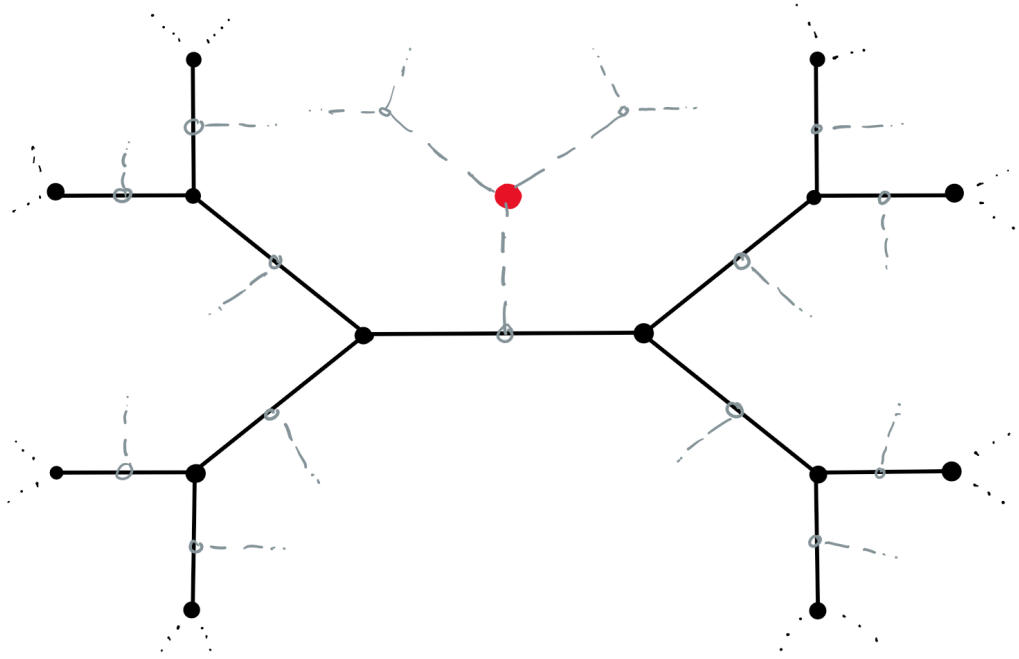
Representations of p-adic groups and applications

Jessica Fintzen

University of Cambridge and Duke University

September 2020



Representations of p-adic groups

Notation: $F/\mathbb{Q}_p$ finite or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \supset \mathfrak{p}$, residue field $\mathbb{F}_q$

Representations of p-adic groups

Notation: $F/\mathbb{Q}_p$ finite or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \supset \mathfrak{p}$, residue field $\mathbb{F}_q$
G (connected) reductive group over F, e.g.
$\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{SO}_n(F), \mathrm{Sp}_{2n}(F), \ldots$

Representations of p-adic groups

Notation: $F/\mathbb{Q}_p$ finite or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \supset \mathfrak{p}$, residue field $\mathbb{F}_q$
G (connected) reductive group over F, e.g.
$\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{SO}_n(F), \mathrm{Sp}_{2n}(F), \ldots$

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex) representations of G.

Representations of p-adic groups

Notation: $F/\mathbb{Q}_p$ finite or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \supset \mathfrak{p}$, residue field $\mathbb{F}_q$
G (connected) reductive group over F, e.g.
$\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{SO}_n(F), \mathrm{Sp}_{2n}(F), \ldots$

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G. (ℓ a prime $\neq p$)

Representations of p-adic groups

Notation: $F/\mathbb{Q}_p$ finite or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \supset \mathfrak{p}$, residue field $\mathbb{F}_q$
G (connected) reductive group over F, e.g.
$\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{SO}_n(F), \mathrm{Sp}_{2n}(F), \ldots$

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G. (ℓ a prime $\neq p$)

Applications to, for example,

- representation theory of p-adic groups

Representations of p-adic groups

Notation: $F/\mathbb{Q}_p$ finite or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \supset \mathfrak{p}$, residue field $\mathbb{F}_q$
G (connected) reductive group over F, e.g.
$\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{SO}_n(F), \mathrm{Sp}_{2n}(F), \ldots$

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G. (ℓ a prime $\neq p$)

Applications to, for example,

- representation theory of p-adic groups
- explicit local Langlands correspondence

Representations of p-adic groups

Notation: $F/\mathbb{Q}_p$ finite or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \supset \mathfrak{p}$, residue field $\mathbb{F}_q$
G (connected) reductive group over F, e.g.
$\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{SO}_n(F), \mathrm{Sp}_{2n}(F), \ldots$

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G. (ℓ a prime $\neq p$)

Applications to, for example,

- representation theory of p-adic groups
- explicit local Langlands correspondence
- automorphic forms (e.g. J.F. and S.W. Shin)

Representations of p-adic groups

Notation: $F/\mathbb{Q}_p$ finite or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \supset \mathfrak{p}$, residue field $\mathbb{F}_q$
G (connected) reductive group over F, e.g.
$\mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{SO}_n(F), \mathrm{Sp}_{2n}(F), \ldots$

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G. (ℓ a prime $\neq p$)

Applications to, for example,

- representation theory of p-adic groups
- explicit local Langlands correspondence
- automorphic forms (e.g. J.F. and S.W. Shin)
- p-adic automorphic forms, p-adic Langlands program
- ...

Representations of p-adic groups

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G.

Representations of p-adic groups

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G.

Building blocks = (irreducible) supercuspidal representations (or cuspidal representations)

Representations of p-adic groups

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G.

Building blocks = (irreducible) supercuspidal representations (or cuspidal representations)

Construction of (super)cuspidal representations:

Representations of p-adic groups

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G.

Building blocks = (irreducible) supercuspidal representations (or cuspidal representations)

Construction of (super)cuspidal representations:

GL_n: R. Howe, A. Moy, ..., (1970s and later)

Representations of p-adic groups

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G.

Building blocks = (irreducible) supercuspidal representations (or cuspidal representations)

Construction of (super)cuspidal representations:

GL_n: R. Howe, A. Moy, ..., (1970s and later)
C. Bushnell and P. Kutzko (1993),

Representations of p-adic groups

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G.

Building blocks = (irreducible) supercuspidal representations (or cuspidal representations)

Construction of (super)cuspidal representations:

GL_n: R. Howe, A. Moy, ..., (1970s and later)
C. Bushnell and P. Kutzko (1993),
M.-F. Vigneras (1996)

Representations of p-adic groups

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G.

Building blocks = (irreducible) supercuspidal representations (or cuspidal representations)

Construction of (super)cuspidal representations:

GL_n:	R. Howe, A. Moy, ..., (1970s and later) C. Bushnell and P. Kutzko (1993), M.-F. Vigneras (1996)
classical groups ($p \neq 2$):	..., S. Stevens (2008),

Representations of p-adic groups

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G.

Building blocks = (irreducible) supercuspidal representations (or cuspidal representations)

Construction of (super)cuspidal representations:

GL_n:	R. Howe, A. Moy, ..., (1970s and later) C. Bushnell and P. Kutzko (1993), M.-F. Vigneras (1996)
classical groups ($p \neq 2$):	..., S. Stevens (2008), R. Kurinzcuk and S. Stevens (2018)

Representations of p-adic groups

Motivation / longterm goal

Want to construct all (irreducible, smooth, complex or $\overline{\mathbb{F}}_\ell$-) representations of G.

Building blocks = (irreducible) supercuspidal representations (or cuspidal representations)

Construction of (super)cuspidal representations:

GL_n:	R. Howe, A. Moy, ..., (1970s and later) C. Bushnell and P. Kutzko (1993), M.-F. Vigneras (1996)
classical groups ($p \neq 2$):	..., S. Stevens (2008), R. Kurinzcuk and S. Stevens (2018)
inner forms of GL_n:	..., V. Sécherre and S. Stevens (2008)

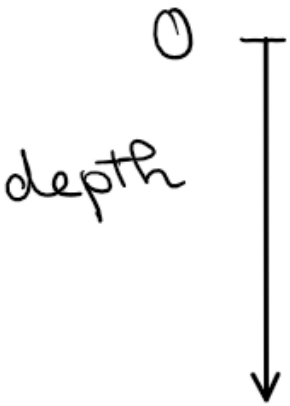
Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

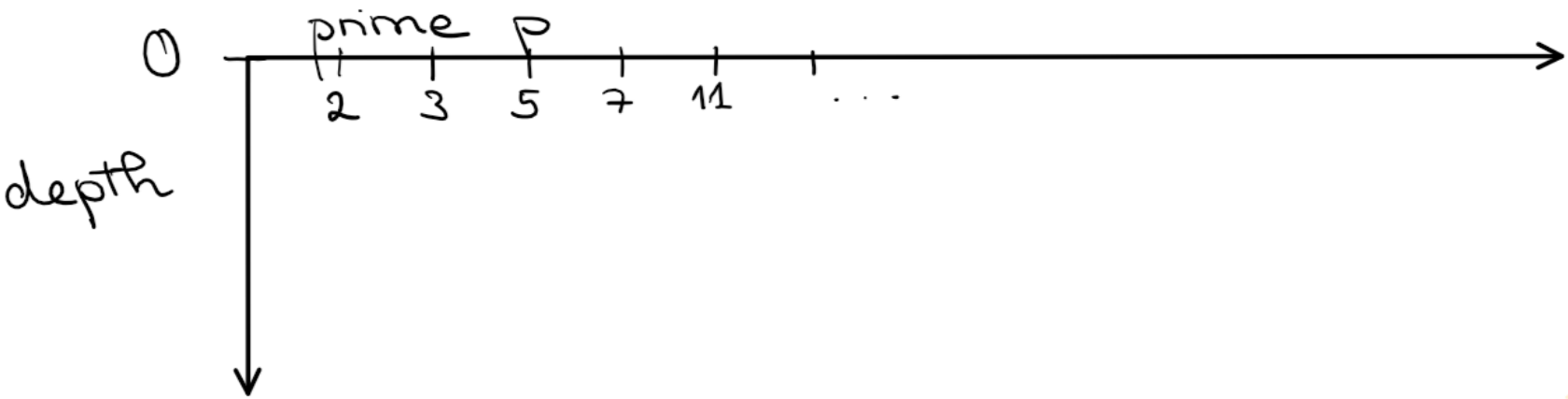
1994/96 A. Moy and G. Prasad



Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

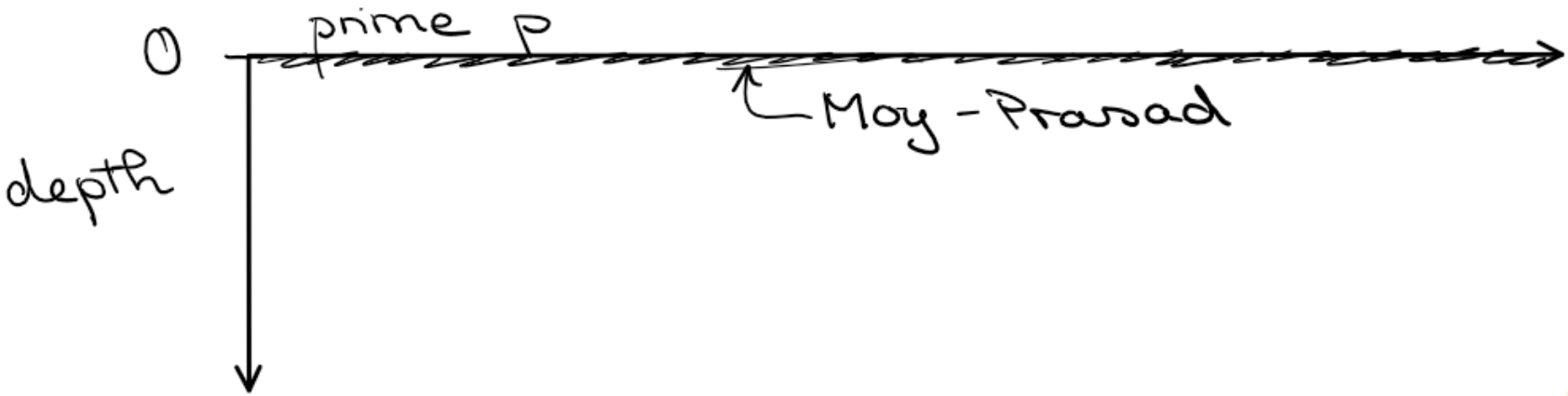
1994/96 A. Moy and G. Prasad



Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

1994/96 A. Moy and G. Prasad (L. Morris: 1993/99)

