On the locally analytic vectors of completed cohomology of modular curves

holomorphic modular forms

$SL_2(\mathbb{R}) \curvearrowright \mathcal{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

$f : \mathcal{H} \longrightarrow \mathbb{C}$ m.f. of wt $k \in \mathbb{Z}$

(1) f is holomorphic, i.e. $\frac{d}{d\bar{z}} f = 0$.

(2) $f(\gamma(z)) = (cz+d)^k f(z)$

$\forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ z \in \mathcal{H}$

(3) holomorphic at infinity

automorphic form

$\rightsquigarrow \varphi_f : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / GL_2(\hat{\mathbb{Z}}) \xrightarrow{\text{smooth}} \mathbb{C}$

$\circlearrowleft GL_2(\mathbb{R})$

(1) $X = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{C})$

$X \cdot \varphi_f = 0$

(2) $\begin{pmatrix} a & -bi \\ bi & a \end{pmatrix} \cdot \varphi_f = k(a+b)\, \varphi_f$, $\begin{pmatrix} a & -bi \\ bi & a \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{C})$

$$\mathfrak{b}_{\mathbb{C}} = \mathbb{C}\cdot X + \left\{ \begin{pmatrix} a & -bi \\ bi & a \end{pmatrix} \right\} \subseteq \mathfrak{gl}_2(\mathbb{C})$$

Borel subalgebra

$\mathfrak{b}_{\mathbb{C}}$ acts on φ_f via a char. $\mu_{k,\mathbb{C}}$

holomorphic modular forms of wt. k = $\mu_{k,\mathbb{C}}$ - isotypic part of automorphic forms

---

Today : p-adic analogue

$$GL_2(\mathbb{R}) \rightsquigarrow GL_2(\mathbb{Q}_p)$$

$$\mathbb{C} \rightsquigarrow C = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$$

automorphic forms $\rightsquigarrow$ completed cohomology

overconvergent modular forms = μ_k - isotypic part of completed cohomology

Completed cohomology

- $K \subseteq GL_2(\mathbb{A}_f)$ open compact
- $Y_K \;/\mathbb{Q}$

$$Y_K(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash \mathcal{H}^{\pm} \times GL_2(\mathbb{A}_f) / K$$

- $K = K^p K_p$
  - $K^p \subseteq GL_2(\mathbb{A}_f^p)$
  - $K_p \subseteq GL_2(\mathbb{Q}_p)$

Fix $K^p \subseteq GL_2(\mathbb{A}_f^p)$

Def'n

$$\tilde{H}^i(K^p, \mathbb{Z}_p) := \varprojlim_n \varinjlim_{K_p \subseteq GL_2(\mathbb{Q}_p)} H^i(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^n)$$

$\uparrow$ dual

$$\tilde{H}_i(K^p, \mathbb{Z}_p) := \varprojlim_{K_p} H_i(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}_p)$$

$$\underset{GL_2(\mathbb{Q}_p)}{\circlearrowright}\tilde{H}^i(K^p, C) := \tilde{H}^i(K^p, \mathbb{Z}_p) \hat{\otimes}_{\mathbb{Z}_p} C$$

$\cup\!|$ Banach space $/C$

$\tilde{H}^i(K^p, C)^{la}$: $GL_2(\mathbb{Q}_p)$-locally analytic vectors

$\circlearrowright$

$\mathfrak{gl}_2(\mathbb{Q}_p)$

- $\mathfrak{g} = \mathfrak{gl}_2(C)$

$\cup\!|$ $\subseteq GL_2(\mathbb{Q}_p)$

- $\mathfrak{b} = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$, $B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$

- $k \in \mathbb{Z}$, $\mu_k : \mathfrak{b} \to C$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto kd$$

Determine $\tilde{H}^i(K^p, C)^{la}_{\underline{k}}$: μ_k-isotypic part

$$\tilde{H}^i(C)^{la}_k := \varinjlim_{K^p} \tilde{H}^i(K^p, C)^{la}_k$$

$\circlearrowright$

$GL_2(\mathbb{A}_f^p) \times B \times G_{\mathbb{Q}_p}$

- $Y_k / \mathbb{Q}_p$
- $G_{\mathbb{Q}_p} \curvearrowright C = \widehat{\overline{\mathbb{Q}}_p}$

Theorem 1 $k \neq 1$

$$H^1(C)^{la}_k = N_{k,1} \oplus N_{k,w}(k-1)$$

Hodge-Tate decomposition

(1) $N_{k,w} = M^{\dagger}_{2-k} \otimes \chi_{k,w}$ $\leftarrow$ a char of $GL_2(\mathbb{A}_f^p) \times B$

$(k \neq 2)$

$M^{\dagger}_{2-k}$: overconvergent modular forms of wt $2-k$

(2) $k \geq 2$, $N_{k,1} = M^{\dagger}_k / M_k \otimes \chi_{k,1}$

M_k: classical modular forms of wt k.

