The orbit method, microlocal analysis and applications to L-functions

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Objectives

- Analyze L-functions on $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ and related groups.
- Determine asymptotics of "special functions" attached to representations of such groups.
- $n = 1$: Bernstein–Reznikov, Michel–Venkatesh, $(\dots)$: analysis via explicit formulas, complicated in higher rank.
- Develop the orbit method in the spirit of microlocal analysis, giving a soft approach that applies in higher rank.
- Standard problems (moments, subconvexity).

Contents

Special values of L-functions as branching coefficients

The orbit method

Microlocal analysis

Local applications

Global applications

Let (G, H) be one of

$$(\mathrm{GL}_{n+1}(\mathbb{R}), \mathrm{GL}_n(\mathbb{R})),$$

$$(\mathrm{U}(p+1,q), \mathrm{U}(p,q)), \quad (\mathrm{SO}(p+1,q), \mathrm{SO}(p,q)).$$

(G, H) is a strong Gelfand pair:[1] irreducible representations of G have multiplicity-free restriction to H.

[1] Sun–Zhu, Aizenbud–Gourevitch

Some notation

- Arithmetic lattices

$$\Gamma \leq G, \quad \Gamma_H = \Gamma \cap H \leq H.$$

- Quotients

$$[G] := \Gamma \backslash G, \quad [H] := \Gamma_H \backslash H.$$

  Assume compact.

- Irreducible subrepresentations

$$\pi \subseteq L^2([G]), \quad \sigma \subseteq L^2([H]).$$

Global restriction

Let $v \in \pi \subseteq L^2([G])$ be a smooth vector.

Restrict to $[H]$, spectrally decompose:

$$v\big|_{[H]} = \sum_{\sigma \subseteq L^2([H])} \text{"global" projection of } v \text{ to } \sigma.$$

Local restriction

For $\pi \in \mathrm{Irr}(G)$,

$$\pi|_H = \text{direct integral of some } \sigma \in \mathrm{Irr}(H).$$

This means: there are "local" projection maps $\pi \to \sigma$ such that

$$\|v\|^2 = \int_{\sigma \in \mathrm{Irr}(H)} \|\,\text{"local" projection of } v \text{ to } \sigma\|^2 \, d\sigma.$$

We may normalize so that $d\sigma = \text{Plancherel}$.

Branching coefficients

Strong Gelfand property implies

$$\text{global projection} = \text{scalar multiple of local projection.}$$

$\rightsquigarrow$ branching coefficient $\mathcal{L}(\pi, \sigma) \in \mathbb{R}_{\geq 0}$: for all $v \in \pi$,

$$\|\text{global projection of } v \text{ to } \sigma\|^2 =$$

$$\mathcal{L}(\pi, \sigma) \, \|\text{local projection of } v \text{ to } \sigma\|^2.$$

Quantifies how automorphic forms in π correlate with those in σ.

Branching coefficients and L-values

Refined Gan–Gross–Prasad conjecture, known in many cases:[2]

$$\mathcal{L}(\pi, \sigma) = \text{special value of an } L\text{-function.}$$

Example

In the $\mathrm{GL}_n(\mathbb{Z})$ case,[3]

$$\mathcal{L}(\pi, \sigma) \approx \frac{1}{4} \left| \frac{L(\pi \times \bar{\sigma}, 1/2)}{L(\mathrm{Ad}, \cdots, 1)} \right|^2 .$$

[2] Hecke, Waldspurger, Ichino–Ikeda, N. Harris, Jacquet–Rallis, W. Zhang, Beuzart-Plessis, Beuzart-Plessis–Liu–Zhang–Zhu, Beuzart-Plessis–Chaudouard–Zydor

[3] Jacquet–Piatetski-Shapiro–Shalika

Motivation for studying L-values

Problem: estimate special values of L-functions,

- individually ("subconvexity problem") or
- on average over a family ("moment problem").

Considered fundamental nowadays.