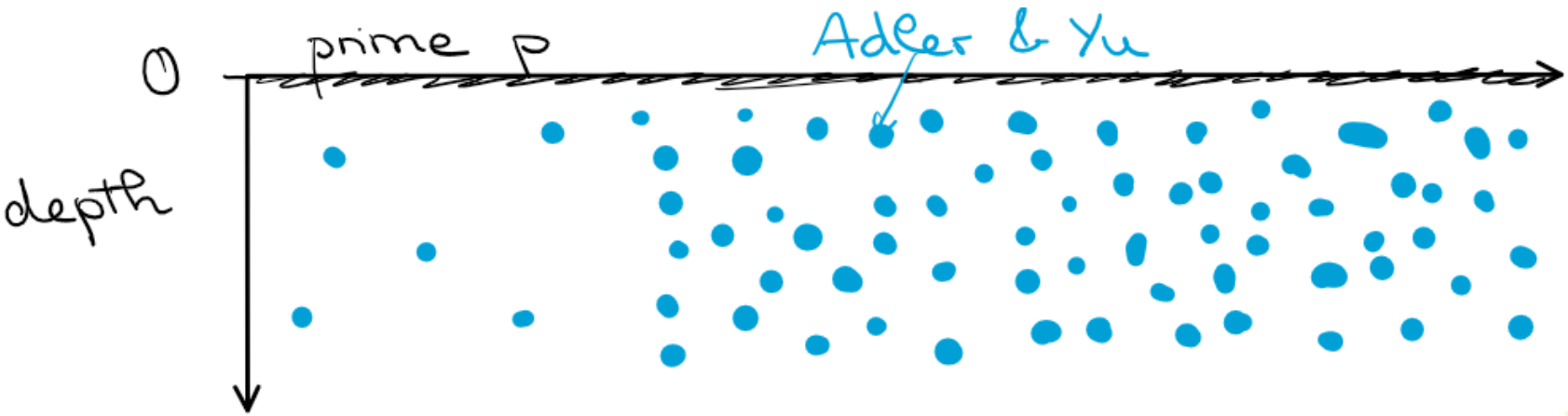


Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

1994/96 A. Moy and G. Prasad (L. Morris: 1993/99)

1998, 2001 J. Adler, J.-K. Yu



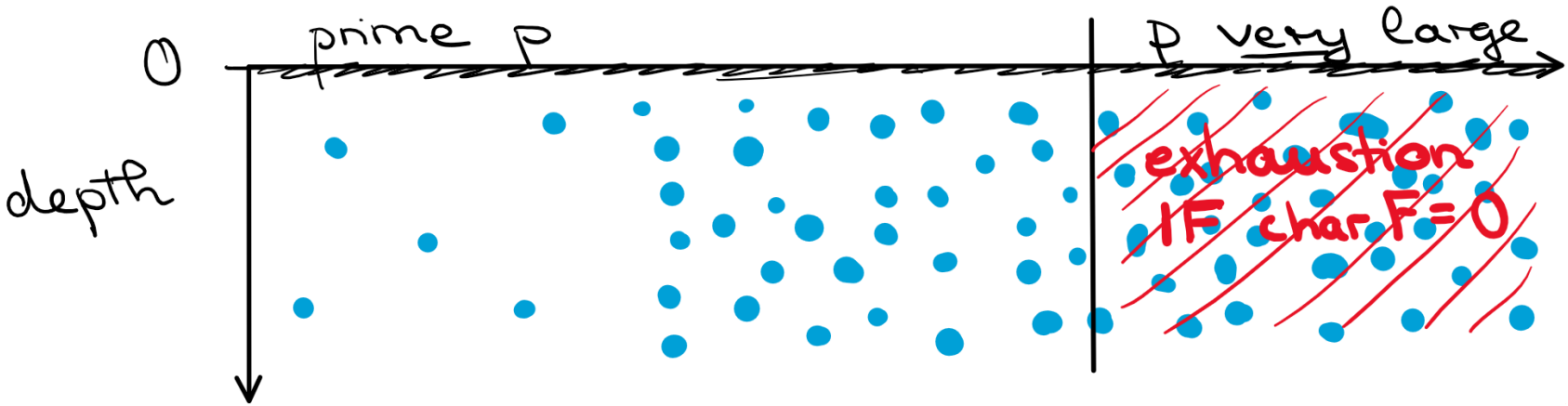
Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

1994/96 A. Moy and G. Prasad (L. Morris: 1993/99)

1998, 2001 J. Adler, J.-K. Yu

2007 J.-L. Kim: Yu's construction yields all supercuspidal representations **if p is very large and char $F = 0$**



Construction of supercuspidal representations

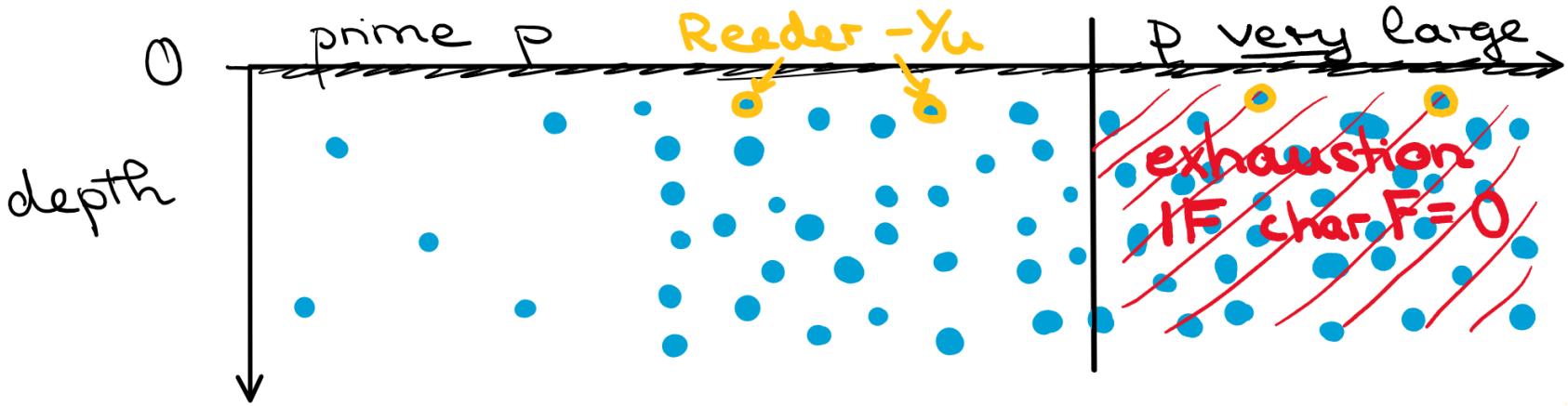
Constructions of supercuspidal representations for general G:

1994/96 A. Moy and G. Prasad (L. Morris: 1993/99)

1998, 2001 J. Adler, J.-K. Yu

2007 J.-L. Kim: Yu's construction yields all supercuspidal representations **if p is very large and char $F = 0$**

2014 M. Reeder and J.-K. Yu: epipelagic representations



Epipelagic representations

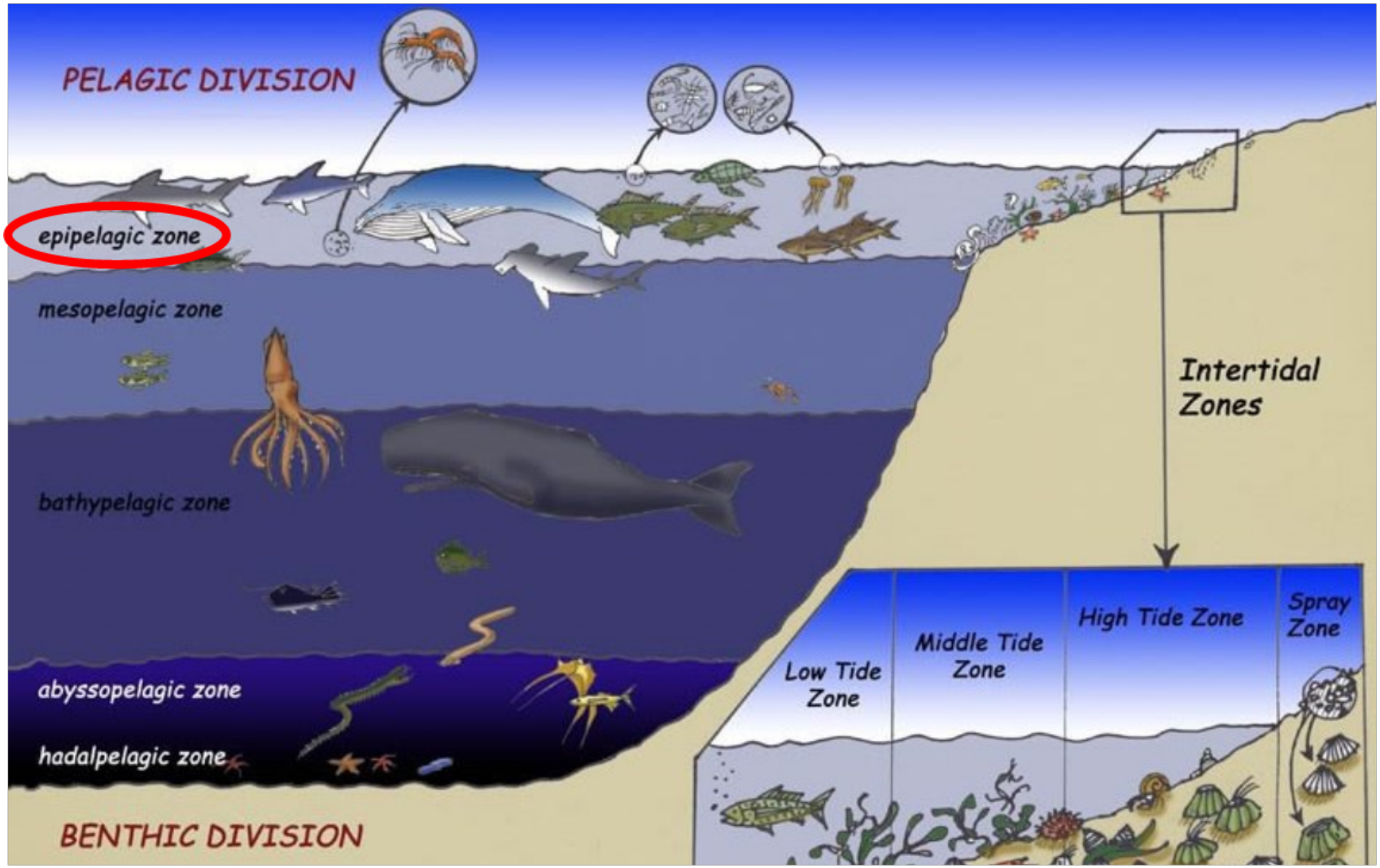


Figure: The epipelagic zone of the ocean;
source: Sheri Amsel. Glossary (what words mean) with pictures!. 2005-2015. April 2, 2015, `http://www.exploringnature.org/db/detail.php?dbID=13&detID=406`