$$M_k = \varinjlim_{K \subseteq GL_2(\mathbb{A}_f)} M_k(K)$$

$$M_k(K) = H^0(X_{K,C}, \omega^k)$$

$Y_K \subseteq X_K$, $X_{K,C} = X_K \times C$

$k \leq -1$,

$$0 \to H^1(\omega^k) \otimes \mathcal{X}_{k,1} \to N_{k,1} \to M_k^{\dagger} \otimes \mathcal{X}_{k,1} \to 0$$

$$H^1(\omega^k) := \varinjlim_{K} H^1(X_{K,c}, \omega^k)$$

$$M_k^{\dagger} = \varinjlim_{K \subseteq GL_2(\mathbb{A}_f)} M_k^{\dagger}(K)$$

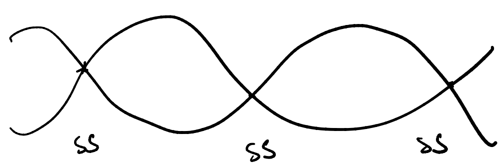
$\mathcal{X}_K$: rigid analytic space $/ \mathbb{C}$ of $X_{K,\mathbb{C}}$

$\cup$ open

$\mathcal{X}_{K,c}$: canonical locus

$M_k^{\dagger}(K)$: sections of ω^k defined on a strict nbhd of $\mathcal{X}_{K,c}$

- $K = K^p \Gamma(p^n)$, $\Gamma(p^n) = 1 + p^n M_2(\mathbb{Z}_p)$
- Katz - Mazur : $\mathfrak{X}_K / \mathcal{O}_C$



Irred. components $\longrightarrow$ $\varphi : (\mathbb{Z}/p^n)^2 \twoheadrightarrow \mathbb{Z}/p^n$

$$C \longmapsto \varphi_C$$

$\mathcal{X}_{K,c}$: tubular nbhd of

$$\bigcup_{C,\ \varphi_C(1,0)=0} C^{\circ}$$

remove all ss points

<u>Thm 2</u> <u>$k=1$</u>

$$\tilde{H}^1(C)^{la}_1 = \begin{array}{c} M_1^+ \otimes \chi_1 \\ \hline H^1(\omega) \otimes \chi_1 \\ \hline M_1^+/M_1 \otimes \chi_1 \end{array}$$

Sen operator

sub $\rightarrow$ M_1^+/M_1

$$\begin{array}{c} M_1^+/M_1 \\ \hline M_1 \\ \hline H^1(\omega) \\ \hline M_1^+/M_1 \end{array}$$

Sen operator

$$\tilde{H}^1(C)^{la,0}_1 := \begin{array}{c} M_1 \\ \hline H^1(\omega) \\ \hline M_1^+/M_1 \end{array}$$

"Hodge-Tate wt 0 part"

$\cdot \quad (\tilde{H}^1(C)_1^{la})^{G_{\mathbb{Q}_p}} \subseteq \tilde{H}^1(C)_1^{la,0}$

Applications

I. classicality

$f \in M_1^{\dagger}(K)$ is an eigenform with evs $\in \bar{\mathbb{Q}}_p$

$\circlearrowleft$

T_ℓ, U_ℓ, U_p

$\rightsquigarrow \rho_f : G_{\mathbb{Q}} \longrightarrow GL_2(\bar{\mathbb{Q}}_p)$

If $f \in M_1(K)$,

Deligne - Serre: ρ_f has finite image.

Theorem 3

$f \in M_1(K) \iff \rho_f(G_{\mathbb{Q}})$ is finite

i.e. f is classical

$\iff \rho_f(I_{\mathbb{Q}_p})$ is finite

(Sen: $\rho_f|_{G_{\mathbb{Q}_p}}$ is
Hodge-Tate of wts $0, 0$)

II. characterization of $M_k^\dagger(K)$

Theorem 4

$f \in M_k^\dagger(K)$ eigenform

evs $\in \overline{\mathbb{Q}}_p$

Then ρ_f has Hodge-Tate-Sen wts

$0, k-1$.

Conversely, if

$\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$

(1) $\rho|_{G_{\mathbb{Q}_\ell}}$ unramified a.a. ℓ

(2) odd

(3) $\rho|_{G_{\mathbb{Q}_p}}$ has Hodge-Tate-Sen wts

$0, k-1$, irreducible

(4) mild hypothesis $\bar{\rho}$.

then ρ arises from an overconvergent eigenform.

III. Fontaine - Mazur conjecture in the irregular case.

Key construction

Scholze: $\mathcal{X}_{K^p} \sim \varprojlim_{K_p} \mathcal{X}_{K^pK_p}$

perfectoid space

. $\pi_{HT} : \mathcal{X}_{K^p} \longrightarrow \mathcal{F}\ell = \mathbb{P}^1$

$GL_2(\mathbb{Q}_p) \curvearrowright \mathcal{O}_{K^p} := \pi_{HT *} \mathcal{O}_{\mathcal{X}_{K^p}}$

$\tilde{H}^i(K^p, C) \cong H^i(\mathcal{F}\ell, \mathcal{O}_{K^p})$

. $\mathcal{O}_{K^p}^{la} \subseteq \mathcal{O}_{K^p}$ Locally analytic section

<u>Fact</u> $\quad \tilde{H}^i(K^p, C)^{la} \simeq H^i(\mathcal{F}\ell, \mathcal{O}^{la}_{K^p})$

· $\mathcal{O}^{la}_{K^p}$ satisfies a 1-st order differential eq'n (p-adic Cauchy-Riemann eq'n)

horizontal nilpotent subalgebra on $\mathcal{F}\ell$.