Motivated by questions/applications discovered in the 1980's and 1990's:

- random matrix heuristics for moments of families[4]
- existence and distribution of integral solutions of $Q(v) = n$ for a ternary quadratic form Q[5]
- arithmetic quantum unique ergodicity: $|\varphi_j|^2 \, d\mu \sim (?)$[6]

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[4] Keating–Snaith, Conrey–Farmer–Keating–Rubinstein-Snaith, ...

[5] Iwaniec, Duke, Duke–Schulze-Pillot, Duke–Friedlander–Iwaniec, ...

[6] Rudnick–Sarnak, Lindenstrauss, Holowinsky–Soundararajan, ...

Problems

To study $\mathcal{L}(\pi, \sigma)$ using the defining property[7]

$$\|\text{global projection of } v \text{ to } \sigma\|^2 =$$

$$\mathcal{L}(\pi, \sigma) \, \|\text{local projection of } v \text{ to } \sigma\|^2$$

we must understand the local and global projections.

Sample local problem

Let $\mathcal{F} \subseteq \mathrm{Irr}(H)$ be a nice family.
Can we construct a nice vector $v \in \pi$ so that

$$\|\text{local projection of } v \text{ to } \sigma\|^2 \approx \begin{cases} 1 & \text{if } \sigma \in \mathcal{F}, \\ 0 & \text{if } \sigma \notin \mathcal{F}? \end{cases}$$

Can we then estimate the matrix coefficients $\langle gv, v \rangle$?

[7] following Bernstein–Reznikov, ...

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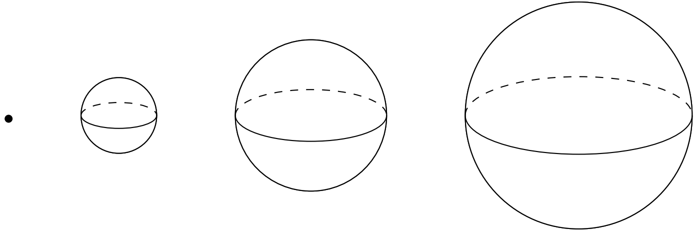
G: Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}^\wedge = \mathrm{Hom}(\mathfrak{g}, i\mathbb{R})$

Definition

A coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^\wedge$ is a G-orbit.

Example

$G = \mathrm{SO}(3)$, $\mathfrak{g}, \mathfrak{g}^\wedge \cong \mathbb{R}^3$, $\{\text{coadjoint orbits}\} = \{\text{spheres}\}$.



Theorem (Kirillov)

Lie algebra structure on $\mathfrak{g}$ defines a symplectic structure on $\mathcal{O}$, hence a symplectic volume form ω.

Orbit method: heuristic based on an (approximate) bijection

$$\mathrm{Irr}(G) \approx \{\text{coadjoint orbits}\}$$

$$\pi \longleftrightarrow \mathcal{O}_\pi$$

compatible with natural operations.

$\mathcal{O}_\pi$ should describe the character χ_π: for small $x \in \mathfrak{g}$,

$$\chi_\pi(\exp(x)) = j^{-1/2}(x) \int_{\xi \in \mathcal{O}_\pi} e^{\langle x, \xi \rangle} \, d\omega(\xi),$$

where $j = \mathrm{Jac}(\exp : \mathfrak{g} \to G)$, $j(0) = 1$.

Such an identity is called the Kirillov formula for π.
Valid for

- G nilpotent (Kirillov)
- G reductive, π tempered (Rossmann)

In particular,

$$\dim(\pi) = \chi_\pi(\exp(0)) = \mathrm{vol}(\mathcal{O}_\pi, d\omega).$$

Example

$G = \mathrm{SO}(3)$. Classify by highest weight:

$$\mathrm{Irr}(G) = \{\pi_T : T \in \mathbb{Z}_{\geq 0}\}$$

Kirillov formula for $\pi = \pi_T$ holds with

$$\mathcal{O}_\pi = \text{sphere of radius } T + \tfrac{1}{2},$$

$$\mathrm{vol}(\mathcal{O}_\pi, d\omega) = 2T + 1 = \dim(\pi).$$

Microlocal analysis on $\mathbb{R}^n$

The art of decomposing functions into pieces that are "localized" in both position and frequency, e.g.,

$$v_{(x_0,\xi_0)}(x) := \exp(-|x - x_0|^2 + i\xi_0 \cdot x).$$

Implemented using pseudodifferential operators $\mathrm{Op}(a)$ attached to suitable functions ("symbols") a on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$:

$$\mathrm{Op}(a) v_{(x_0,\xi_0)} \approx a(x_0, \xi_0) v_{(x_0,\xi_0)}.$$

Microlocal analysis on a representation π of a Lie group G

The art of decomposing vectors into pieces that are "localized" in a suitable sense: they are "approximate eigenvectors" for small elements of G.