Construction of supercuspidal representations

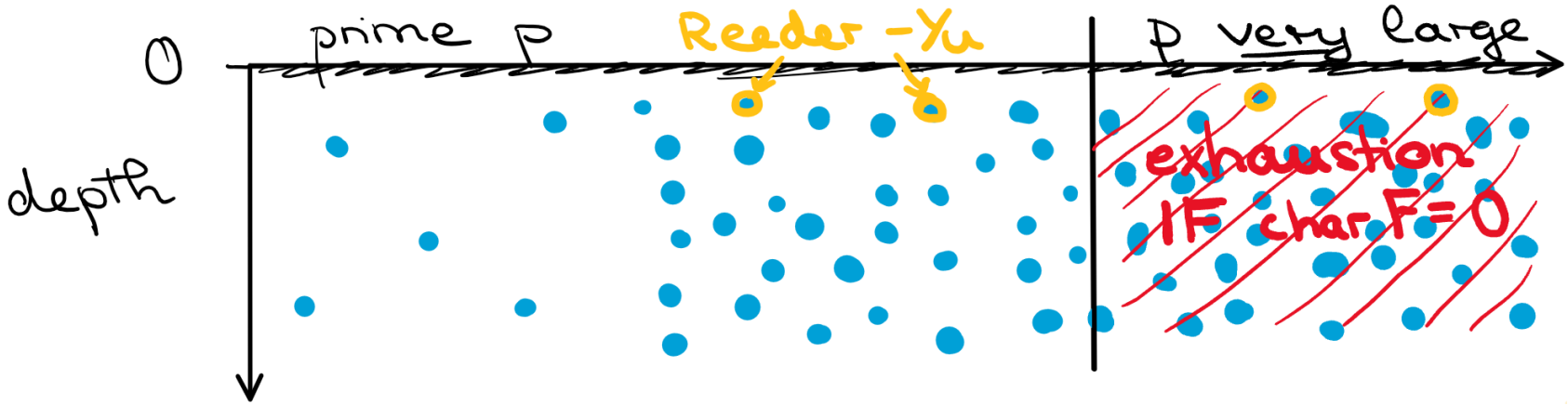
Constructions of supercuspidal representations for general G:

1994/96 A. Moy and G. Prasad (L. Morris: 1993/99)

1998, 2001 J. Adler, J.-K. Yu

2007 J.-L. Kim: Yu's construction yields all supercuspidal representations **if p is very large and char $F = 0$**

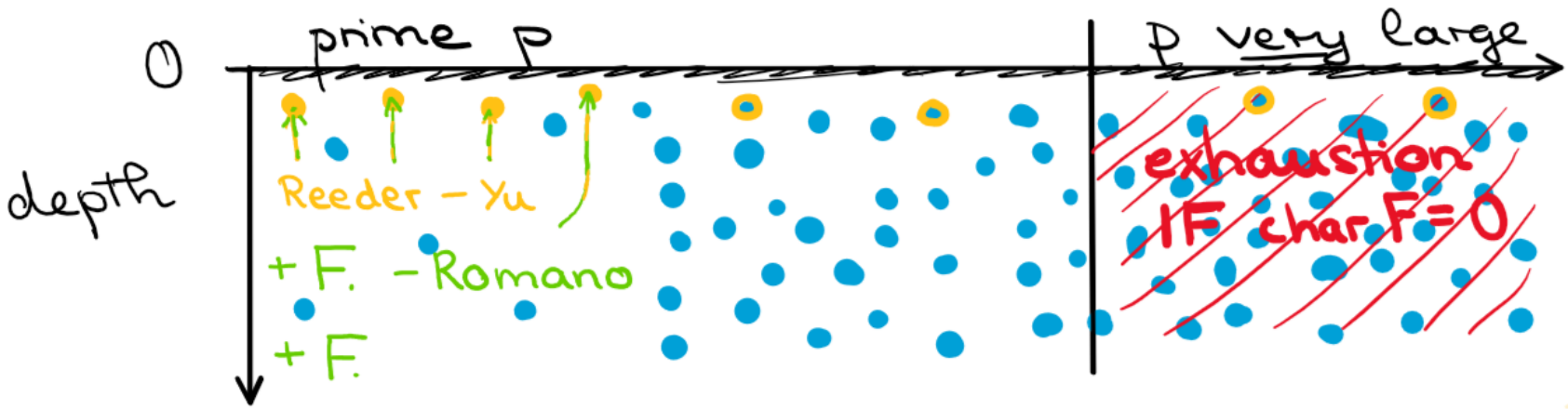
2014 M. Reeder and J.-K. Yu: epipelagic representations



Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

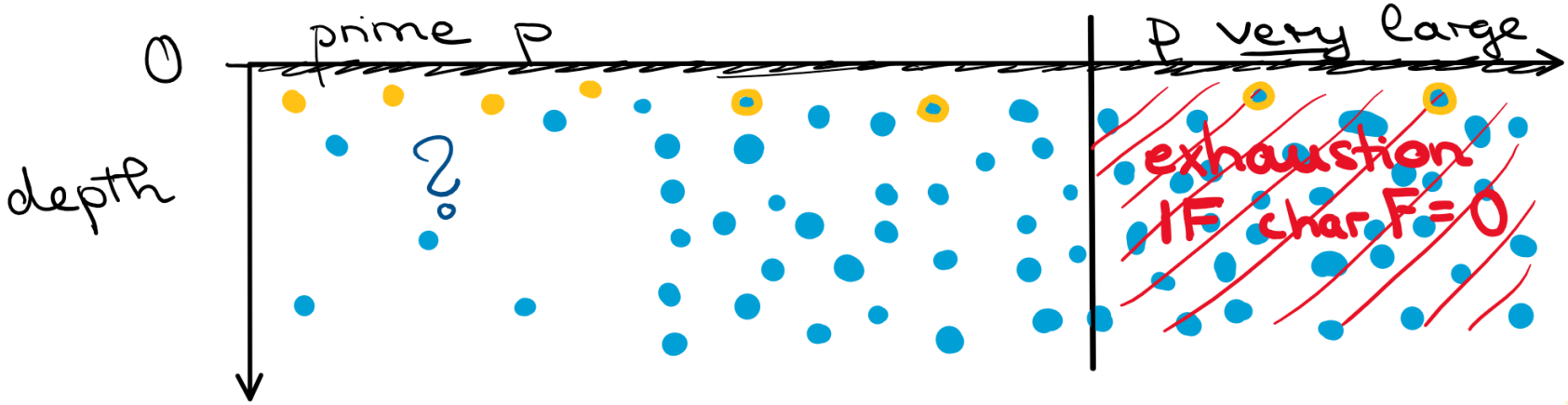
1994/96	A. Moy and G. Prasad (L. Morris: 1993/99)
1998, 2001	J. Adler, J.-K. Yu
2007	J.-L. Kim: Yu's construction yields all supercuspidal representations **if p is very large and char $F = 0$**
2014	M. Reeder and J.-K. Yu: epipelagic representations
2017, 2020?	**J. F.** and B. Romano (special case), **J. F.** (general case): input for Reeder–Yu exists also for small p



Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

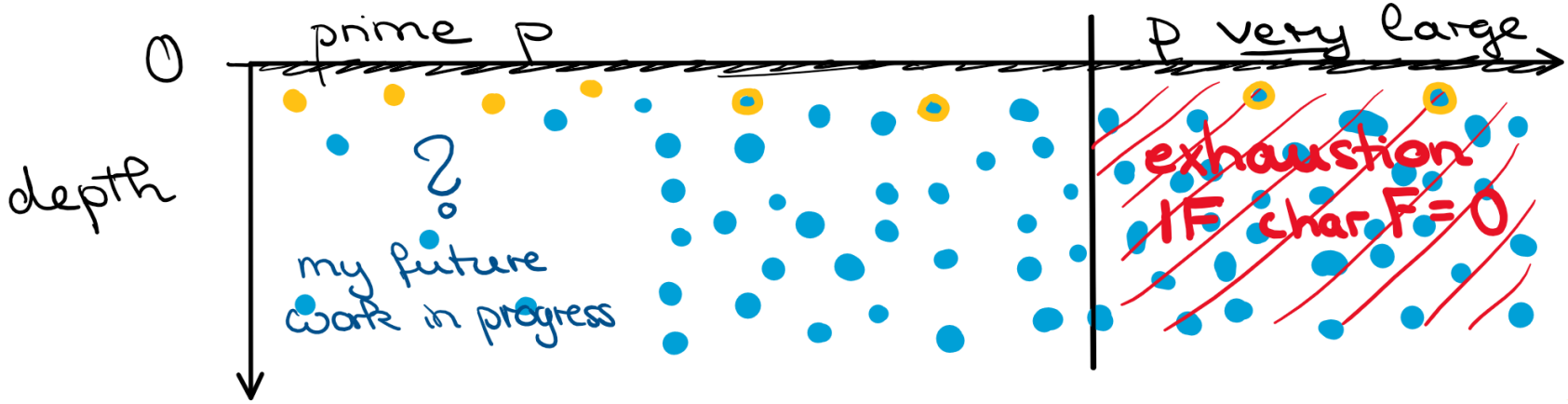
1994/96	A. Moy and G. Prasad (L. Morris: 1993/99)
1998, 2001	J. Adler, J.-K. Yu
2007	J.-L. Kim: Yu's construction yields all supercuspidal representations **if p is very large and char $F = 0$**
2014	M. Reeder and J.-K. Yu: epipelagic representations
2017, 2020?	**J. F.** and B. Romano (special case), **J. F.** (general case): input for Reeder–Yu exists also for small p



Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

1994/96	A. Moy and G. Prasad (L. Morris: 1993/99)
1998, 2001	J. Adler, J.-K. Yu
2007	J.-L. Kim: Yu's construction yields all supercuspidal representations **if p is very large and char $F = 0$**
2014	M. Reeder and J.-K. Yu: epipelagic representations
2017, 2020?	**J. F.** and B. Romano (special case), **J. F.** (general case): input for Reeder–Yu exists also for small p



Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

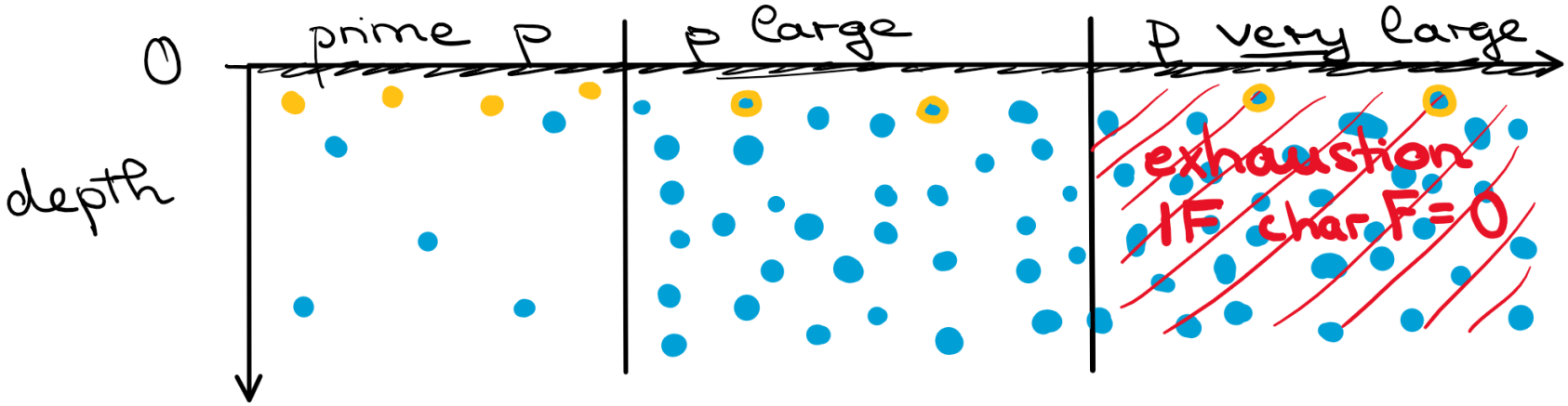
1994/96 A. Moy and G. Prasad (L. Morris: 1993/99)

