On the geometric connected components of moduli of p-adic shtukas.
(Work in progress)

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Local Shimura datum

- $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{un}}$. σ lift of Frobenious. $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$.
- Formalism conjectured by Rapoport-Viehmann. Materialized by Scholze-Weinstein.
- p-adic shtuka datum is a triple: (G, b, μ).
- $G/\mathbb{Q}_p$ reductive group.
- $b \in G(\breve{\mathbb{Q}}_p)$.
- $\mu \in \{\mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}\}/G$ conjugacy class of cocharacters.
- If μ is minuscule $\implies$ (G, b, μ) is local Shimura datum.
- In analogy with Shimura datum (G, ν).
- Just as $\nu \rightsquigarrow$ Hodge structure. $(b, \mu) \rightsquigarrow$ "p-adic Hodge structure".
- $b \rightsquigarrow$ "isocrystal". $\mu \rightsquigarrow$ "filtration".

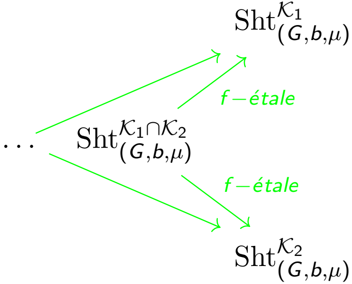
Some gadgets

We can associate:

1. $E = E(\mu)$ reflex field (of conj. class).
   - Consider the Weil group W_E.
2. Reductive group J_b.
   - σ-centralizer of b.
   - $J_b(\mathbb{Q}_p) = \{g \in G(\breve{\mathbb{Q}}_p) \mid g^{-1}b\sigma(g) = b\}$
   - Inner form of a Levi subgroup.
   - $J_b(\mathbb{Q}_p)$ is "Automorphism group of isocrystal".
3. A p-adic period domain $Gr_{B_{dR}}^{\leq\mu}/\mathbb{C}_p$.
   - Some geometric object of p-adic analytic geometry. (Scholze's diamonds).
   - If μ is minuscule $\mathscr{F}\ell_\mu = G/P_\mu$. In particular, it is a rigid-analytic space.
   - Open subset $\mathscr{F}\ell_\mu^b \subseteq \mathscr{F}\ell_\mu$. Called admissible locus.
   - Comes equipped with $J_b(\mathbb{Q}_p) \times W_E$-action.

Local Shimura Varieties/moduli of p-adic shtukas.

- Tower parametrized by compact open subgroups $\mathcal{K} \subseteq G(\mathbb{Q}_p)$:
- $J_b(\mathbb{Q}_p) \times W_E$-equivariant tower of analytic spaces:



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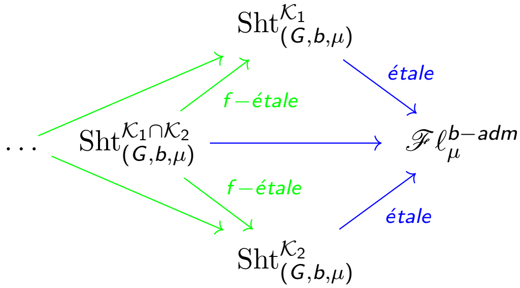

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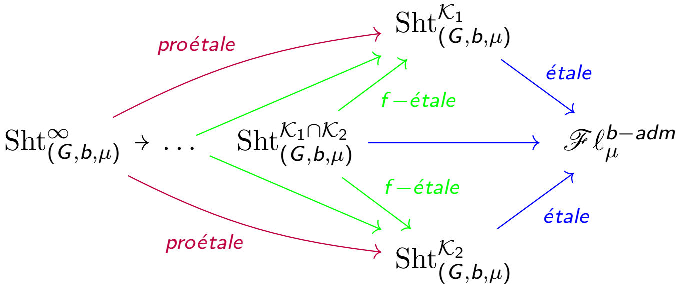


- $\mathrm{Sht}^{\infty}_{(G,b,\mu)} \to \mathscr{F}\ell^{b-adm}_{\mu}$. A $G(\mathbb{Q}_p)$ proétale Galois cover.
- $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ acts on $\mathrm{Sht}^{\infty}_{(G,b,\mu)}$. Geometric incarnation of LLC and JLC. (Kottwitz conjecture).

Example

- If $G = GL_n$.
- $b = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 1 \\ p & 0 & 0 & \dots & 0 \end{pmatrix}$, $\mu = (1, 0, \dots, 0)$,
- Then $\mathrm{Sht}_{(GL_n, b, \mu)}$ is the Lubin-Tate tower. Used on Harris-Taylor's proof of LLC.
- $\mathscr{F}\ell_\mu^b = \mathbb{P}^{n-1}$.
- $\mathrm{Sht}_{(GL_n, b, \mu)}^{GL_n(\mathbb{Z}_p)} = \mathcal{M}_\eta$ it is the Raynaud generic fiber of a formal scheme.
- $\mathcal{M}$ is deformation space of 1-dimensional formal p-divisible group of height n.