Microlocal calculus

- Work with localized vectors implicitly via operators

$$\pi(f) = \int_{g \in G} f(g)\pi(g)\, dg$$

attached to $f \in C_c^\infty(G)$ supported near the identity.

- Describe f by pulling back to the Lie algebra and taking Fourier transform. Call the result $a \in \mathcal{S}(\mathfrak{g}^\wedge)$, and write

$$\pi(f) =: \mathrm{Op}(a).$$

- Kirillov formula (ignoring $j^{-1/2}$):

$$\mathrm{trace}(\mathrm{Op}(a)) \approx \int_{\mathcal{O}_\pi} a\, d\omega.$$

Heuristic. For each partition

$$\mathcal{O}_\pi = \sqcup \mathcal{P}_\tau$$

where $\mathcal{P}_\tau$ has volume one and is concentrated near $\tau \in \mathcal{O}_\pi$, there corresponds an orthonormal basis

$$\pi = \oplus \mathbb{C} v_\tau$$

such that

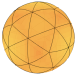
$$\mathrm{Op}(a) v_\tau \approx a(\tau) v_\tau$$

if a is essentially constant on $\mathcal{P}_\tau$.

We call v_τ localized at τ and $\mathcal{P}_\tau$ its microlocal support.

Example

$G = \mathrm{SO}(3)$, $\pi = \pi_T$ as above, $\mathcal{P}_\tau \approx \mathcal{O}_\pi \cap (\tau + \mathrm{O}(T^{1/2}))$



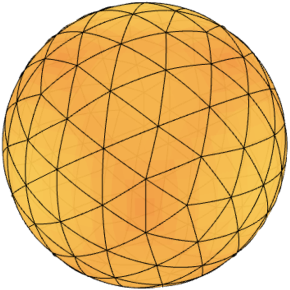


Figure: Localized partitions $\mathcal{O}_\pi = \sqcup \mathcal{P}_\tau$ for $T \approx 40, 160$

If

$$\partial^\alpha a(\xi) \ll_\alpha T^{(-1/2-\varepsilon)|\alpha|}.$$

then a varies by $\ll T^{-\varepsilon}$ on each $\mathcal{P}_\tau$.

Theorem (NV 2018, N 2020)

Let $\pi = \pi_T$ be a unitary representation of a Lie group G. The assignment $a \mapsto \mathrm{Op}(a)$, restricted to functions $a = a_T$ satisfying estimates as above, enjoys a reasonable microlocal calculus, e.g.:

1. If a is supported on elements of size $\asymp T$ and satisfies

$$\partial^\alpha a(\xi) \ll_\alpha T^{(-1/2-\varepsilon)|\alpha|},$$

   then $\mathrm{Op}(a)$ has operator norm $\mathrm{O}(1)$.
2. If G is reductive and π is irreducible, then $\mathrm{Op}(a)$ has trace norm $\mathrm{O}(T^d)$, where $2d = \dim(\mathcal{O}_\pi) = \dim(G) - \mathrm{rank}(G)$.
3. $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(a \star b)$, where $a \star b \sim ab$.
4. Similar results for polynomials, functions with less regularity transverse to the coadjoint orbits, etc.