1998, 2001 J. Adler, J.-K. Yu

2007 J.-L. Kim: Yu's construction yields all supercuspidal representations **if p is very large and char $F = 0$**

2014 M. Reeder and J.-K. Yu: epipelagic representations

2017, 2020? **J. F.** and B. Romano (special case), **J. F.** (general case): input for Reeder–Yu exists also for small p



Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

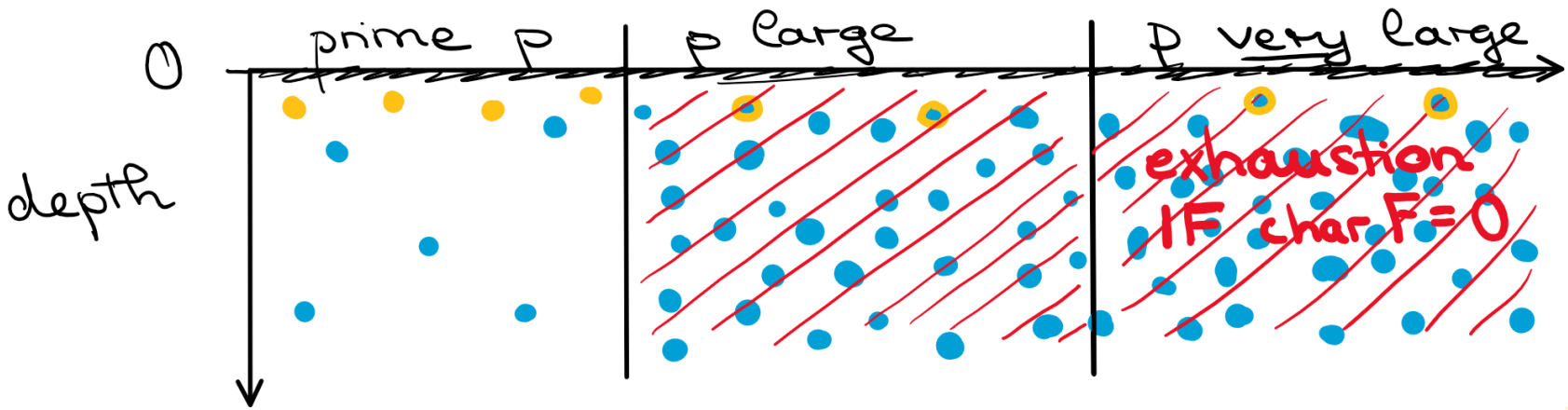
1994/96 A. Moy and G. Prasad (L. Morris: 1993/99)

1998, 2001 J. Adler, J.-K. Yu

2007 J.-L. Kim: Yu's construction yields all supercuspidal representations **if p is very large and char $F = 0$**

2014 M. Reeder and J.-K. Yu: epipelagic representations

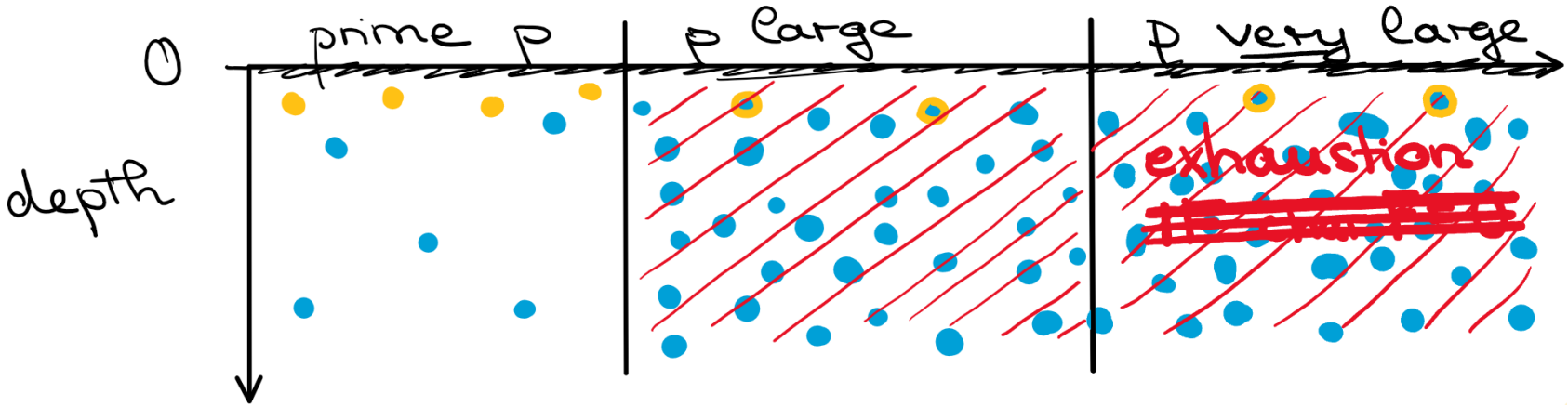
2017, 2020? **J. F.** and B. Romano (special case), **J. F.** (general case): input for Reeder–Yu exists also for small p



Construction of supercuspidal representations

Constructions of supercuspidal representations for general G:

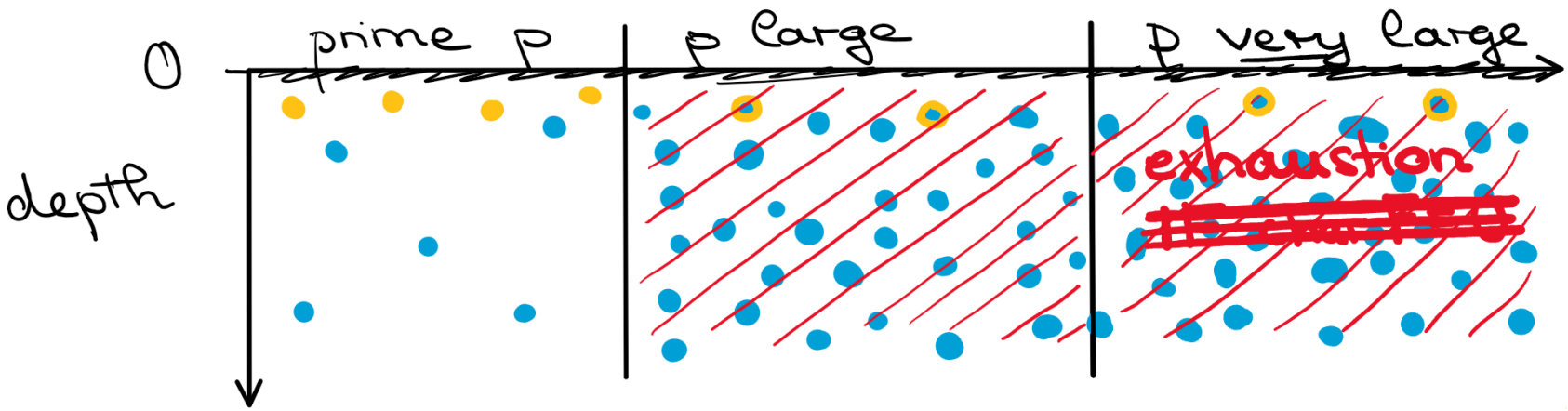
1994/96	A. Moy and G. Prasad (L. Morris: 1993/99)
1998, 2001	J. Adler, J.-K. Yu
2007	J.-L. Kim: Yu's construction yields all supercuspidal representations **if p is very large and char $F = 0$**
2014	M. Reeder and J.-K. Yu: epipelagic representations
2017, 2020?	**J. F.** and B. Romano (special case), **J. F.** (general case): input for Reeder–Yu exists also for small p



Results

Theorem 1 (F., Oct 2018)

Suppose G splits over a tame extension of F and $p \nmid |W|$, then Yu's construction yields all supercuspidal representations.



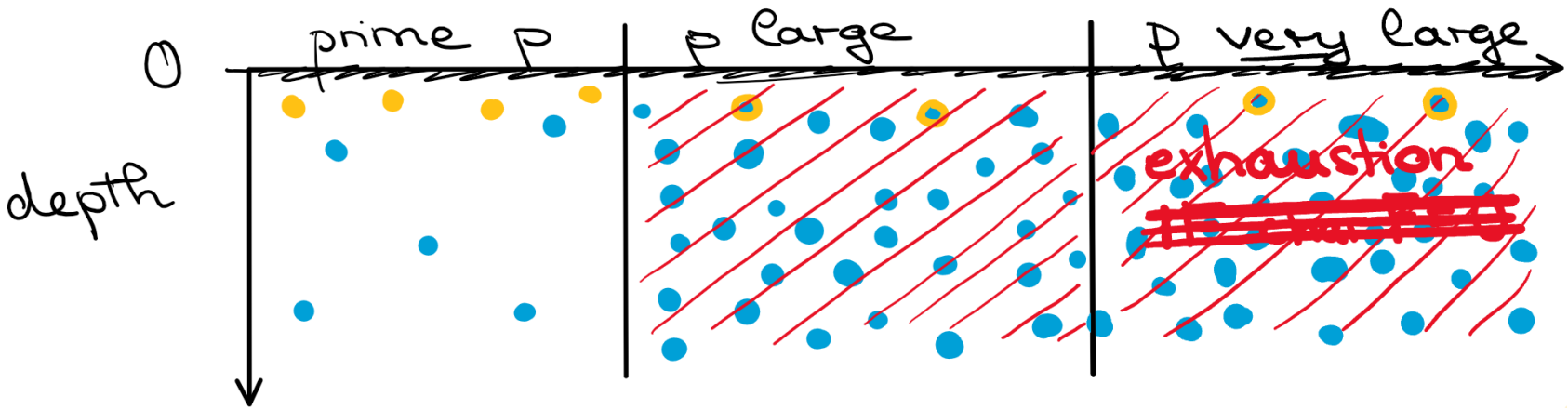
Results

Theorem 1 (F., Oct 2018)

Suppose G splits over a tame extension of F and $p \nmid |W|$, then Yu's construction yields all supercuspidal representations.

type	$A_n\ (n \geqslant 1)$	$B_n, C_n\ (n \geqslant 2)$	$D_n\ (n \geqslant 3)$	E_6
$\lvert W \rvert$	$(n+1)!$	$2^n \cdot n!$	$2^{n-1} \cdot n!$	$2^7 \cdot 3^4 \cdot 5$

type	E_7	E_8	F_4	G_2
$\lvert W \rvert$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^2$	$2^2 \cdot 3$