Main Goal

- Describe connected components. $\pi_0(\mathrm{Sht}^{\infty}_{(G,b,\mu)}) \Leftrightarrow \pi_0(\mathit{Tower})$.
- Understand $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$-action.
- Consequences:
  1. Explicit computation of $H^0(\mathrm{Sht}_{(G,b,\mu)})$.
  2. $H^0(\mathrm{Sht}_{(G,b,\mu)})$ acts on rest of cohomology through "cup product". Allows twists by characters.
  3. Expresses geometrically: LLC and JLC are compatible with character twists.

Case of tori

If G is a torus:

- $\mathscr{F}\ell_\mu = \{*\}$.
- $\mathrm{Sht}^\infty_{(G,b,\mu)} \cong G(\mathbb{Q}_p) \times \{*\}$. Free $G(\mathbb{Q}_p)$-torsor.
- Let $x \in \pi_0(\mathrm{Sht}^\infty_{(G,b,\mu)})$.
- $J_b(\mathbb{Q}_p) = G(\mathbb{Q}_p)$ canonically. Take $j \in J_b(\mathbb{Q}_p)$ and $\gamma \in W_E$ then:

$$x \cdot_{J_b} j = x \cdot_G j^{-1}$$

$$x \cdot_{W_E} \gamma = x \cdot_{G(\mathbb{Q}_p)} [Nm \circ \mu \circ Art_E(\gamma)^{-1}]$$

$$W_E \xrightarrow{Art_E} E^\times \xrightarrow{(-1)} E^\times \xrightarrow{\mu} G(E) \xrightarrow{Nm} G(\mathbb{Q}_p)$$

Main Result

Theorem (G.)

Assume G unramified, (b, μ) HN-irreducible and G^{der} is simply connected, then $det : (G, b, \mu) \to (G^{ab}, det(b), det(\mu))$ gives

$$\det : \mathrm{Sht}^{\infty}_{(G,b,\mu)} \to \mathrm{Sht}^{\infty}_{(G^{ab},det(b),det(\mu))}$$

and induces a $G(\mathbb{Q}_p) \times \mathrm{J}_b(\mathbb{Q}_p) \times W_E$-equivariant bijection

$$\pi_0(\det) : \pi_0(\mathrm{Sht}_{(G,b,\mu)}) \xrightarrow{=} \pi_0(\mathrm{Sht}_{(G^{ab},det(b),det(\mu))})$$

- Canonical map $G(\mathbb{Q}_p) \times \mathrm{J}_b(\mathbb{Q}_p) \times W_E \to G^{ab}(\mathbb{Q}_p) \times G^{ab}(\mathbb{Q}_p) \times W_E$.

Remarks

1. De Jong (1994): Lubin-Tate case over $\mathbb{Q}_p$.
2. Strauch(2006): Lubin-Tate case over arbitrary local field (including ramification).
3. M. Chen(2012): Computation for tori. Introduced determinant map.
4. M. Chen(2014): Unramified Rapoport-Zink space of EL or PEL type.
5. New for local Shimura varieties associated to exceptional reductive groups. We handle μ non-minuscule. We handle G^{der} non-simply connected.
6. Central strategy is the same, but techniques and difficulties are very different.

Road map for de Jong + M. Chen

$$
\begin{array}{ccccc}
\mathcal{M}_\infty & & \mathcal{M} & \longleftarrow & \mathcal{M}^{red}_{\mathbb{F}_p} & & \pi_0(\mathcal{M}^{\mathcal{K}}_\eta) \\
\downarrow & \nearrow & & \overset{\cong_{\pi_0}}{\nearrow} & \downarrow \cong & & \downarrow \cong \\
\mathcal{M}^{\mathcal{K}}_\eta & & & & X^{\mathcal{K}}_{\leq\mu}(b) & & \pi_0(X^{\mathcal{K}}_{\leq\mu}(b))
\end{array}
$$

Blue From infinite to hyperspecial level.

- Group theoretic techniques.
- M. Chen's work on "generic" crystalline representations.

Green Theory of Rapoport-Zink spaces.

Magenta Formal smoothness of unramified RZ spaces.
$\Gamma(\mathcal{M}^{\mathcal{K}}_\eta, \mathcal{O}^+) = \Gamma(\mathcal{M}, \mathcal{O})$

Brown Dieudonné Theory

Red Computation of connected components of minuscule affine Deligne Lusztig varieties . (Chen-Kisin-Viehmann).

