An infinitesimal variant of Guo-Jacquet trace formulae and its comparison

based on my thesis supervised by P.-H. Chaudouard

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Relative Langlands programme

- Aim: periods of automorphic forms $\longleftrightarrow$ special values of L-functions
- In certain cases: obtained by comparison of periods (i.e. integrals over subgroups) on different groups $\longrightarrow$ reduced to known cases
- Relative trace formula: reduced to comparison of relative orbital integrals
- Important steps: fundamental lemma and transfer
- Examples: Gan-Gross-Prasad conjecture, Guo-Jacquet conjecture (today's topic), etc.

An example: linear periods and the Guo-Jacquet conjecture

Notations:

- E/F: a quadratic extension of number fields
- η: the quadratic character $\mathbb{A}^\times/F^\times \to \{\pm 1\}$ associated to E/F
- $G := GL_{2n}$
- $H := GL_n \times GL_n$
- $Z := \mathrm{Cent}(G)$
- π: a cuspidal representation of $G(\mathbb{A})$ with trivial central character

We define the periods by the linear forms:

$$\mathcal{P}_H : V_\pi \to \mathbb{C}, \phi \mapsto \int_{H(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)dh$$

and

$$\mathcal{P}_{H,\eta} : V_\pi \to \mathbb{C}, \phi \mapsto \int_{H(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)\eta(\det(h))dh$$

An example: linear periods and the Guo-Jacquet conjecture

We say that π is *H-distinguished* if both of $\mathcal{P}_H$ and $\mathcal{P}_{H,\eta}$ are not identically zero.

Friedberg-Jacquet: this property is directly related to the non-vanishing of central L-values

Theorem (Friedberg-Jacquet, 1993)

π is H-distinguished if and only if the following conditions are satisfied:

- the exterior square L-function $L(s, \pi, \Lambda^2)$ of π has a pole at $s = 1$;
- $L(\frac{1}{2}, \pi_E) \neq 0$ where π_E is the quadratic base change of π.

Conjectural generalisation of Waldspurger's formula in higher ranks: relate (the non-vanishing of) $L(\frac{1}{2}, \pi_E)$ to (that of) other periods $\longrightarrow$ motivation of the Guo-Jacquet conjecture

An example: linear periods and the Guo-Jacquet conjecture

Notations:

- D: a quaternion algebra over F containing E
- $G' := GL_n(D)$
- $H' := GL_n(E)$
- we identify $\mathrm{Cent}(G')$ with Z
- π': a cuspidal representation of $G'(\mathbb{A})$ with trivial central character

We define the period by the linear form:

$$\mathcal{P}_{H'} : V_{\pi'} \to \mathbb{C}, \phi \mapsto \int_{H'(F)Z(\mathbb{A})\backslash H'(\mathbb{A})} \phi(h)dh$$

We say that π' is H'-distinguished if $\mathcal{P}_{H'}$ is not identically zero.

An example: linear periods and the Guo-Jacquet conjecture

Conjecture (Guo-Jacquet, 1996)

Suppose that π' and π are associated by the Jacquet-Langlands correspondence. If π' is H'-distinguished, then π is H-distinguished.

Remarks:

- a converse is also expected at least for odd n;
- the case $n = 1$ was first discovered by Waldspurger (1985), and then reproved by Jacquet (1986) via the relative trace formula;
- some cases were obtained by Feigon-Martin-Whitehouse (2018) via a simple relative trace formula.

Relative trace formula approach (formal description)

We consider the right involution of $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$ on $L^2(G(F)\backslash G(\mathbb{A}))$. We obtain an integral operator $R(f)$ of kernel

$$k_f(x,y) := \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

The relative trace formula for the case of (G,H) is an equality between two (geometric and spectral) developments of

$$\int_{[H]}\int_{[H]} k_f(x,y)\eta(\det(x))dxdy,$$

where $[H] := H(F)\backslash H(\mathbb{A}) \cap G(\mathbb{A})^1$.