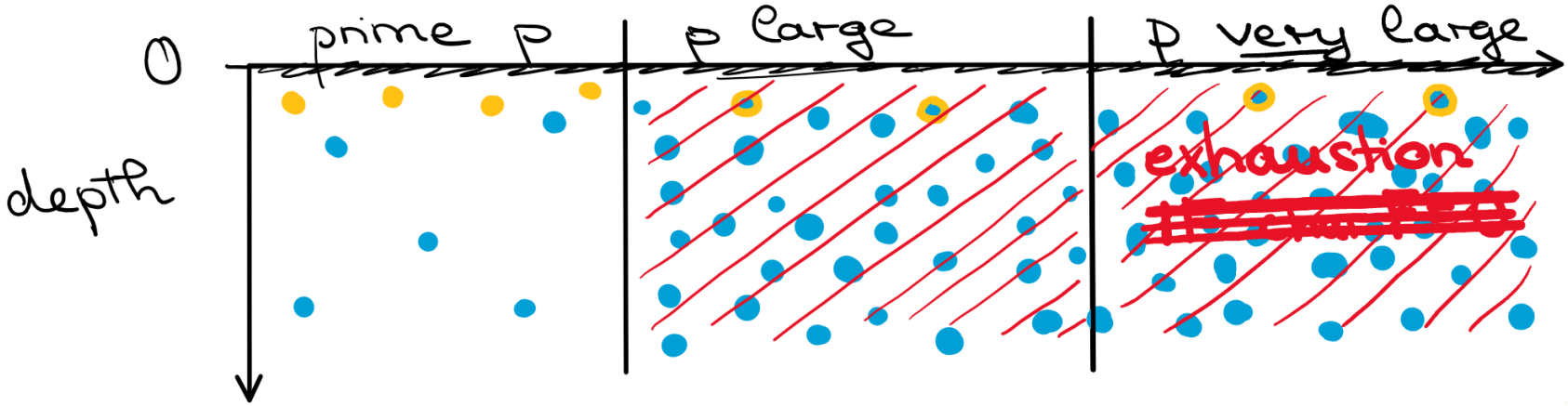
Results

Theorem 1 (F., Oct 2018)

Suppose G splits over a tame extension of F and $p \nmid |W|$, then Yu's construction yields all supercuspidal representations.

Theorem 2 (F., May 2019)

A construction analogous to Yu's construction yields all cuspidal $\overline{\mathbb{F}}_\ell$-representations if $p \nmid |W|$ (and G is tame).



Results

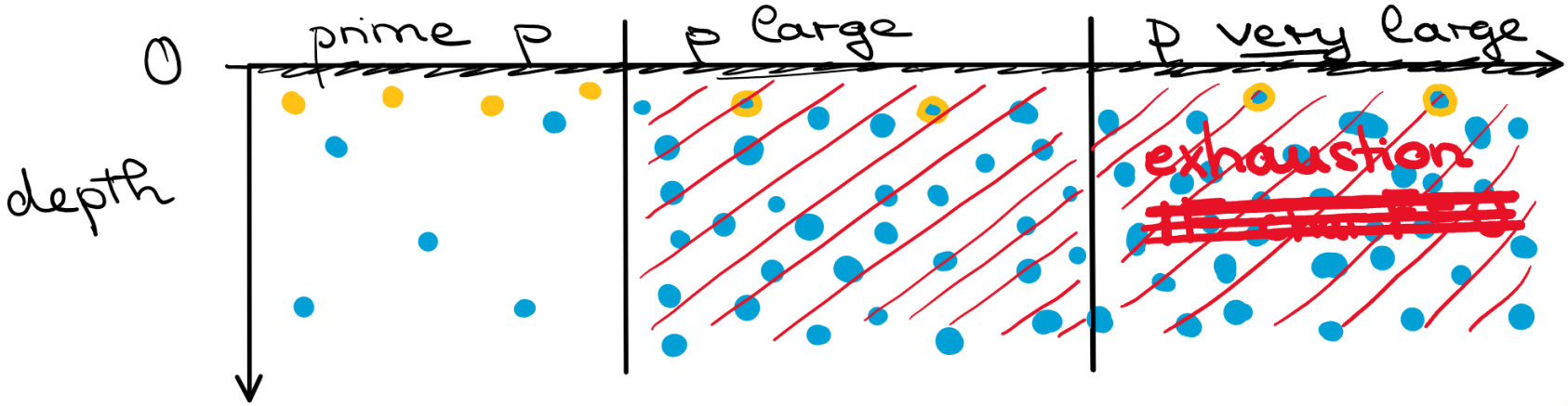
Theorem 1 (F., Oct 2018)

Suppose G splits over a tame extension of F and $p \nmid |W|$, then Yu's construction yields all supercuspidal representations.

Theorem 2 (F., May 2019)

A construction analogous to Yu's construction yields all cuspidal $\overline{\mathbb{F}}_\ell$-representations if $p \nmid |W|$ (and G is tame).

The condition $p \nmid |W|$ is optimal in general*. (F., Jan 2018)



Results

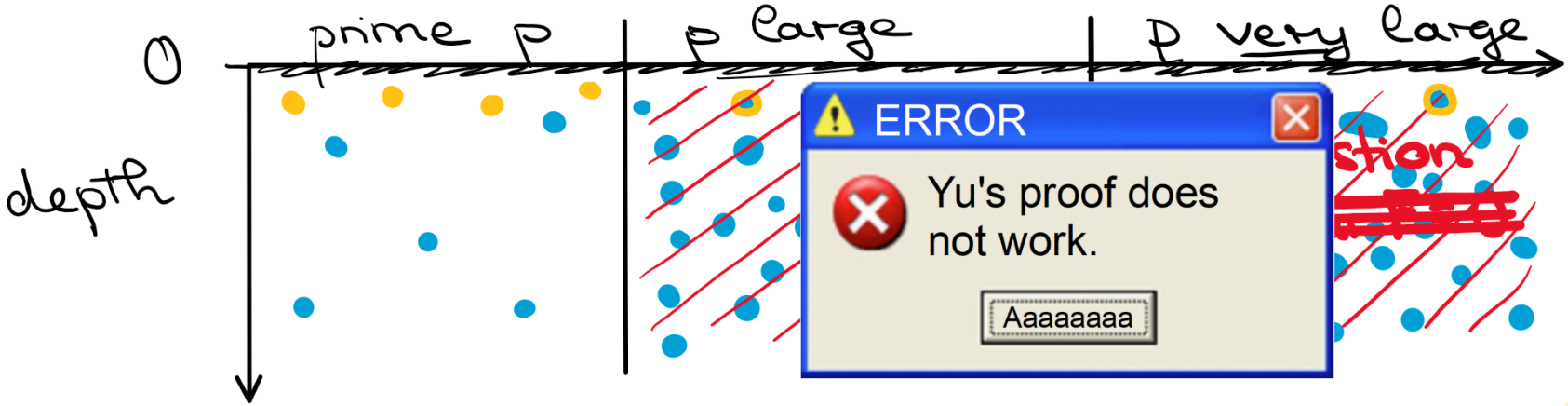
Theorem 1 (F., Oct 2018)

Suppose G splits over a tame extension of F and $p \nmid |W|$, then Yu's construction yields all supercuspidal representations.

Theorem 2 (F., May 2019)

A construction analogous to Yu's construction yields all cuspidal $\overline{\mathbb{F}}_\ell$-representations if $p \nmid |W|$ (and G is tame).

The condition $p \nmid |W|$ is optimal in general*. (F., Jan 2018)



Results

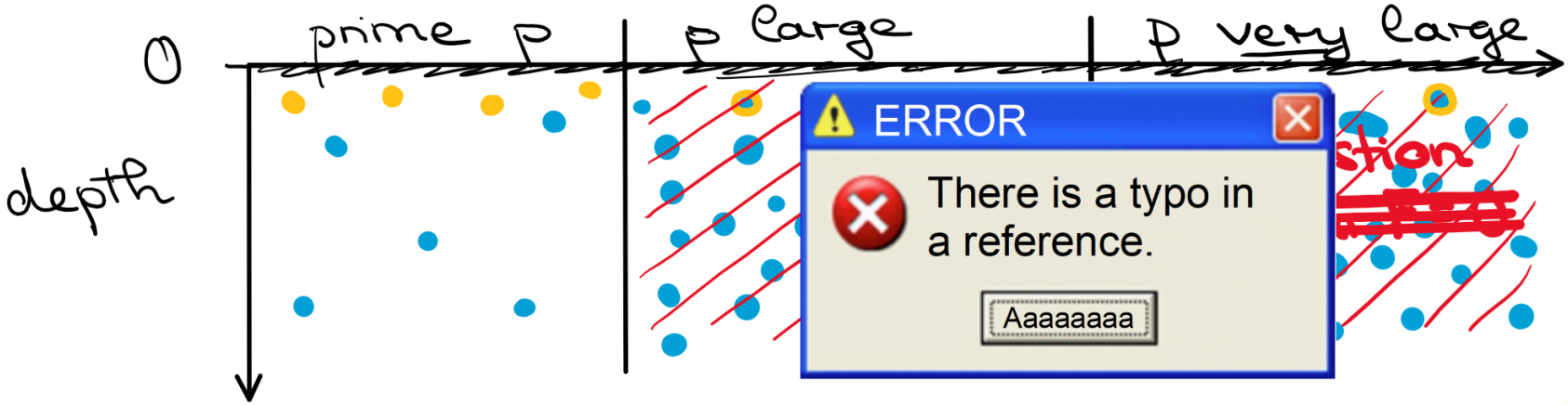
Theorem 1 (F., Oct 2018)

Suppose G splits over a tame extension of F and $p \nmid |W|$, then Yu's construction yields all supercuspidal representations.

Theorem 2 (F., May 2019)

A construction analogous to Yu's construction yields all cuspidal $\overline{\mathbb{F}}_\ell$-representations if $p \nmid |W|$ (and G is tame).

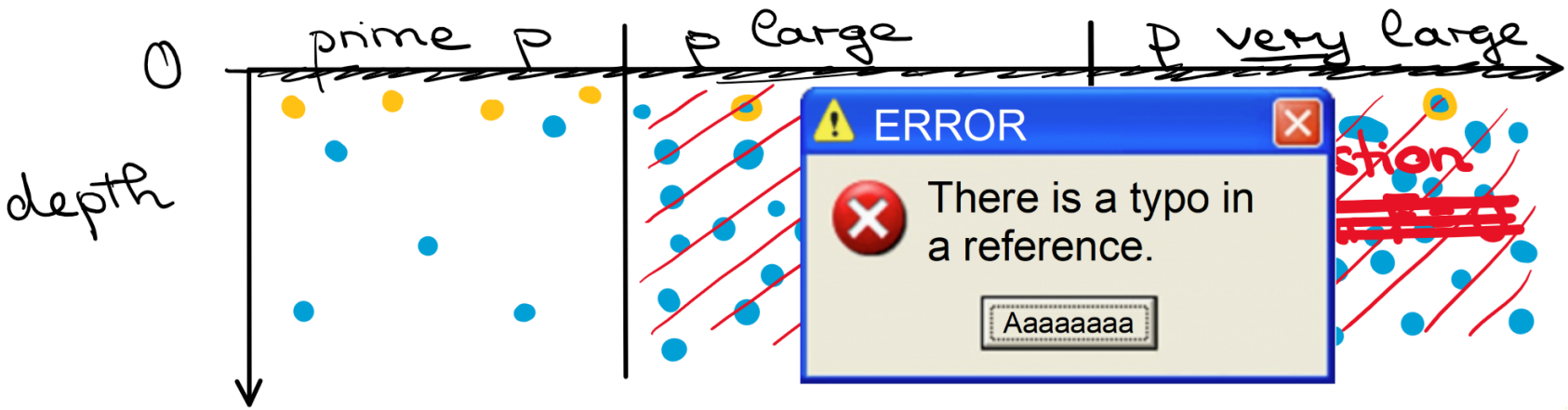
The condition $p \nmid |W|$ is optimal in general*. (F., Jan 2018)



Results continued

Proposition 3 (F., Aug 2019)

There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).