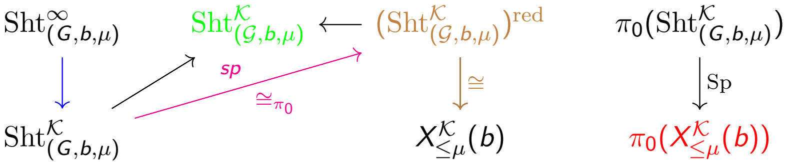
Integral models

- Main difficulty: What are integral models for diamonds? We make a modest attack to this question.
- Solve analogy:

$$\textit{Rigid analytic spaces} \rightsquigarrow \textit{Diamonds}$$
$$\textit{Formal schemes} \rightsquigarrow ?$$

- First approximation to "?": Kimberlites

Road map for us



- **Blue** Adapt Chen's strategy to diamonds.
- **Green** Scholze-Weinstein propose a model.
- **Magenta** Specialization maps for kimberlites. Prove SW are kimberlites.
- **Brown** Reduction functor for kimberlites.
- **Red** Computation of connected components of general ADLV. (Chen-Kisin-Viehmann, He-Zhou, Nie).

Main technical theorem

Theorem (G.)

Let $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ a hyperspecial subgroup. There is a specialization map $\mathrm{Sp} : |\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}| \to |X^{\mathcal{K}}_{\leq\mu}(b)|$ *that satisfies the following properties:*

- *Continuous and spectral (of locally spectral spaces).*
- $\mathrm{J}_b(\mathbb{Q}_p)$*-equivariant.*
- *A quotient map.*
- *Induces a bijection* $\pi_0(\mathrm{Sp}) : \pi_0(\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}) \cong \pi_0(X^{\mathcal{K}}_{\leq\mu}(b))$.

Kimberlites

- $\{Diamonds\} \subseteq \{v - sheaves\} \supseteq \{Kimberlites\}$.
- (Lourenço) For "nice" formal schemes $\mathfrak{X}/\mathbb{Z}_p \mapsto (X_\eta, \mathfrak{X}^{\mathrm{red}}, \mathrm{Sp})$ is a fully-faithful embedding.
- Axioms on a v-sheaf $\mathcal{F}$ that allow us to construct a triple!!! $\mathcal{F} \mapsto (\mathcal{F}^{an}, \mathcal{F}^{\mathrm{red}}, \mathrm{Sp})$

How to construct $\mathcal{F}^{\mathrm{red}}$ and $\mathcal{F}^{an}$?

- Analogy: $$\begin{array}{c}\{\text{Reduced Schemes}\}\\ \iota\downarrow \quad \uparrow(-)^{\mathrm{red}}\\ \{\text{Formal Schemes}\}\end{array}$$ Adjunction!
- Scholze constructs fully-faithful functor:

$$\Diamond : \{\textit{Perfect Schemes}\} \to \{v - \textit{sheaves}\}$$

- Observation: $\Diamond$ admits a right adjoint $(-)^{\mathrm{red}}$ (sort of). Not always a scheme!!!.
- If $[(\mathcal{F})^{\mathrm{red}}]^{\Diamond} \to \mathcal{F}$ is closed immersion, then $\mathcal{F}^{an} := \mathcal{F} \setminus [(\mathcal{F})^{\mathrm{red}}]^{\Diamond}$. Complementary open subsheaf.

How to construct *Sp*?

- Descend it from a known case: Let (R, R^+) be a Tate Huber pair, $\varpi \in R$ is pseudo-uniformizer.
- $\mathrm{Spa}(R, R^+) \subseteq \mathrm{Spa}(R^+, R^+)$. Also, one computes $\mathrm{Spd}(R^+, R^+)^{\mathrm{red}} = \mathrm{Spec}(R^+/\varpi)^{perf}$.
- There is a specialization map $Sp : |\mathrm{Spa}(R^+, R^+)| \to |\mathrm{Spec}(R^+/\varpi)|$. Formula:

$$x \mapsto \mathfrak{p}_x = \{r \in R^+ \mid |r|_x < 1\}$$

$$\begin{array}{ccc} |\,\mathrm{Spd}(R^+, R^+)\,| & \xrightarrow{\text{cover}} & |\,\mathcal{F}\,| \\ \downarrow \mathrm{sp}_{R^+} & & \downarrow \mathrm{sp}_{\mathcal{F}} \\ |\,\mathrm{Spd}(R^+, R^+)^{\mathrm{red}}\,| & \longrightarrow & |\,\mathcal{F}^{\mathrm{red}}\,| \end{array}$$

The axioms:

Definition

$\mathcal{F}$ is a kimberlite if:

