Isometries of lattices and Hasse principles

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Dick Gross and Curt McMullen (2002)

Characteristic polynomials of isometries of even, unimodular lattices.

DEFINITIONS

Lattice : (L, q)

- L is a free $\mathbf{Z}$-module of finite rank,

- $q : L \times L \rightarrow \mathbf{Z}$ is a symmetric bilinear form.

unimodular : $\det(q) = \pm 1.$

DEFINITIONS

Lattice : (L, q)

- L is a free **Z**-module of finite rank,
- $q : L \times L \to \mathbf{Z}$ is a symmetric bilinear form.

unimodular : $\det(q) = \pm 1.$

even : $q\,(x, x) \equiv 0 \pmod 2$

for all $x \in L$.

SIGNATURE

Well-known :

(L, q) an even, unimodular lattice

signature

$$(r, s)$$

$$r \equiv s \pmod{8}$$

DEFINITIONS

Isometry of (L, q) : element of $\mathrm{SO}(L, q)$

$$t : L \to L$$

$$q(t(x), t(y)) = q(x, y).$$

$\det(t) = 1.$

GEOMETRY : K3 SURFACES

X complex analytic $K3$ surface

$L = H^2(X, \mathbf{Z})$, $\quad q : L \times L \to \mathbf{Z}$: intersection form.

(L, q) is an even, unimodular lattice.

signature $(3, 19)$

$(L, q) \simeq (-E_8) \oplus (-E_8) \oplus H \oplus H \oplus H$.

GEOMETRY : K3 SURFACES

Automorphism of X induces an isometry of (L, q)

Characteristic polynomial $F \in \mathbf{Z}[X]$ symmetric, product of a

Salem polynomial and of cyclotomic polynomials.

Real root > 1 of the Salem polynomial $=$ dynamical degree of the automorphism (McMullen, Gross-McMullen, Brandhorst,...).

GEOMETRY - KNOTS

$K \subset S^3$ knot, A Seifert matrix. We have $\det(A - A^t) = 1$.

Set

$$q = A + A^t.$$

signature of K : signature of q

The signature is a knot invariant.

Alexander polynomial of K :

$$\Delta(X) = \det(AX - A^t).$$

Δ is symmetric, $\Delta(1) = 1$.

The Alexander polynomial is a knot invariant.

A THEOREM OF SEIFERT

A square integral matrix with $\det(A - A^t) = 1$

Then there exists a knot with Seifert matrix A.

GEOMETRY - KNOTS

Assume that $\det(A) = 1$. Set $t = A^{-1}A^t$.

t is an isometry of q.

$$\det(A).\det(IX - t) = \det(AX - A^t) = \Delta(X)$$

Δ is the characteristic polynomial of t.

Assume that

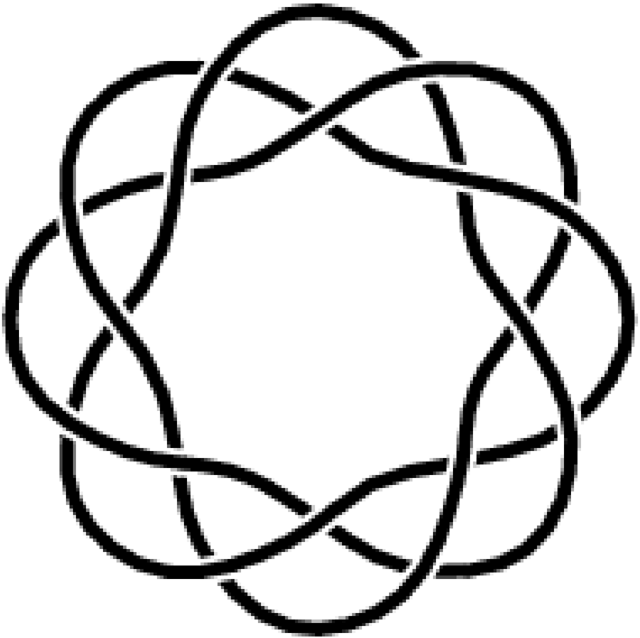
q is unimodular : $\Delta(-1) = \pm 1$

EXAMPLE

$$\Delta_{u,v} = \frac{(X^{uv} - 1)(X - 1)}{(X^{u} - 1)(X^{v} - 1)},$$

$u, v > 1$ odd integers with u and v prime to each other.

Alexander polynomial of the (u, v) torus knot.



GROSS and McMULLEN

Let $F \in \mathbf{Z}[X]$, monic, symmetric, irreducible, $\deg(F) = 2n$.

(r, s) integers $\geqslant 0$ such that

$$r \equiv s \pmod{8},$$

$$r + s = 2n.$$

GROSS and McMULLEN

If there exists an even, unimodular lattice of signature (r, s) having an isometry with characteristic polynomial F, then :

(C 1) $|F(1)|$, $|F(-1)|$ and $(-1)^n F(1) F(-1)$ are all squares.