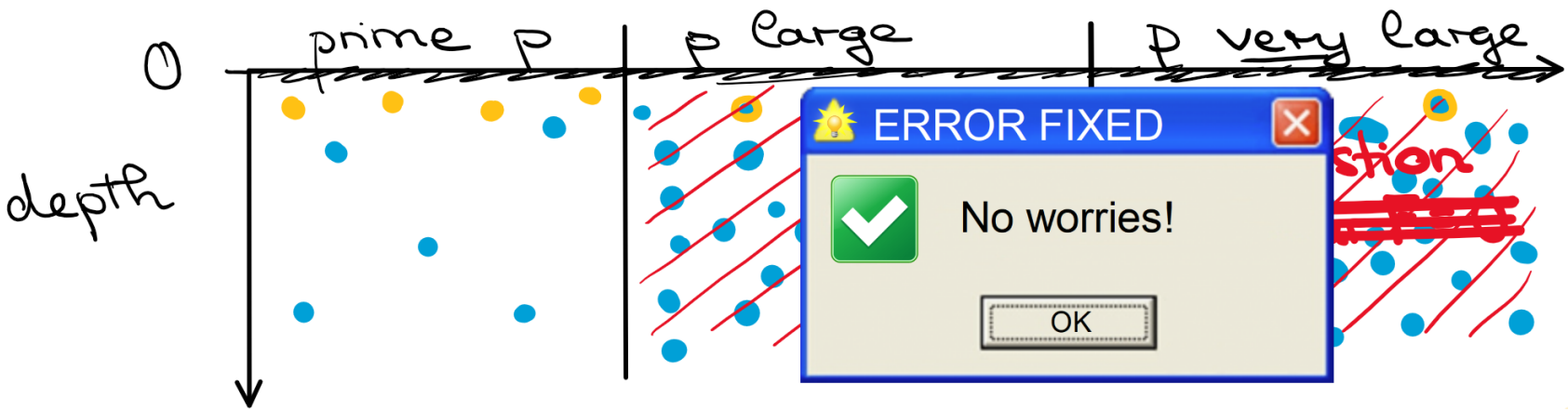
Results continued

Proposition 3 (F., Aug 2019)

There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).

Theorem 4 (F., Aug 2019)

Yu's construction yields indeed supercuspidal representations.



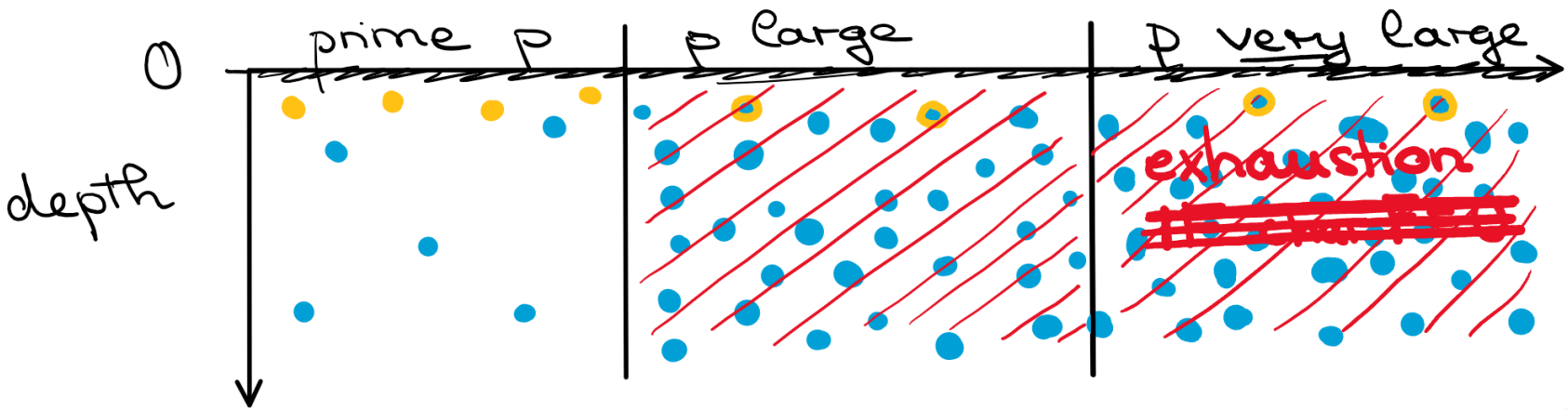
Results continued

Proposition 3 (F., Aug 2019)

There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).

Theorem 4 (F., Aug 2019)

Yu's construction yields indeed supercuspidal representations.



Results continued

Proposition 3 (F., Aug 2019)

There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).

Theorem 4 (F., Aug 2019)

Yu's construction yields indeed supercuspidal representations.

Approach to construct supercuspidal representations

1. Construct a representation ρ_K of a compact (mod center) subgroup $K \subset G$ (e.g. $K = \mathrm{SL}_n(\mathbb{Z}_p)$ inside $G = \mathrm{SL}_n(\mathbb{Q}_p)$).

Results continued

Proposition 3 (F., Aug 2019)

There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).

Theorem 4 (F., Aug 2019)

Yu's construction yields indeed supercuspidal representations.

Approach to construct supercuspidal representations

1. Construct a representation ρ_K of a compact (mod center) subgroup $K \subset G$ (e.g. $K = \mathrm{SL}_n(\mathbb{Z}_p)$ inside $G = \mathrm{SL}_n(\mathbb{Q}_p)$).
2. Build a representation of G from the representation ρ_K (keyword: compact-induction).

Example of a supercuspidal representation

$G = \mathrm{SL}_2(F)$,

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$$\rho_K : \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix}$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$G_{x,0.5}$

$$\rho_K : \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix}$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$G_{x,0.5}$ $\qquad$ $G_{x,0.5}$ / $G_{x,0.5+}$

$$\rho_K : \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$G_{x,0.5}$ $\qquad$ $G_{x,0.5}$ / $G_{x,0.5+}$

$$\rho_K : \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$$

$$\simeq \begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix}$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$G_{x,0.5}$ $\qquad$ $G_{x,0.5}$ / $G_{x,0.5+}$

$$\rho_K : \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$$

$$\simeq \begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mapsto a + b$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$G_{x,0.5}$ $\qquad$ $G_{x,0.5}$ / $G_{x,0.5+}$

$$\rho_K : \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$$

$$\simeq \begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix} \to \mathbb{F}_q$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mapsto a + b$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$G_{x,0.5}$ $\qquad$ $G_{x,0.5}$ / $G_{x,0.5+}$

$$\rho_K : \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$$

$$\simeq \begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix} \to \mathbb{F}_q \to \mathbb{C}^*$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mapsto a + b$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$$\rho_K : \underbrace{\begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix}}_{G_{x,0.5}} \twoheadrightarrow \underbrace{\begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}}_{G_{x,0.5} \,/\, G_{x,0.5+}}$$

$$\simeq \begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix} \to \mathbb{F}_q \to \mathbb{C}^*$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mapsto a + b$$

Supercuspidal representation:

$$\text{c-ind}_K^G \rho_K = \left\{ f : G \to \mathbb{C} \;\middle|\; f(kg) = \rho_K(k) f(g)\ \forall g \in G, k \in K \right\}$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$G_{x,0.5}$ $\qquad$ $G_{x,0.5}$ / $G_{x,0.5+}$

$$\rho_K : \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$$

$$\simeq \begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix} \to \mathbb{F}_q \to \mathbb{C}^*$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mapsto a + b$$

Supercuspidal representation:

$$\text{c-ind}_K^G \rho_K = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(kg) = \rho_K(k) f(g)\ \forall g \in G, k \in K \\ f \text{ compactly supported} \end{array} \right\}$$

Example of a supercuspidal representation

$$G = \mathrm{SL}_2(F),\ K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$$

$$\rho_K : K \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*,\ \rho_K : \{\pm 1\} \to 1 \in \mathbb{C}^*$$

$$\rho_K : \underbrace{\begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix}}_{G_{x,0.5}} \twoheadrightarrow \underbrace{\begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}}_{G_{x,0.5} \,/\, G_{x,0.5+}}$$

$$\simeq \begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix} \to \mathbb{F}_q \to \mathbb{C}^*$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mapsto a + b$$

Supercuspidal representation:

$$\text{c-ind}_K^G \rho_K = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(kg) = \rho_K(k) f(g)\ \forall g \in G, k \in K \\ f \text{ compactly supported} \end{array} \right\}$$

G-action: $g.f(\star) = f(\star \cdot g)$

Results continued continued

Proposition 3 (F., Aug 2019)

There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).

Theorem 4 (F., Aug 2019)

Yu's construction yields indeed supercuspidal representations:

Results continued continued

Proposition 3 (F., Aug 2019)

There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).

Theorem 4 (F., Aug 2019)

Yu's construction yields indeed supercuspidal representations: $\text{c-ind}_{K_{\text{Yu}}}^{G} \rho_{K_{\text{Yu}}}$.

Results continued continued

Proposition 3 (F., Aug 2019)

There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).

Theorem 4 (F., Aug 2019)

Yu's construction yields indeed supercuspidal representations: $\text{c-ind}_{K_{\text{Yu}}}^{G} \rho_{K_{\text{Yu}}}$.

Theorem 5 (F.–Kaletha–Spice, 2019/2020)

There exists a character $\epsilon : K_{\text{Yu}} \to \{\pm 1\}$ such that Yu's Prop 14.1 and Thm 14.2 are satisfied for the twisted representation $\epsilon\rho_{K_{\text{Yu}}}$ of K_{Yu}.

Results continued continued

Proposition 3 (F., Aug 2019)

There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).

Theorem 4 (F., Aug 2019)

Yu's construction yields indeed supercuspidal representations: $\text{c-ind}_{K_{\text{Yu}}}^{G} \rho_{K_{\text{Yu}}}$.

Theorem 5 (F.–Kaletha–Spice, 2019/2020)

There exists a character $\epsilon : K_{\text{Yu}} \to \{\pm 1\}$ such that Yu's Prop 14.1 and Thm 14.2 are satisfied for the twisted representation $\epsilon\rho_{K_{\text{Yu}}}$ of K_{Yu}. In particular, $\text{c-ind}_{K_{\text{Yu}}}^{G} \epsilon\rho_{K_{\text{Yu}}}$ is supercuspidal.

Results continued continued

Theorem 5 (F.–Kaletha–Spice, 2019/2020)

There exists a character $\epsilon : K \to \{\pm 1\}$ such that Yu's Prop 14.1 and Thm 14.2 are satisfied for the twisted representation $\epsilon\rho_{K_{\mathrm{Yu}}}$ of K. In particular, c-ind$_{K_{\mathrm{Yu}}}^{G} \epsilon\rho_{K_{\mathrm{Yu}}}$ is supercuspidal.

Applications of Theorem 5

Results continued continued

Theorem 5 (F.–Kaletha–Spice, 2019/2020)

There exists a character $\epsilon : K \to \{\pm 1\}$ such that Yu's Prop 14.1 and Thm 14.2 are satisfied for the twisted representation $\epsilon\rho_{K_{\mathrm{Yu}}}$ of K. In particular, c-ind$_{K_{\mathrm{Yu}}}^{G} \epsilon\rho_{K_{\mathrm{Yu}}}$ is supercuspidal.