Relative trace formula approach (formal description)

Roughly, we should have

$$\sum_{\gamma \in H(F)\backslash G(F)/H(F)} J_\gamma(f) = \sum_{\text{cusp. rep. } \pi} J_\pi(f) + \dots$$

where $J_\gamma(f)$ is the relative orbital integral, and $J_\pi(f)$ is related to the periods $\mathcal{P}_H$ and $\mathcal{P}_{H,\eta}$.

Similarly, we also expect the relative trace formula (but without η involved) for the case of (G', H'):

$$\sum_{\gamma' \in H'(F)\backslash G'(F)/H'(F)} J_{\gamma'}(f') = \sum_{\text{cusp. rep. } \pi'} J_{\pi'}(f') + \dots$$

Relative trace formula approach (formal description)

Recall:

- The Guo-Jacquet conjecture concerns the comparison of periods.
- Relative trace formula: it suffices to compare relative orbital integrals
- Known: fundamental lemma (Guo) and smooth transfer (C. Zhang)

Fundamental lemma (Guo, 1996)

At almost all unramified places v, the basic functions $\mathbf{1}_{G(\mathcal{O}_{F_v})}$ and $\mathbf{1}_{G'(\mathcal{O}_{F_v})}$ have "matching" local orbital integrals.

Transfer for orbital integrals (C. Zhang, 2015)

At any non-archimedean place v, every $f' \in \mathcal{C}_c^\infty(G'(F_v))$ "matches" some $f \in \mathcal{C}_c^\infty(G(F_v))$ and vice versa.

Relative trace formula approach (formal description)

Remark: we have omitted the important vanishing condition for non-matching orbits in the above statements.

Problem

The double integral

$$\int_{[H]}\int_{[H]} k_f(x,y)\eta(\det(x))dxdy$$

is divergent!

This is because $[H]$ is not compact and $k_f(x,y)$ is not rapidly decreasing. Similar problems arise for $J_\gamma(f)$, $J_\pi(f)$ and the case of (G',H') as well.

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An infinitesimal variant

An infinitesimal variant of the problem: the symmetric spaces G/H and G'/H' replaced by the tangent spaces at the neutral element

Why is it interesting?

- close to the geometric side of the original formula
- useful in the comparison of geometric sides (cf. C. Zhang's proof)

Remark: reduction to Lie algebras is also a feature of classical works of Harish-Chandra's and Waldspurger's

An infinitesimal variant: the case of (G, H)

We discuss the case of $(G, H) = (GL_{2n}, GL_n \times GL_n)$ in details. Let $\mathfrak{s}$ be the tangent space of the symmetric space G/H at the neutral element:

$$\mathfrak{s} = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : A, B \in \mathfrak{gl}_n \right\} \simeq \mathfrak{gl}_n \oplus \mathfrak{gl}_n.$$

Then H acts on $\mathfrak{s}$ by conjugation:

$$(h_1, h_2) \cdot (A, B) = (h_1 A h_2^{-1}, h_2 B h_1^{-1}).$$

We define a relation of equivalence on $\mathfrak{s}(F)$ by the categorical quotient:

$$\pi : \mathfrak{s} \to \mathfrak{s}//H \simeq \mathbf{A}^n, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mapsto \text{Char. poly. of } AB$$

$$X \sim Y \in \mathfrak{s}(F) \overset{\text{def}}{\Longleftrightarrow} \pi(X) = \pi(Y)$$

Denote by $\mathcal{O}$ the set of classes of equivalence.

An infinitesimal variant: the case of (G, H)

Examples

If $\mathfrak{o} \in \mathcal{O}$ is a regular semi-simple class (i.e. $\pi(\mathfrak{o})$ is a separable polynomial and its constant term is nonzero), then $\mathfrak{o}$ is an $H(F)$-orbit.

Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ be a Bruhat-Schwartz function and $x \in H(\mathbb{A})$. We consider the "kernel"

$$k_f(x) := \sum_{\gamma \in \mathfrak{s}(F)} f(x^{-1}\gamma x) = \sum_{\mathfrak{o} \in \mathcal{O}} k_{f,\mathfrak{o}}(x),$$

where $k_{f,\mathfrak{o}}(x) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma x)$.