Relative characters

For tempered (π, σ), consider the hermitian form

$$\mathcal{H}_\sigma : \pi \otimes \bar{\pi} \to \mathbb{C}$$

$$\mathcal{H}_\sigma(v \otimes v) = \|\text{local projection of } v \text{ to } \sigma\|^2.$$

For a nice function a on $\mathfrak{g}^\wedge$, define the relative character

$$\mathcal{H}_{\pi,\sigma}(a) := \sum_{v \in \mathcal{B}(\pi)} \mathcal{H}_\sigma(\mathrm{Op}(a)v \otimes v).$$

Example

If $a|_{\mathcal{O}_\pi} \approx$ characteristic function of $\mathcal{P}_\tau$, then

$$\mathcal{H}_{\pi,\sigma}(a) \approx \|\text{local projection of } v_\tau \text{ to } \sigma\|^2.$$

Question

For $a = a_T$ as in the microlocal calculus, what is the asymptotic behavior of $\mathcal{H}_{\pi,\sigma}(a)$? Here $(\pi, \sigma) = (\pi_T, \sigma_T)$.

Stability

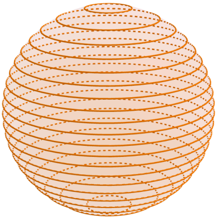
Set $\mathcal{O}_{\pi,\sigma} := \mathcal{O}_\pi \cap \text{preimage}(\mathcal{O}_\sigma)$.
Call (π, σ) stable if $\mathcal{O}_{\pi,\sigma}$: *H*-torsor (closed orbit, trivial stabilizer).

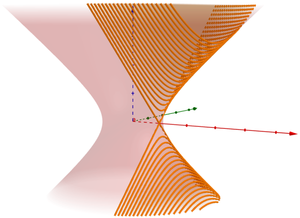
Fact

This is a generic condition, depending only upon the infinitesimal characters $(\lambda_\pi, \lambda_\sigma)$, equivalent to

$$\{\text{eigenvalues of } \lambda_\pi\} \cap \{\text{eigenvalues of } \lambda_\sigma\} = \emptyset.$$

Some $\mathcal{O}_{\pi,\sigma}$'s for $(\mathrm{SO}(3), \mathrm{SO}(2))$ and $(\mathrm{PGL}_2(\mathbb{R}), \mathrm{GL}_1(\mathbb{R}))$:





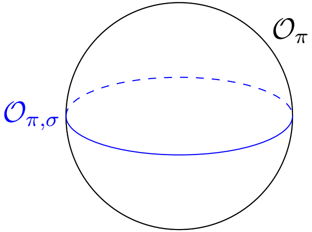
Relative character asymptotics in the stable case

Theorem (NV 2018)

Let $(\pi, \sigma) = (\pi_T, \sigma_T)$ be tempered and "uniformly stable": $T^{-1}(\lambda_\pi, \lambda_\sigma)$ lies in a fixed compact collection of stable pairs. Then for $a = a_T$ as in the microlocal calculus,

$$\mathcal{H}_{\pi,\sigma}(a) \sim \int_{\mathcal{O}_{\pi,\sigma}} a,$$

where we integrate by transporting Haar on H to its torsor $\mathcal{O}_{\pi,\sigma}$.



Informally, $\|\text{local projection of } v_\tau \text{ to } \sigma\|^2 \approx \mathrm{vol}(\mathcal{O}_{\pi,\sigma} \cap \mathcal{P}_\tau)$.

Sketch of proof

We first write

$$\mathcal{H}_{\pi,\sigma}(a) = \int_{h\in H} \operatorname{trace}(\pi(h)\operatorname{Op}(a))\overline{\chi_\sigma(h)}\,dh.$$

Then:

- Choose a localized partition $\mathcal{O}_\pi = \sqcup \mathcal{P}_\tau$.
  Key case: a concentrated near some $\mathcal{P}_\tau$ with $\tau \in \mathcal{O}_{\pi,\sigma}$.
- $\operatorname{trace}(\pi(h)\operatorname{Op}(a)) \approx \langle h v_\tau, v_\tau \rangle =$ negligible unless $h\tau \approx \tau$.
  Happens only if h is small, since stability $\implies H_\tau = \{1\}$.
- For small h, apply Taylor expansion and Kirillov formula.

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Overview

We determine the asymptotics of the following averages of branching coefficients:

1. (N–Venkatesh 2018)

$$\sum_{\sigma \in \mathcal{F}} \mathcal{L}(\pi, \sigma), \quad \mathcal{F} : \text{“long” family}$$

   with π fixed. Proof uses Ratner theory.