Applications of Theorem 5

- Formula for Harish-Chandra character of these supercuspidal representations (Spice, in progress)

Results continued continued

Theorem 5 (F.–Kaletha–Spice, 2019/2020)

There exists a character $\epsilon : K \to \{\pm 1\}$ such that Yu's Prop 14.1 and Thm 14.2 are satisfied for the twisted representation $\epsilon\rho_{K_{\mathrm{Yu}}}$ of K. In particular, c-ind$_{K_{\mathrm{Yu}}}^{G} \epsilon\rho_{K_{\mathrm{Yu}}}$ is supercuspidal.

Applications of Theorem 5

- Formula for Harish-Chandra character of these supercuspidal representations (Spice, in progress)
- Candidate for local Langlands correspondence for non-singular representations (Kaletha, Dec 2019)

Results continued continued

Theorem 5 (F.–Kaletha–Spice, 2019/2020)

There exists a character $\epsilon : K \to \{\pm 1\}$ such that Yu's Prop 14.1 and Thm 14.2 are satisfied for the twisted representation $\epsilon\rho_{K_{\mathrm{Yu}}}$ of K. In particular, c-ind$_{K_{\mathrm{Yu}}}^{G} \epsilon\rho_{K_{\mathrm{Yu}}}$ is supercuspidal.

Applications of Theorem 5

- Formula for Harish-Chandra character of these supercuspidal representations (Spice, in progress)
- Candidate for local Langlands correspondence for non-singular representations (Kaletha, Dec 2019)
- Character identities for the LLC for regular supercuspidal representations (in progress)

Results continued continued

Theorem 5 (F.–Kaletha–Spice, 2019/2020)

There exists a character $\epsilon : K \to \{\pm 1\}$ such that Yu's Prop 14.1 and Thm 14.2 are satisfied for the twisted representation $\epsilon\rho_{K_{\mathrm{Yu}}}$ of K. In particular, c-ind$_{K_{\mathrm{Yu}}}^{G} \epsilon\rho_{K_{\mathrm{Yu}}}$ is supercuspidal.

Applications of Theorem 5

- Formula for Harish-Chandra character of these supercuspidal representations (Spice, in progress)
- Candidate for local Langlands correspondence for non-singular representations (Kaletha, Dec 2019)
- Character identities for the LLC for regular supercuspidal representations (in progress)
- Hecke-algebra identities (hope)

Types

Theory of types: Obtain information about representations of G by studying representations of compact open subgroups of G.

Types

Theory of types: Obtain information about representations of G by studying representations of compact open subgroups of G.

Definition

A pair (K, ρ) of a compact open subgroup $K \subset G$ and a finite dimensional representation ρ is a **supercuspidal type** if

Types

Theory of types: Obtain information about representations of G by studying representations of compact open subgroups of G.

Definition

A pair (K, ρ) of a compact open subgroup $K \subset G$ and a finite dimensional representation ρ is a **supercuspidal type** if for every irreducible representation π:

$$\mathrm{Hom}_K(\rho, \pi) \neq \{0\} \Rightarrow \pi \text{ is supercuspidal}$$

Types

Theory of types: Obtain information about representations of G by studying representations of compact open subgroups of G.

Definition

A pair (K, ρ) of a compact open subgroup $K \subset G$ and a finite dimensional representation ρ is a **supercuspidal type** if for every irreducible representation π:

$$\mathrm{Hom}_K(\rho, \pi) \neq \{0\} \Rightarrow \pi \text{ is supercuspidal}$$

Example: $G = \mathrm{SL}_2(F)$: $K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix}$, $\rho = \rho_K : K \to \mathbb{F}_q \to \mathbb{C}^*$

Types

Theory of types: Obtain information about representations of G by studying representations of compact open subgroups of G.

Definition

A pair (K, ρ) of a compact open subgroup $K \subset G$ and a finite dimensional representation ρ is a **supercuspidal type** if for every irreducible representation π:

$$\mathrm{Hom}_K(\rho, \pi) \neq \{0\} \Rightarrow \pi \text{ is supercuspidal}$$

Example: $G = \mathrm{SL}_2(F)$: $K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix}$, $\rho = \rho_K : K \to \mathbb{F}_q \to \mathbb{C}^*$

Definition

An **omni-supercuspidal type** of level p^m is a pair (U, λ), where U is an open compact subgroup of G and $\lambda : U \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}$ such that $(U, \psi \circ \lambda)$ is a supercuspidal type for every nontrivial character $\psi : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{C}^*$.

Omni-supercuspidal types - results

Definition

An **omni-supercuspidal type** of level p^m is a pair (U, λ), where U is an open compact subgroup of G and $\lambda : U \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}$ such that $(U, \psi \circ \lambda)$ is a supercuspidal type for every nontrivial character $\psi : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{C}^*$.

Omni-supercuspidal types - results

Definition

An **omni-supercuspidal type** of level p^m is a pair (U, λ), where U is an open compact subgroup of G and $\lambda : U \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}$ such that $(U, \psi \circ \lambda)$ is a supercuspidal type for every nontrivial character $\psi : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{C}^*$.

Theorem 6 (F.–Shin, Sep 2020, draft available at `https://www.dpmms.cam.ac.uk/~jf457/research.html`)

Assume G splits over a tame extension, $\mathrm{char}(F) = 0$ and $p > \mathrm{Cox}(G)$. Then there exists a sequence $\{(U_m, \lambda_m)\}_{m \geqslant 1}$ such that

- each (U_m, λ_m) is an omni-supercuspidal type of level p^m,
- $U_1 \rhd U_2 \rhd \cdots$, and $\{U_m\}_{m \geqslant 1}$ forms a basis of open neighborhoods of 1.

Omni-supercuspidal types - applications

Theorem 6 (F.–Shin, Sep 2020, draft available at `https://www.dpmms.cam.ac.uk/~jf457/research.html`)

Assume G splits over a tame extension, $\mathrm{char}(F) = 0$ and $p > \mathrm{Cox}(G)$. Then there exists a sequence $\{(U_m, \lambda_m)\}_{m \geqslant 1}$ such that

- each (U_m, λ_m) is an omni-supercuspidal type of level p^m,
- $U_1 \rhd U_2 \rhd \cdots$, and $\{U_m\}_{m \geqslant 1}$ forms a basis of open neighborhoods of 1.

Omni-supercuspidal types - applications

Theorem 6 (F.–Shin, Sep 2020, draft available at `https://www.dpmms.cam.ac.uk/~jf457/research.html`)

Assume G splits over a tame extension, $\mathrm{char}(F) = 0$ and $p > \mathrm{Cox}(G)$. Then there exists a sequence $\{(U_m, \lambda_m)\}_{m \geqslant 1}$ such that

- each (U_m, λ_m) is an omni-supercuspidal type of level p^m,
- $U_1 \rhd U_2 \rhd \cdots$, and $\{U_m\}_{m \geqslant 1}$ forms a basis of open neighborhoods of 1.

Applications of Theorem 6

1. congruences between arbitrary algebraic automorphic forms and those that are supercuspidal at p
   (for groups that are compact-mod-center at infinity)

Omni-supercuspidal types - applications

Theorem 6 (F.–Shin, Sep 2020, draft available at `https://www.dpmms.cam.ac.uk/~jf457/research.html`)

Assume G splits over a tame extension, $\mathrm{char}(F) = 0$ and $p > \mathrm{Cox}(G)$. Then there exists a sequence $\{(U_m, \lambda_m)\}_{m \geqslant 1}$ such that

- each (U_m, λ_m) is an omni-supercuspidal type of level p^m,
- $U_1 \rhd U_2 \rhd \cdots$, and $\{U_m\}_{m \geqslant 1}$ forms a basis of open neighborhoods of 1.

Applications of Theorem 6

1. congruences between arbitrary algebraic automorphic forms and those that are supercuspidal at p
   (for groups that are compact-mod-center at infinity)
2. (classical) automorphic forms with fixed weight that are supercuspidal at p are dense in the spectrum of the Hecke algebra (based on work by Emerton–Paškūnas)

Omni-supercuspidal types - applications continued

Applications of Theorem 6

1. congruences between arbitrary algebraic automorphic forms and those that are supercuspidal at p
   (for groups that are compact-mod-center at infinity)
2. (classical) automorphic forms with fixed weight that are supercuspidal at p are dense in the spectrum of the Hecke algebra
   (based on work of Emerton–Paškūnas)

Omni-supercuspidal types - applications continued

Applications of Theorem 6

1. congruences between arbitrary algebraic automorphic forms and those that are supercuspidal at p
   (for groups that are compact-mod-center at infinity)
2. (classical) automorphic forms with fixed weight that are supercuspidal at p are dense in the spectrum of the Hecke algebra
   (based on work of Emerton–Paškūnas)

Applications of 1 or 2:
"May assume that an automorphic form is supercuspidal at p."

Omni-supercuspidal types - applications continued

Applications of Theorem 6

1. congruences between arbitrary algebraic automorphic forms and those that are supercuspidal at p
   (for groups that are compact-mod-center at infinity)
2. (classical) automorphic forms with fixed weight that are supercuspidal at p are dense in the spectrum of the Hecke algebra
   (based on work of Emerton–Paškūnas)

Applications of 1 or 2:
"May assume that an automorphic form is supercuspidal at p."

Example: Global Langlands correspondence for GL_n:

Omni-supercuspidal types - applications continued

Applications of Theorem 6

1. congruences between arbitrary algebraic automorphic forms and those that are supercuspidal at p
   (for groups that are compact-mod-center at infinity)
2. (classical) automorphic forms with fixed weight that are supercuspidal at p are dense in the spectrum of the Hecke algebra
   (based on work of Emerton–Paškūnas)

Applications of 1 or 2:
"May assume that an automorphic form is supercuspidal at p."

Example: Global Langlands correspondence for GL_n:

nice autom rep Π of GL_n
such that Π is (ess) square integrable at some prime

Omni-supercuspidal types - applications continued

Applications of Theorem 6

1. congruences between arbitrary algebraic automorphic forms and those that are supercuspidal at p
   (for groups that are compact-mod-center at infinity)
2. (classical) automorphic forms with fixed weight that are supercuspidal at p are dense in the spectrum of the Hecke algebra
   (based on work of Emerton–Paškūnas)

Applications of 1 or 2:
"May assume that an automorphic form is supercuspidal at p."

Example: Global Langlands correspondence for GL_n:

nice autom rep Π of GL_n
such that Π is (ess) square integrable at some prime

Omni-supercuspidal types - applications continued

Applications of Theorem 6

1. congruences between arbitrary algebraic automorphic forms and those that are supercuspidal at p
   (for groups that are compact-mod-center at infinity)
2. (classical) automorphic forms with fixed weight that are supercuspidal at p are dense in the spectrum of the Hecke algebra
   (based on work of Emerton–Paškūnas)

Applications of 1 or 2:
"May assume that an automorphic form is supercuspidal at p."

Example: Global Langlands correspondence for GL_n:

nice autom rep Π of GL_n such that Π is (ess) square integrable at some prime

Galois representation with $\overline{\mathbb{Q}}_p$-coefficients

Omni-supercuspidal types - applications continued

Applications of Theorem 6

1. congruences between arbitrary algebraic automorphic forms and those that are supercuspidal at p
   (for groups that are compact-mod-center at infinity)
2. (classical) automorphic forms with fixed weight that are supercuspidal at p are dense in the spectrum of the Hecke algebra
   (based on work of Emerton–Paškūnas)

Applications of 1 or 2:
"May assume that an automorphic form is supercuspidal at p."