Problem

The development

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{[H]} k_{f,\mathfrak{o}}(x)\eta(\det(x))dx$$

is divergent!

An infinitesimal variant: the case of (G, H)

A naïve solution: replace $k_{f,\mathfrak{o}}(x)$ by a truncated kernel

$$F^G(x, T)k_{f,\mathfrak{o}}(x),$$

where $F^G(x, T)$ is the characteristic function of some compact truncated Siegel domain with a truncation parameter $T \in \mathbb{R}^{2n}$.

Proposition&Definition

For all regular parameter $T \in \mathbb{R}^{2n}$ and all $N > 0$, there exists an exponential polynomial $J_{\mathfrak{o}}^T(\eta, f)$ in T such that

$$\int_{[H]} F^G(x, T)k_{f,\mathfrak{o}}(x)\eta(\det(x))dx = J_{\mathfrak{o}}^T(\eta, f) + O(e^{-N\|T\|}).$$

Denote by $J_{\mathfrak{o}}(\eta, f)$ the constant term of $J_{\mathfrak{o}}^T(\eta, f)$.

An infinitesimal variant: the case of (G, H)

Remark: the above proposition/definition has masked the difficulty. In fact, the naïve truncation depends on T and is not delicate enough for applications. We actually need to first "define" $J_{\mathfrak{o}}^{T}(\eta, f)$ via a so-called mixed truncation: an analogue of Arthur's truncation in the classical case which is compatible with both of H and $\mathfrak{s}$.

Let Ψ be a nontrivial continuous character of $\mathbb{A}/F$. We define the Fourier transform $\hat{f}$ of $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ by

$$\forall X \in \mathfrak{s}(\mathbb{A}), \hat{f}(X) := \int_{\mathfrak{s}(\mathbb{A})} f(Y)\Psi(\mathrm{Tr}(YX))dY.$$

Theorem (infi. global RTF for (G, H))

We have

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(\eta, f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(\eta, \hat{f}).$$

An infinitesimal variant: the case of (G, H)

Remarks:

- its simple form has been used in C. Zhang's proof of smooth transfer;
- it results from the Poisson summation formula;
- at the infinitesimal level, the Fourier transform of the geometric side plays the role of the original spectral side.

In addition, we can write most terms $J_{\mathfrak{o}}(\eta, f)$ as explicit weighted orbital integrals. Besides orbital integrals, these weighted analogues are the first new terms that we want to study.

An infinitesimal variant: the case of (G, H)

Theorem

If $\mathfrak{o}$ is regular semi-simple, we have

$$J_{\mathfrak{o}}(\eta, f) = \mathrm{vol}([H_{X_1}]) \int_{H_{X_1}(\mathbb{A})\backslash H(\mathbb{A})} f(x^{-1}X_1 x) v_{P_1}(x)\eta(\det(x))dx,$$

where P_1 and $X_1 \in (\mathfrak{o} \cap \mathfrak{p}_1)(F)$ are some elliptic data associated to $\mathfrak{o}$, and $v_{P_1}(x)$ is the volume of some explicit convex envelope.

Remark: the weight $v_{P_1}(x)$ is equal to the weight in the twisted trace formula for $(GL_n \times GL_n) \rtimes \theta$, where θ exchanges two copies of GL_n.

An infinitesimal variant: the case of (G', H')

We have similar (in fact simpler) constructions and results for the other case (G', H') in spite of some rationality issues. Here are some remarks:

- after the base change to an algebraic closure of F, two cases are the same;
- an infinitesimal variant concerns the twisted conjugation of $H' = GL_n(E)$ on $\mathfrak{s}' = \mathfrak{gl}_n(E)$;
- the weight for regular semi-simple terms is equal to the weight in the twisted trace formula for $GL_n(E) \rtimes \theta'$, where θ' is the Galois involution;
- we actually study a more general case inspired by the related local conjecture of Prasad and Takloo-Bighash (2011), which is necessary for the converse of the Guo-Jacquet conjecture for even n. Precisely, let G' be the set of invertible elements in a central simple algebra $\mathfrak{g}'$ over F containing E, and H' be the centraliser of $E^\times$ in G'.

