) = k[\eta]$-equivariant.

Conjecture 3: The association $M \mapsto \check{M}$ is involutive, when you switch the roles of G and $\check{G}$. Here, $M = T^*X$, or a symplectic space, or some more general "affine coisotropic Hamiltonian space" satisfying certain conditions.

[I'm told that this should reflect a correspondence of "boundary conditions under S-duality for the Kapustin–Witten 4D–TQFT" (see also Gaiotto–Witten).]

Examples:

attribution/name	M or (G, M)	$\check{M}$ or $(\check{G}, \check{M})$	attribution/name
Tate	$(\mathbb{G}_m, T^*\mathbb{A}^1)$	$(\mathbb{G}_m, T^*\mathbb{A}^1)$	Tate
Godement-Jacquet	$(\mathrm{GL}_n \times \mathrm{GL}_n, T^*M_n)$	$T^*(\mathbb{A}^n \times^{\mathrm{GL}_n} \check{G})$	Rankin-Selberg
Hecke	$T^*(\mathbb{G}_m \backslash \mathrm{PGL}_2)$	$(\mathrm{SL}_2, T^*\mathbb{A}^2)$	normalized Eisenstein series
Gross–Prasad	$\mathrm{SO}_{2n} \backslash \mathrm{SO}_{2n} \times \mathrm{SO}_{2n+1}$	$(\mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}, \mathrm{std} \otimes \mathrm{std})$	θ-correspondence
group	$\Delta H \backslash (H \times H)$	$\Delta'(\check{H}) \backslash (\check{H} \times \check{H})$	(twisted) group
point	$G \backslash G$	$(\check{N}, \psi) \backslash \check{G}$	Whittaker

Remark: We expect a more general story, where we can drop the "smooth affine" condition on one side and the "spherical/coisotropic" condition on the other, e.g., this is suggested by the example of toric varieties. Part of our interest in our conjectures is the possibility of studying non-unique periods; but this has not been our focus for now.

Recipe for building $\check{M}$ out of M: Would have to talk about structure of spherical varieties and their associated Hamiltonian spaces. No time today, see [S.–Jonathan Wang], or my older paper on "Spherical functions on spherical varieties".

3 Global conjecture

There is a whole hierarchy of conjectures, according to the paradigm of TQFT which associates to manifolds of different dimensions objects of different categorical depth:

Dim	"Manifold"	(G, M)-theory $\in G$-theory	$(\check{G}, \check{M})$-theory $\in \check{G}$-theory
3	global field k	X-theta series $\Theta_X \in C^\infty(G(k)\backslash G(\mathbb{A}))$	L-value
2	geometric function field $k = \mathbb{F}(C)$	Period sheaf $P_X \in \mathrm{Shv}(\mathrm{Bun}_G^C)$	L-sheaf $L_X \in \mathrm{QC}(\mathrm{Loc}_{\check{G}}^C)$
2	local field $F = k_v$	$\mathrm{Fns}(\mathrm{X}(\mathrm{F})) \in \mathrm{Rep}(\mathrm{G}(\mathrm{F}))$	(...)
1	the unramified closure $\bar{F}$	$\mathrm{Shv}(X(\bar{F})) \in (G(\bar{F})$-module cat.)	$\mathrm{QC}(\mathrm{Loc}^{\check{X}}) \in (\mathrm{QC}(\mathrm{Loc}_{\check{G}}^{D^*})$-module cat.)

Hence, there is a local conjecture relating the $G(F)$-category $\mathrm{Shv}(X(\bar{F}))$ to a $\mathrm{QC}(\mathrm{Loc}_{\check{G}}^{D^*})$-category associated to $\check{M}$ — *although: "spectral quantization" missing!*, see below. For Tate's thesis, this was proven recently by Sam Raskin and Justin Hilburn.

I will describe the global conjecture (over a curve C over an algebraically closed field $\mathbb{F}$, although you should be thinking of Frobenius actions when $\mathbb{F} = \overline{\mathbb{F}_q}$; no ramification). However, I need to know that

$$\check{M} = T^{*,\psi}\underbrace{(S_+ \times^{\check{G}_X \check{U}} \check{G})}_{\check{X}}, \text{ a } \check{G}\text{-polarization,}$$

and in fact I'll take $(\check{U}, \psi)$ to be trivial (for simplicity), so $\check{M} = T^*\check{X}$, for an honest $\check{G}$-space $\check{X}$ living over $\check{G}_X\backslash\check{G}$.

By Koszul duality the local conjecture can be rewritten (dropping boundedness of sheaves from now on)

$$\mathrm{Shv}(LX/L^+G) \simeq \mathrm{QC}^!(T[-1]\check{X}/\check{G}),$$

where $T[-1]\check{X}$, the shifted tangent bundle of $\check{X}$, is the derived self-intersection of $\check{X}$ (the derived fixed-point scheme of $\check{X}$ under the trivial action of $\check{G}$).