Example: Global Langlands correspondence for GL_n:

nice autom rep Π of GL_n such that Π is (ess) square integrable at some prime

Galois representation with $\overline{\mathbb{Q}}_p$-coefficients

1 or 2 $\Rightarrow$ remove square integrability assumption
(F–Shin, Paškūnas)

Congruences of algebraic automorphic forms

$\mathcal{G}$ (connected) reductive group over $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R})$ is compact,

Congruences of algebraic automorphic forms

$\mathcal{G}$ (connected) reductive group over $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R})$ is compact, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$,

Congruences of algebraic automorphic forms

$\mathcal{G}$ (connected) reductive group over $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R})$ is compact, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$, $U^p \subset \prod_{\ell \neq p}{}' \mathcal{G}(\mathbb{Q}_l)$ compact open,

Congruences of algebraic automorphic forms

$\mathcal{G}$ (connected) reductive group over $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R})$ is compact, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$, $U^p \subset \prod_{\ell \neq p}{}' \mathcal{G}(\mathbb{Q}_l)$ compact open, $U_p \subset \mathcal{G}(\mathbb{Q}_p)$

Congruences of algebraic automorphic forms

$\mathcal{G}$ (connected) reductive group over $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R})$ is compact, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$, $U^p \subset \prod_{\ell \neq p}{}' \mathcal{G}(\mathbb{Q}_l)$ compact open, $U_p \subset \mathcal{G}(\mathbb{Q}_p)$

Definition (algebraic automorphic forms)

a. $M(U_p U^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}) / U_p U^p \mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$

Congruences of algebraic automorphic forms

$\mathcal{G}$ (connected) reductive group over $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R})$ is compact, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$, $U^p \subset \prod_{\ell \neq p}{}' \mathcal{G}(\mathbb{Q}_l)$ compact open, $U_p \subset \mathcal{G}(\mathbb{Q}_p)$

Definition (algebraic automorphic forms)

a. $M(U_p U^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}) / U_p U^p \mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$

b. $m \in \mathbb{N}$, $A_m := \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m - 1})$,

Congruences of algebraic automorphic forms

$\mathcal{G}$ (connected) reductive group over $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R})$ is compact, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$, $U^p \subset \prod_{\ell \neq p}{}' \mathcal{G}(\mathbb{Q}_l)$ compact open, $U_p \subset \mathcal{G}(\mathbb{Q}_p)$

Definition (algebraic automorphic forms)

a. $M(U_p U^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}) / U_p U^p \mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$

b. $m \in \mathbb{N}$, $A_m := \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m - 1})$, suppose U_p acts smoothly on A_m smoothly:

Congruences of algebraic automorphic forms

$\mathcal{G}$ (connected) reductive group over $\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R})$ is compact, $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$, $U^p \subset \prod_{\ell \neq p}{}' \mathcal{G}(\mathbb{Q}_l)$ compact open, $U_p \subset \mathcal{G}(\mathbb{Q}_p)$

Definition (algebraic automorphic forms)

a. $M(U_p U^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}) / U_p U^p \mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$

b. $m \in \mathbb{N}$, $A_m := \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m - 1})$,
suppose U_p acts smoothly on A_m smoothly:
$M(U_p U^p, A_m) = \{\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}) / U^p \mathcal{G}(\mathbb{R})^\circ \to A_m \mid$
$f(g u_p) = u_p^{-1} f(g)\ \forall g \in \mathcal{G}(\mathbb{A}), u_p \in U_p\}$

Congruences of algebraic automorphic forms

Definition (algebraic automorphic forms)

a. $M(U_pU^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U_pU^p\mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$

b. $m \in \mathbb{N}$, $A_m := \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m-1})$,
suppose U_p acts smoothly on A_m smoothly
$M(U_pU^p, A_m) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U^p\mathcal{G}(\mathbb{R})^\circ \to A_m \mid$
$f(gu_p) = u_p^{-1}f(g)\ \forall g \in \mathcal{G}(\mathbb{A}), u_p \in U_p\}$

Congruences of algebraic automorphic forms

Definition (algebraic automorphic forms)

a. $M(U_pU^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U_pU^p\mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$

b. $m \in \mathbb{N}$, $A_m := \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m-1})$,
suppose U_p acts smoothly on A_m smoothly
$M(U_pU^p, A_m) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U^p\mathcal{G}(\mathbb{R})^\circ \to A_m \mid$
$f(gu_p) = u_p^{-1}f(g)\ \forall g \in \mathcal{G}(\mathbb{A}), u_p \in U_p\}$

Theorem 7 (F.–Shin, Sep 2020, draft available on my homepage, special case due to Scholze, 2018)

Let $p > \mathrm{Cox}(\mathcal{G})$. Then $\exists U_{p,m} \subset \mathcal{G}(\mathbb{Q}_p)$ with $U_{p,m} \circlearrowright A_m$

2. $M(U_{p,m}U^p, \mathbb{Z}_p)/(p^m) \simeq M(U_{p,m}U^p, A_m)/(1 - T)$

Congruences of algebraic automorphic forms

Definition (algebraic automorphic forms)

a. $M(U_pU^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U_pU^p\mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$

b. $m \in \mathbb{N}$, $A_m := \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m-1})$,
suppose U_p acts smoothly on A_m smoothly
$M(U_pU^p, A_m) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U^p\mathcal{G}(\mathbb{R})^\circ \to A_m \mid$
$f(gu_p) = u_p^{-1}f(g)\ \forall g \in \mathcal{G}(\mathbb{A}), u_p \in U_p\}$

Theorem 7 (F.–Shin, Sep 2020, draft available on my homepage, special case due to Scholze, 2018)

Let $p > \mathrm{Cox}(\mathcal{G})$. Then $\exists U_{p,m} \subset \mathcal{G}(\mathbb{Q}_p)$ with $U_{p,m} \curvearrowright A_m$

2. $M(U_{p,m}U^p, \mathbb{Z}_p)/(p^m) \simeq M(U_{p,m}U^p, A_m)/(1-T)$

3. If $\Pi = \prod' \Pi_\ell \otimes \Pi_\infty$ is an automorphic rep contributing to $M(U_{p,m}U^p, A_m) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$, then Π_p is supercuspidal.

Congruences of algebraic automorphic forms

Definition (algebraic automorphic forms)

a. $M(U_pU^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U_pU^p\mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$

b. $m \in \mathbb{N}$, $A_m := \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m-1})$,
suppose U_p acts smoothly on A_m smoothly
$M(U_pU^p, A_m) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U^p\mathcal{G}(\mathbb{R})^\circ \to A_m \mid$
$f(gu_p) = u_p^{-1}f(g)\ \forall g \in \mathcal{G}(\mathbb{A}), u_p \in U_p\}$

Theorem 7 (F.–Shin, Sep 2020, draft available on my homepage, special case due to Scholze, 2018)

Let $p > \mathrm{Cox}(\mathcal{G})$. Then $\exists U_{p,m} \subset \mathcal{G}(\mathbb{Q}_p)$ with $U_{p,m} \curvearrowright A_m$

1. $U_{p,1} \rhd U_{p,2} \rhd \cdots$, and $\{U_{p,m}\}_{m\geqslant 1}$ forms a basis of open neighborhoods of $1 \in \mathcal{G}(\mathbb{Q}_p)$
2. $M(U_{p,m}U^p, \mathbb{Z}_p)/(p^m) \simeq M(U_{p,m}U^p, A_m)/(1-T)$
3. If $\Pi = \prod' \Pi_\ell \otimes \Pi_\infty$ is an automorphic rep contributing to $M(U_{p,m}U^p, A_m) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$, then Π_p is supercuspidal.

Congruences of algebraic automorphic forms

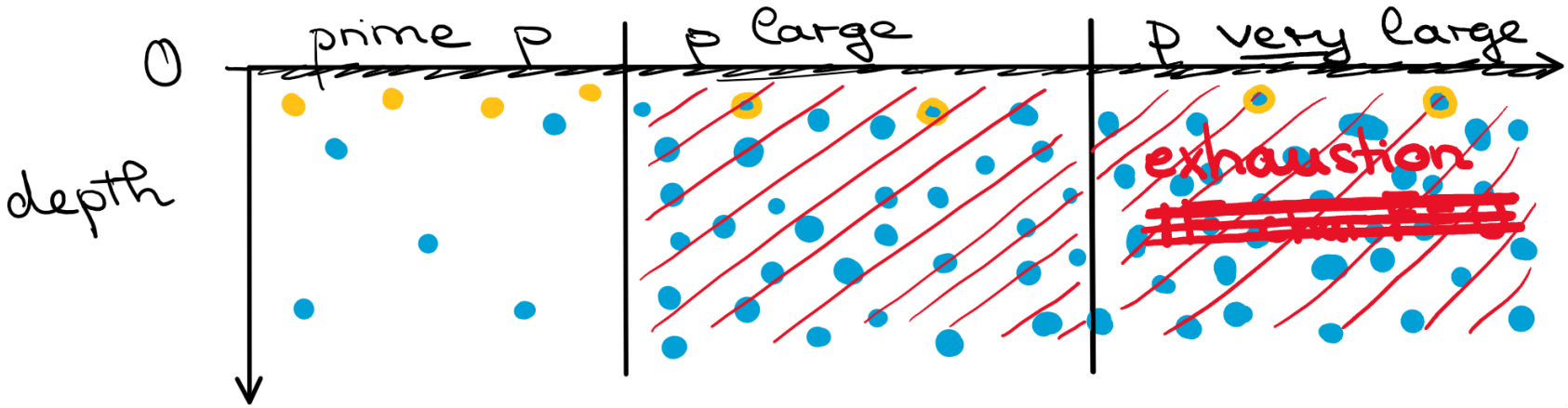
Definition (algebraic automorphic forms)

a. $M(U_pU^p, \mathbb{Z}_p) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U_pU^p\mathcal{G}(\mathbb{R})^\circ \to \mathbb{Z}_p\}$

b. $m \in \mathbb{N}$, $A_m := \mathbb{Z}_p[T]/(1 + T + \ldots + T^{p^m-1})$,
suppose U_p acts smoothly on A_m smoothly
$M(U_pU^p, A_m) = \{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U^p\mathcal{G}(\mathbb{R})^\circ \to A_m \mid$
$f(gu_p) = u_p^{-1}f(g)\ \forall g \in \mathcal{G}(\mathbb{A}), u_p \in U_p\}$

Theorem 7 (F.–Shin, Sep 2020, draft available on my homepage, special case due to Scholze, 2018)

Let $p > \mathrm{Cox}(\mathcal{G})$. Then $\exists U_{p,m} \subset \mathcal{G}(\mathbb{Q}_p)$ with $U_{p,m} \curvearrowright A_m$

1. $U_{p,1} \rhd U_{p,2} \rhd \cdots$, and $\{U_{p,m}\}_{m\geqslant 1}$ forms a basis of open neighborhoods of $1 \in \mathcal{G}(\mathbb{Q}_p)$
2. $M(U_{p,m}U^p, \mathbb{Z}_p)/(p^m) \simeq M(U_{p,m}U^p, A_m)/(1-T)$
(+ Hecke algebra)
3. If $\Pi = \prod' \Pi_\ell \otimes \Pi_\infty$ is an automorphic rep contributing to $M(U_{p,m}U^p, A_m) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$, then Π_p is supercuspidal.

The end of the talk,
but only the beginning of the story ...