- *$\mathcal{F}$ is v-locally formal (covered by $\mathrm{Spd}(R^+, R^+)$ in the sense of v-sheaves).*
- *$\mathcal{F}$ is separated.*
- *$(\mathcal{F}^{\mathrm{red}})^{\Diamond} \to \mathcal{F}$ is a closed immersion.*
- *$\mathcal{F}^{\mathrm{red}}$ is represented by a perfect scheme.*
- *$\mathcal{F}^{an} := \mathcal{F} \setminus (\mathcal{F}^{\mathrm{red}})^{\Diamond}$ is a locally spatial diamond.*

Examples:

- If $\mathfrak{X}/\mathbb{Z}_p$ is a separated formal scheme then $\mathfrak{X}^\diamondsuit$ is a kimberlite.
- Product of kimberlites are kimberlites.
- Exotic example $\mathbb{Z}_p^\diamondsuit \times_{\mathbb{F}_p^\diamondsuit} \mathbb{Z}_p^\diamondsuit$ is a kimberlite.
  $((\mathbb{Z}_p^\diamondsuit \times_{\mathbb{F}_p^\diamondsuit} \mathbb{Z}_p^\diamondsuit)^{\mathrm{red}})^\diamondsuit = \mathbb{F}_p^\diamondsuit$
- Main example: Scholze-Weinstein's integral model $\mathrm{Sht}^{\mathcal{K}}_{(\mathcal{G},b,\mu)}$ is a kimberlite, we have an open immersion $(\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}) \subseteq (\mathrm{Sht}^{\mathcal{K}}_{(\mathcal{G},b,\mu)})^{an}$, and $(\mathrm{Sht}^{\mathcal{K}}_{(\mathcal{G},b,\mu)})^{\mathrm{red}} = X^{\mathcal{K}}_{\leq\mu}(b)$.

Theorem

(G.) Given a kimberlite $\mathcal{F}$, there is a canonical, well-defined specialization map $\mathrm{Sp} : |\mathcal{F}^{an}| \to |\mathcal{F}^{\mathrm{red}}|$*. It is continuous and a spectral map of locally spectral spaces.*

Theorem

(G.) Given a kimberlite $\mathcal{F}$*, there is a canonical, well-defined specialization map* $\mathrm{Sp} : |\mathcal{F}^{an}| \to |\mathcal{F}^{\mathrm{red}}|$*. It is continuous and a spectral map of locally spectral spaces.*

Main technical theorem

Theorem (G.)

There is a specialization map $\mathrm{Sp} : |\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}| \to |X^{\mathcal{K}}_{\leq\mu}(b)|$, *it satisfies the following properties:*

- *Continuous and spectral*
- $\mathrm{J}_b(\mathbb{Q}_p)$*-equivariant.*

Remains:

- *Induces a bijection* $\pi_0(\mathrm{Sp}) : \pi_0(\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}) \cong \pi_0(X^{\mathcal{K}}_{\leq\mu}(b))$.

Ingredients: $\pi_0(\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}) \cong \pi_0(X^{\mathcal{K}}_{\leq\mu}(b))$

- Tubular neighborhoods: given $x \in |\mathcal{F}^{\mathrm{red}}|$ a closed point define:

$$\widehat{\mathcal{F}}_{/x} = \mathrm{Sp}^{-1}(x)^{\circ} \subseteq \mathrm{Sp}^{-1}(x)$$

- Admits moduli interpretation. It is an open subdiamond $\widehat{\mathcal{F}}_{/x} = \mathrm{Sp}^{-1}(x)^{\circ} \subseteq \mathcal{F}^{an}$
- Point-set topology arguments in the constructible topology reduces to prove: $|\widehat{\mathrm{Sht}^{\mathcal{K}}_{(\mathcal{G},b,\mu)}}_{/x}|$ connected for all $x \in |X^{\mathcal{K}}_{\leq\mu}(b)|$.

Ingredients: $\pi_0(\mathrm{Sht}^{\mathcal{K}}_{(G,b,\mu)}) \cong \pi_0(X^{\mathcal{K}}_{\leq\mu}(b))$

- Classic theme: $\mathscr{F}\ell_\mu$ and $\mathcal{M}$ have isomorphic tubular neighborhoods. Prove "Kimberlite version" for $\mathrm{Sht}^{\mathcal{K}}_{(\mathcal{G},b,\mu)}$ and $Gr^{\leq\mu}_{B_{dR}}$.
- This finishes the proof for minuscule cocharacters since $Gr^{\leq\mu}_{B_{dR}}$ is represented by a smooth formal scheme.
- For non-minuscule cocharacters we use a "Kimberlite version" of the Demazure resolution.

This is the end

Thanks!!!