We will globalize it to a matching of objects

$$\mathcal{P}_X \leftrightarrow \mathcal{L}_X$$

under the conjectural geometric Langlands duality

$$\mathrm{Shv}(\mathrm{Bun}_G) \simeq \mathrm{QC}^!(\mathrm{Loc}_{\check{G}})$$

(meaning of "Shv" and "$\mathrm{QC}^!$" interdependent here), expressing the formulas relating theta series and L-functions.

The “period sheaf” (corresponding to the “X-theta series”) is $\mathcal{P}_X = \pi_! k$, where $\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G$.

Here, Bun_G^X is the stack of G-bundles together with a section to X:

$$\mathrm{Bun}_G^X = \mathrm{Map}(C, X/G) \to \mathrm{Bun}_G = \mathrm{Map}(C, BG).$$

The “L-sheaf” (corresponding to the special value of an L-function) is $\check{\pi}_* \omega$, where $\check{\pi} : \mathrm{Loc}_{\check{G}}^{\check{X}} \to \mathrm{Loc}_{\check{G}}$.

Here, $\mathrm{Loc}_{\check{G}}^{\check{X}}$ is the derived stack of $\check{G}$-local systems, together with a flat section to $\check{X}$, e.g., in the Betti setting, $\mathrm{Loc}_{\check{G}}$ classifies representations of the étale fundamental group $\rho : \pi_1(C) \to \check{G}$, and the fiber over ρ is the derived invariant scheme $\check{X}^\rho$. In the setting of de Rham local systems,

$$\mathrm{Loc}_{\check{G}}^{\check{X}} = \mathrm{Map}(C_{dR}, \check{X}/\check{G}) \to \mathrm{Loc}_{\check{G}} = \mathrm{Map}(C_{dR}, B\check{G}).$$

Basic case and numerical conjecture: Assume that a geometric Langlands parameter ($\check{G}$-local system) ρ only has isolated (classical) fixed points x_i on $\check{X}$. Then, the fiber of $\mathrm{Loc}_{\check{G}}^{\check{X}} \to \mathrm{Loc}_{\check{G}}$ over ρ is

$$\sum_i H^1(\rho, T_{x_i}\check{X}).$$

Given that $\check{X}$ is a vector bundle over $\check{G}_X\backslash\check{G}$ (say with fiber V), for the existence of fixed points the local system admits a reduction ρ_X to $\check{G}_X$, and we have $T_{x_i}\check{X} = V \oplus \check{\mathfrak{g}}/\check{\mathfrak{g}}_X$.

Applying the sheaf–function dictionary, suppose that ρ is restriction of a Langlands parameter (denoted by same letter), and f is the automorphic form associated to the skyscraper sheaf δ_ρ , then (assuming that ρ is a smooth point of $\mathrm{Loc}_{\check{G}}$)

$$\langle \Theta_X, f^\vee \rangle = \mathrm{tr}(\mathrm{Fr}, \mathrm{Hom}(\mathcal{L}_X, \delta_\rho)^\vee) = q^{-(g-1)\dim G}\,\mathrm{tr}(\mathrm{Fr}, \bigoplus_i \wedge^\bullet H^1(\rho, T_{x_i}\check{X})) = q^{-(g-1)\dim G} \sum_i L(\rho, T_{x_i}\check{X}).$$

(I will ignore the factor $q^{-(g-1)\dim G}$ from now on; it has to do with choices of measures I haven't explained.)

Examples (a lot of interesting scalars/measures swept under the rug! denoted by $\approx$):

1. Whittaker case: $\check{X} = \mathrm{pt}$, so
$$\langle \Theta_X, f^\vee \rangle \approx 1.$$
(Here, Θ_X is the Poincaré series, and the pairing computes the Whittaker coefficient of $f^\vee$.)

2. Group case: $X = H$, say semisimple, $\check{X} = \check{H}$ with Chevalley-twisted action of $\check{G} = \check{H} \times \check{H}$. Hence, for ρ to have fixed points, it has to be of the form $\rho = \tau \times \tau^\vee$.

   Assume τ to be geometrically elliptic (= its restriction to the geometric fundamental group does not lie in a proper Levi). Then, its centralizer $S_\tau \subset \check{H}$ is discrete, and corresponds to the (classical) fixed points on $\check{H}$. Their tangent space is $\check{\mathfrak{h}}$, and we get
$$\langle \Theta_X, f^\vee \rangle \approx |S_\tau| L(\tau, \check{\mathfrak{h}}, 0).$$

3. There is a version of the conjecture which doesn't require polarization, which arises when we consider endomorphisms of the period sheaf $\mathcal{P}_X$: Normalize the automorphic form $f \leftrightarrow \rho$ so that $\langle f, f \rangle = |S_\pi| L(\tau, \check{\mathfrak{g}}, 1)$, then
$$|\langle \Theta_X, f \rangle|^2 \approx \sum_i L(\rho, T_{x_i} \check{M}) \text{ — the Ichino–Ikeda conjecture.}$$