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Noninvariant comparison

Comparison of orbital integrals: works of Guo's and C. Zhang's

Question

How can we compare the new terms (such as weighted orbital integrals) arising in the general infinitesimal global trace formulae?

Difficulty: these distributions are noninvariant under the conjugation of $H(\mathbb{A})$ or $H'(\mathbb{A})$ (close to the Arthur-Selberg trace formula)

- Arthur's approach in the classical case: to make everything invariant
- Labesse's proposal: to compare directly these noninvariant terms (cf. Chaudouard's works on the stable base change)

Our local results: to provide some evidence for the latter approach in the relative context

Local setting

Notations:

- E/F: a quadratic extension of non-archimedean local fields of characteristic 0
- η: the quadratic character $F^\times/NE^\times \to \{\pm 1\}$ associated to E/F, where $NE^\times$ denotes the norm of $E^\times$
- (G, H) and (G', H') are defined as before with the same dimension
- $\mathfrak{s}_{\mathrm{rs}}$ (or $\mathfrak{s}'_{\mathrm{rs}}$): the set of regular semi-simple elements in $\mathfrak{s}$ (or $\mathfrak{s}'$)

Given the following data:

- an "ω-stable" Levi subgroup M of G (i.e. $\omega := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in M$, thus $\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}} \neq \emptyset$)
- a point $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$
- a special maximal compact subgroup K of G (implicitly choosing the "standard" one)

Local setting

We define the local weighted orbital integral

$$J_M^G(\eta, X, f) := |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X))\eta(\det(x))v_M^G(x)dx$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, where $|D^{\mathfrak{s}}(X)|_F$ is the Weyl discriminant and $v_M^G(x)$ is the corresponding local weight function.

Similarly, given the following data:

- a Levi subgroup M' of H'
- a point $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ (with $\widetilde{M'}$ denoting the unique Levi subgroup of G' such that $\widetilde{M'} \cap H' = M'$)
- a special maximal compact subgroup K of H' (implicitly choosing the "standard" one)

Local setting

We define the local weighted orbital integral

$$J_{M'}^{H'}(Y,f') := |D^{\mathfrak{s}'}(Y)|_F^{1/2} \int_{H'_Y(F)\backslash H'(F)} f'(\mathrm{Ad}(x^{-1})(Y))v_{M'}^{H'}(x)dx$$

for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, where $|D^{\mathfrak{s}'}(Y)|_F$ is the Weyl discriminant and $v_{M'}^{H'}(x)$ is the corresponding local weight function.

Matching of Levi subgroups: there is an injection $M' \mapsto M$ from the set of Levi subgroups of H' into the set of ω-stable Levi subgroups of G.

Matching of orbits: since $\mathfrak{s}//H \simeq \mathbf{A}^n \simeq \mathfrak{s}'//H'$, we say that $X \in \mathfrak{s}_{\mathrm{rs}}(F)$ and $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)$ have matching orbits if they have the same image in $\mathbf{A}^n$.

In general, fix a pair of matching Levi subgroups $M' \leftrightarrow M$ of H' and G respectively. We have the notion of "M-matching orbits" for $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ defined by blocks.

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The weighted fundamental lemma

A basic and important case of the noninvariant comparison is the so-called weighted fundamental lemma. It roughly says that at almost all unramified places, some basic functions on $\mathfrak{s}(F)$ and $\mathfrak{s}'(F)$ should have associated local weighted orbital integrals on matching orbits.

For almost all unramified places, we have $(G'(F), H'(F)) \simeq (GL_{2n}(F), GL_n(E))$ and $\mathfrak{s}'(F) \simeq \mathfrak{gl}_n(E)$, on which $H'(F)$ acts by twisted conjugation. Notations:

- E/F: unramified with odd residue characteristic
- $\mathcal{O}_F$ (resp. $\mathcal{O}_E$): the ring of integers in F (resp. E)
- f_0: the characteristic function of $\mathfrak{s}(\mathcal{O}_F) \simeq (\mathfrak{gl}_n \oplus \mathfrak{gl}_n)(\mathcal{O}_F)$
- f_0': the characteristic function of $\mathfrak{s}'(\mathcal{O}_F) \simeq \mathfrak{gl}_n(\mathcal{O}_E)$
- $\mathrm{vol}(H(\mathcal{O}_F)) = \mathrm{vol}(H'(\mathcal{O}_F)) = 1$