2. (N 2020)

$$\sum_{\pi \in \mathcal{F}} \mathcal{L}(\pi, \sigma), \quad \mathcal{F} : \text{“short” family, amplified.}$$

   Proof uses (amplified) relative trace formula, linear algebra.
   Corollary: subconvex bounds $\mathcal{L}(\pi, \sigma) \ll C(\pi, \sigma)^{1/4 - \delta_n}$.

“long” $\sim [T, 2T]$, “short” $\sim [T, T+1]$.
$\mathcal{F}$ must be uniformly stable; equivalently, $C(\pi, \sigma) \asymp T^{2n(n+1)}$.

Role of relative character asymptotics

Both proofs use

- construction of localized vectors via operator calculus,
- relative character asymptotics, and
- the following consequence of the definition of $\mathcal{L}(\pi, \sigma)$:

$$\sum_{v \in \mathcal{B}(\pi)} \sum_{u \in \mathcal{B}(\sigma)} \left(\int_{[H]} \mathrm{Op}(a) v \cdot \bar{u} \right) \left(\int_{[H]} \bar{v} \cdot u \right) = \mathcal{L}(\pi, \sigma) \mathcal{H}_{\pi,\sigma}(a).$$

We construct $a = a_T$ so that

$$\mathcal{H}_{\pi,\sigma}(a) \sim \int_{\mathcal{O}_{\pi,\sigma}} a$$

approximates the characteristic function of the family $\mathcal{F}$. Then

$$\sum_{\sigma \in \mathcal{F}} \mathcal{L}(\pi, \sigma) \approx \sum_{v \in \mathcal{B}(\pi)} \int_{[H]} \mathrm{Op}(a) v \cdot \bar{v},$$

$$\sum_{\pi \in \mathcal{F}} \mathcal{L}(\pi, \sigma) \approx \sum_{u \in \mathcal{B}(\sigma)} \int_{x, y \in [H]} \overline{u(x)} u(y) \sum_{\gamma \in \Gamma} f(x^{-1} \gamma y) \, dx \, dy$$

if $\mathrm{Op}(a) = \pi(f)$.

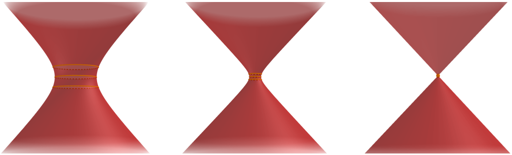
Estimating averages over σ

$$\sum_{\sigma \in \mathcal{F}} \mathcal{L}(\pi, \sigma) \approx \int_{[H]} [a], \quad [a] := \sum_{v \in \mathcal{B}(\pi)} \mathrm{Op}(a) v \cdot \bar{v}$$

Decompose into localized vectors:

$$[a] \approx \int_{\tau \in \mathcal{O}_\pi} a(\tau) |v_\tau|^2 \, d\omega(\tau).$$

For fixed π, since $\mathrm{supp}(a) \to \infty$, we can replace $\mathcal{O}_\pi$ by the nilcone:

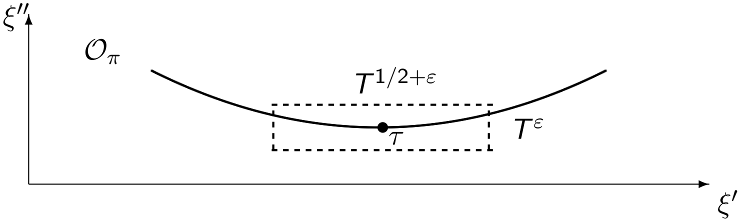


Any weak limit of the $|v_\tau|^2$ then attains unipotent invariance, hence (by Ratner) equidistributes. In particular, $[a]$ equidistributes.