The weighted fundamental lemma

Theorem (weighted FL)

Let $M \leftrightarrow M'$ be a pair of matching Levi subgroups of G and H' respectively. We have

- if $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ have M-matching orbits, then
$$\kappa(X) J_M^G(\eta, X, f_0) = J_{M'}^{H'}(Y, f'_0),$$
where we define the transfer factor $\kappa(X) := \eta(\det(A))$ for $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$;
- if $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ satisfies $\det(AB) \notin NE^\times$, then
$$J_M^G(\eta, X, f_0) = 0.$$

The weighted fundamental lemma

Remarks:

- its non-weighted version has been obtained by Guo (and C. Zhang);
- it is reduced to the weighted fundamental lemmas of full spherical Hecke algebras for the base changes $F \times F/F$ and E/F of GL_n (Kottwitz, 1980; Arthur-Clozel, 1989; Labesse, 1995).

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An infinitesimal local trace formula

In order to go further, we need some preparation in local harmonic analysis. Precisely, we prove an infinitesimal variant of invariant local trace formulae in this relative setting following classical works of Waldspurger, 1995; Arthur, 1991.

We shall focus on the case of (G, H) for illustration. Our formula is deduced from the noninvariant local trace formula whose starting point is as follows. For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we denote

$$K(\eta, f, f') := \int_{Z(F)\backslash H(F)} \int_{\mathfrak{s}(F)} f(X)f'(\mathrm{Ad}(x^{-1})(X))\eta(\det(x))dXdx.$$

Then the Plancherel formula formally implies

$$K(\eta, f, \hat{f}') = K(\eta, \hat{f}, f').$$

However, $K(\eta, f, f')$ is divergent and we need a local truncation process to make it valid.

An infinitesimal local trace formula

During the proof, we also obtain some other results of independent interest (non-weighted analogues used by C. Zhang):

- Howe's finiteness for weighted orbital integrals
- Representability of the Fourier transform of weighted orbital integrals: there exists a locally constant function $\hat{j}_M^G(\eta, X, \cdot)$ on $\mathfrak{s}_{\mathrm{rs}}(F)$ such that

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), J_M^G(\eta, X, \hat{f}) = \int_{\mathfrak{s}(F)} f(Y)\hat{j}_M^G(\eta, X, Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2}dY$$

- Limit at "infinity": if $M \neq G$, then $\lim_{\mu\to\infty} \hat{j}_M^G(\eta, \mu X, Y) = 0$

Similar results are obtained for the case of (G', H').

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Rough idea

Recall:

- Classical case (Waldspurger, 1997) : fundamental lemma (Ngô, 2010) $\Longrightarrow$ transfer commutes with Fourier transform $\Longrightarrow$ transfer.
- A weighted annalogue: Chaudouard, 2005&2007 (for inner forms and the stable base change)
- A relative analogue: C. Zhang, 2015 (for the case of Guo-Jacquet)

Very vaguely, we prove a relative and weighted analogue of the commutativity (under certain acceptable conditions) with the help of the weighted fundamental lemma in this case.

Identities between Fourier transforms of weighted orbital integrals

Let M be an ω-stable Levi subgroup of G and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. We deduce the (H, η)-invariant local weighted orbital integral $I_M^G(\eta, X, \cdot)$ from the noninvariant one $J_M^G(\eta, X, \cdot)$ by Arthur's standard process.

Notations:

- $\hat{j}_M^G(\eta, X, \cdot)$: loc. const. function on $\mathfrak{s}_{\mathrm{rs}}(F)$ representing $\hat{J}_M^G(\eta, X, \cdot)$
- $\hat{i}_M^G(\eta, X, \cdot)$: loc. const. function on $\mathfrak{s}_{\mathrm{rs}}(F)$ representing $\hat{I}_M^G(\eta, X, \cdot)$

Remarks:

- $\hat{i}_M^G(\eta, X, U)$ is (H, η)-invariant on U and $(M \cap H, \eta)$-invariant on X;
- $\hat{j}_M^G$ is decomposed as $\hat{i}_M^G$ and weight functions v_M^G.