Estimating averages over π

$$\sum_{\pi \in \mathcal{F}} \mathcal{L}(\pi, \sigma) \approx \sum_{u \in \mathcal{B}(\sigma)} \int_{x,y \in [H]} \overline{u(x)} u(y) \sum_{\gamma \in \Gamma} f(x^{-1} \gamma y) \, dx \, dy$$

$\sum_{\gamma \in \Gamma_H} \rightsquigarrow$ main term $\asymp T^{n(n+1)/2}$; apply amplification method.
Remains to show that $\Gamma - \Gamma_H$ gives a smaller contribution.
We may take a supported near some $\tau \in \mathcal{O}_{\pi,\sigma}$ with $|\tau| \asymp T$:



Key case: u localized at $\tau_H :=$ (restriction of τ) $\in \mathfrak{h}^\wedge$, otherwise $\int_{[H]} \mathrm{Op}(a) v \cdot \bar{u} =$ negligible.

- Microlocalization of u implies approximate equivariance under the centralizer H_{τ_H} of τ_H. Key problem: estimate

$$\max_{y \in H} \int_{x \in H} \int_{z \in H_{\tau_H}} |f(x^{-1}\gamma y z)| \, dz \, dx$$

- f concentrates on G_τ, so $f(x^{-1}\gamma y z)$ detects when $x\tau \approx \gamma y z \tau$.
- Key case: $y = 1$ and γ centralizes τ.
- Key problem: exhibit some transversality between the subvarieties $H\tau$ and $\gamma H_{\tau_H} \tau$ of $\mathcal{O}_\pi$.

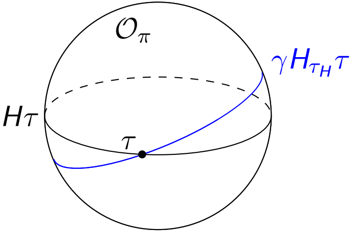


Figure: The "toy case" $(G, H) = (\mathrm{U}(2), \mathrm{U}(1))$, in which $H_{\tau_H} = H$.

- Numerology:

$$\dim(H_{\tau_H}\tau) = n, \quad \dim(H\tau) = n^2, \quad \dim(\mathcal{O}_\pi) = n^2 + n.$$

  Maybe $\gamma H_{\tau_H}\tau$ and $H\tau$ literally transverse for generic $\gamma \in \Gamma$? Need just a bit of transversality, but for every $\gamma \notin \Gamma_H$.
- Key idea: suffices to establish transversality, but with $H_{\tau_H}\tau$ (n-dimensional) replaced by $Z_H\tau$ (1-dimensional).
- Passing to the Lie algebra, reduce to a linear algebra problem:

Theorem (N 2020)

If $\tau \in \mathfrak{g}^\wedge$ is H-stable, $\gamma \in \mathfrak{g}_\tau - \mathfrak{z}$ and $z \in \mathfrak{z}_H - \{0\}$, then

$$[\gamma, [z, \tau]] \notin [\mathfrak{h}, \tau].$$

Theorem (N 2020)

Let M_n denote the space of $n \times n$ complex matrices. Embed $M_n \hookrightarrow M_{n+1}$ via

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Set $z := \mathrm{diag}(1, \dots, 1, 0) \in M_n$.
Let $\tau \in M_{n+1}$. Write

$$\tau = \begin{pmatrix} \tau_H & b \\ c & d \end{pmatrix}.$$

with $\tau_H \in M_n$. Let $\gamma \in M_{n+1}$. Assume that

- *no eigenvalue of τ is also an eigenvalue of τ_H,*
- $[\gamma, \tau] = 0$, *and*
- $[\gamma, [z, \tau]] = [y, \tau]$ *for some $y \in M_n$.*

Then γ is a scalar matrix.

Summary

- We develop the orbit method in analytic form as a microlocal calculus for Lie group representations, sharp up to ε's.
- We apply that calculus to determine relative character asymptotics in the stable regime.
- We deduce moment estimates and subconvex bounds in higher rank via some additional (local and global) arguments.

Theorem (NV 2018)

Assume $[G]$, $[H]$: compact.
Fix π: tempered, generic.
Let $T \to \infty$. Set

$$\mathcal{F}_T := \left\{ \sigma \subseteq L^2([H]) \middle| \begin{matrix} m_\pi(\sigma) = 1 \\ \lambda_\sigma \in T \cdot \Lambda \end{matrix} \right\},$$

with Λ: nice fixed compact collection of infinitesimal characters with all eigenvalues nonzero.
Then

$$\sum_{\sigma \in \mathcal{F}_T} \mathcal{L}(\pi, \sigma) \sim \frac{|\mathcal{F}_T|}{\mathrm{vol}(\Gamma \backslash G)}.$$

Remark

Translates to (average L-value) $\sim 2 \prod_p (\cdots)$, as predicted via random matrix heuristics.

Theorem (N 2020)

Let $(G, H) = (\mathrm{U}_{n+1}, \mathrm{U}_n)$. Assume $[H]$: compact.
Let $T \to \infty$. Let $(\pi, \sigma) = (\pi_T, \sigma_T)$ be Hecke-irreducible and tempered, with $T^{-1}(\lambda_\pi, \lambda_\sigma)$ in a fixed compact stable collection. Then

$$\mathcal{L}(\pi, \sigma) \ll T^{n(n+1)/2-\delta}$$

for some fixed $\delta = \delta_n > 0$.

Context

- Stability condition equivalent to "no conductor dropping":

$$C(\pi, \sigma) \asymp T^{2n(n+1)}.$$

- Translates to a subconvex bound for the corresponding L-function:

$$\mathcal{L}(\pi, \sigma) \ll C(\pi, \sigma)^{1/4-\delta}.$$

  In the everywhere-tempered case, $\delta = 1/(16n^5 + \mathrm{O}(n^4))$.

Example

$(G, H) = (\mathrm{SO}(3), \mathrm{SO}(2))$
$\pi \in \mathrm{Irr}(G)$ of highest weight $T \in \mathbb{Z}_{\geq 0}$, $\dim(\pi) = 2T + 1$
Weight space decomposition:

$$\pi|_H = \oplus_{\ell=-T}^{T} \sigma_\ell,$$

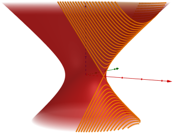
$\sigma_\ell =$ the subspace where H acts by $\theta \mapsto e^{i\ell\theta}$.

Example

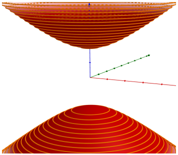
$G = \mathrm{PGL}_2(\mathbb{R}) \cong \mathrm{SO}(1,2)$

$$\mathfrak{g}^\wedge \ni i \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}$$

- one-sheeted hyperboloid $\mathcal{O}^+(r) = \{x^2 + y^2 - z^2 = r^2\}$



- two-sheeted hyperboloid $\mathcal{O}^-(k) = \{x^2 + y^2 - z^2 = -k^2\}$



Tempered irreducible representations:

- principal series $\pi(r, \varepsilon)$
- discrete series $\pi(k)$ for $k \in \mathbb{Z}_{\geq 1}$

$$\mathcal{O}_{\pi(r,\varepsilon)} = \mathcal{O}^+(r), \quad \mathcal{O}_{\pi(k)} = \mathcal{O}^-(k - 1/2)$$

Example

Take $(G, H) = (\mathrm{GL}_3(\mathbb{R}), \mathrm{GL}_2(\mathbb{R}))$, coadjoint orbits = nilcones:

$$\mathcal{O}_\pi = \mathcal{N} \subset \mathfrak{g}^\wedge, \quad \mathcal{O}_\sigma = \mathcal{N}_H \subseteq \mathfrak{h}^\wedge.$$

Then

$$\begin{aligned} \mathcal{O}_{\pi,\sigma} &= \mathcal{N} \cap \mathrm{preimage}(\mathcal{N}_H) \\ &= \left\{ \xi = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{R}) : \xi, A \text{ are nilpotent} \right\}. \end{aligned}$$

Very far from stable: $\#\{H\text{-orbits on } \mathcal{O}_{\pi,\sigma}\} = \infty$! Representatives

$$\begin{pmatrix} 0 & 1 & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$$

with bc the only invariant. Behavior of $\mathcal{H}_{\pi,\sigma}(a)$ remains unclear.