Let M' be a Levi subgroup of H' and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. We similarly obtain locally constant functions $\hat{j}_{M'}^{H'}(Y, \cdot)$ and $\hat{i}_{M'}^{H'}(Y, \cdot)$ on $\mathfrak{s}'_{\mathrm{rs}}(F)$. We have similar properties for $\hat{i}_{M'}^{H'}$ and decomposition for $\hat{j}_{M'}^{H'}$.

Identities between Fourier transforms of weighted orbital integrals

Part of the noninvariant comparison: relations between $\hat{j}_M^G$ and $\hat{j}_{M'}^{H'}$
Our final result: relations between $\hat{i}_M^G$ and $\hat{i}_{M'}^{H'}$
Remark: combined with the comparison of weights, one could expect relations between $\hat{j}_M^G$ and $\hat{j}_{M'}^{H'}$

Fix a pair of matching Levi subgroups $M' \leftrightarrow M$ of H' and G respectively. For $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}_{\mathrm{rs}}(F)$, we denote $\kappa(X) := \eta(\det(A))$ and $\eta(X) := \eta(\det(AB))$.

Identities between Fourier transforms of weighted orbital integrals

Theorem (Ids between FT of WOI)

- Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ have M-matching orbits. Let $U \in \mathfrak{s}_{\mathrm{rs}}(F)$ and $V \in \mathfrak{s}'_{\mathrm{rs}}(F)$ have matching orbits. Then we have

$$\gamma_\psi(\mathfrak{h}(F))^{-1}\kappa(X)\kappa(U)\hat{i}_M^G(\eta, X, U) = \gamma_\psi(\mathfrak{h}'(F))^{-1}\hat{i}_{M'}^{H'}(Y, V),$$

where γ_ψ's are Weil constants.
- Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $U \in \mathfrak{s}_{\mathrm{rs}}(F)$. If $\eta(X) \neq \eta(U)$, then

$$\hat{i}_M^G(\eta, X, U) = 0.$$

Some ingredients in the proof

Its analogue for ordinary orbital integrals has essentially been obtained by C. Zhang. As in C. Zhang's proof, we use Waldspurger's global method in the seminal work "Le lemme fondamental implique le transfert" (1997) to show the first statement, and a local method to show the second one. The proof involves almost all of our previous results:

- An infinitesimal global trace formula
- The weighted fundamental lemma
- Limit formulae for $\lim_{\mu\to\infty} \hat{i}_M^G(\eta, \mu X, U)$ and $\lim_{\mu\to\infty} \hat{i}_{M'}^{H'}(\mu Y, V)$

Main technical difficulty: compared to the case of inner forms (Chaudouard, 2005), the vanishing condition of our weighted fundamental lemma is more subtle $\longrightarrow$ we need abelian Galois cohomology (Labesse, 1999 *et al.*)

Thank you!

Definition of the mixed truncation: the case of (G, H)

For $x \in H(F)\backslash H(\mathbb{A})$ and $T \in \mathbb{R}^{2n}$, define

$$k^T_{f,\mathfrak{o}}(x) := \sum_{\text{rel. std. } P} (-1)^{\dim(A_P/A_G)} \sum_{\delta \in (P\cap H)(F)\backslash H(F)} \widehat{\tau}^G_P(H_P(\delta x) - T_P) \cdot k_{f,P,\mathfrak{o}}(\delta x),$$

where $\widehat{\tau}^G_P$ is the characteristic function of some cone in the coroot space, and

$$k_{f,P,\mathfrak{o}}(x) := \sum_{X \in \mathfrak{m}_P(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})} f(x^{-1}(X+U)x)dU.$$

Let

$$J^T_{\mathfrak{o}}(\eta, f) := \int_{[H]} k^T_{f,\mathfrak{o}}(x)\eta(\det(x))dx.$$