Set

$$m(F) = \text{number of roots } z \text{ of } F \text{ with } |z| > 1.$$

(C 2)

$$m(F) \leqslant r,\ m(F) \leqslant s,\ m(F) \equiv r \equiv s \pmod 2.$$

GROSS and McMULLEN

$F \in \mathbf{Z}[X]$ monic, symmetric, IRREDUCIBLE, $\deg(F) = 2n$, and (r, s) such that

$$r \equiv s \pmod 8,$$

$$r + s = 2n.$$

Theorem. (Gross-McMullen, 2002) Assume that

$$|F(1)| = |F - 1)| = 1.$$

Then there exists an even, unimodular lattice of signature (r, s) having an isometry with characteristic polynomial $F \iff$ conditions (C 1) and (C 2) hold.

GROSS and McMULLEN

$F \in \mathbf{Z}[X]$ monic, symmetric, IRREDUCIBLE, $\deg(F) = 2n$, and (r, s) such that

$$r \equiv s \pmod{8},$$

$$r + s = 2n.$$

Gross and McMullen (2002) speculate that

conditions (C 1) and (C 2)

may be sufficient for the existence of an even, unimodular lattice of signature (r, s) having an isometry with characteristic polynomial F.

GROSS and McMULLEN

$$F(X) = (X^6 - 3X^5 - X^4 + 5X^3 - X^2 - 3X + 1)(X^4 - X^2 + 1).$$

Gross-McMullen (2002)

Conditions (C 1) and (C 2) hold for $(r, s) = (9, 1)$, but

$$E_8 \oplus H$$

does not have an isometry with characteristic polynomial F.

E.B. and TAELMAN

$F \in \mathbf{Z}[X]$ monic, symmetric, IRREDUCIBLE, $\deg(F) = 2n$, and (r, s) such that

$$r \equiv s \pmod 8,$$

$$r + s = 2n.$$

Theorem. (E.B.-Taelman, 2020) There exists an even, unimodular lattice of signature (r, s) having an isometry with characteristic polynomial $F \iff$ conditions (C 1) and (C 2) hold.

K3 SURFACES

Corollary. Let F be a Salem polynomial of degree 22, and assume that $|F(1)|$, $|F(-1)|$ and $(-1)^n F(1)F(-1)$ are all squares.

Then there exists a complex analytic $K3$ surface X and an automorphism T of X such that the characteristic polynomial of T^* on $H^2(X, \mathbf{Z})$ is F.

Corollary. Let $\Delta \in \mathbf{Z}[X]$ be a monic, symmetric, irreducible polynomial such that $\Delta(1) = 1$. Set $\deg(\Delta) = 2n$, and assume that $\Delta(-1) = (-1)^n$.

Let (r, s) such that $r \equiv s \pmod 8$, $r + s = \deg(\Delta) = 2n$.

There exists a knot with Alexander polynomial equal to Δ and signature (r, s) if and only if condition (C 2) holds.

Follows from Gross-McMullen, since $|\Delta(1)| = |\Delta(-1)| = 1$.

E.B. - TAELMAN : PROOF

Theorem. Let $F \in \mathbf{Z}[X]$ be a monic, symmetric, irreducible polynomial and (r, s) such that $r \equiv s \pmod 8$, $r + s = \deg(F)$.

There exists an even, unimodular lattice of signature (r, s) having an isometry with characteristic polynomial $F \iff$ the conditions (C 1) and (C 2) hold.

Strategy to prove this result :

- Prove it everywhere locally.

- Local-global principle.

LOCAL RESULTS

$F \in \mathbf{Z}[X]$ monic, symmetric, $\deg(F) = 2n$, irrreducible.

(r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$.

Assume that conditions (C 1) and (C 2) hold.

• p odd prime : There exists a unimodular $\mathbf{Z}_p$-lattice having an isometry with characteristic polynomial F.

• $p = 2$: There exists an even, unimodular $\mathbf{Z}_2$-lattice having an isometry with characteristic polynomial F.

• There exists a non-degenerate quadratic form over $\mathbf{R}$ of signature (r, s) having an isometry with characteristic polynomial F.

REDUCIBLE POLYNOMIALS ?

$$F = \prod_{f \in I} f$$

$f \in \mathbf{Z}[X]$ distinct, irreducible, symmetric, of even degree.

- Local results hold.

Conditions (C 1) and (C 2) are local conditions.

- Local-global principle does not always hold.

LOCAL RESULTS

$F \in \mathbf{Z}[X]$ monic, symmetric, $\deg(F) = 2n$, as above.

(r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$.

Assume that conditions (C 1) and (C 2) hold.

• p odd prime : There exists a unimodular $\mathbf{Z}_p$-lattice having an isometry with characteristic polynomial F.

• $p = 2$: There exists an even, unimodular $\mathbf{Z}_2$-lattice having an isometry with characteristic polynomial F.

• There exists a non-degenerate quadratic form over $\mathbf{R}$ of signature (r, s) having an isometry with characteristic polynomial F.

LOCAL-GLOBAL

Example.

$F(X) = (X^6 - 3X^5 - X^4 + 5X^3 - X^2 - 3X + 1)(X^4 - X^2 + 1)$.

$F(1) = -1$, $F(-1) = 1$.

Condition (C 1) holds.

$m(F) = 1$.

$(r, s) = (9, 1)$ or $(5, 5)$ or $1, 9)$.

Condition (C 2) holds for all three.

LOCAL-GLOBAL

$F(X) = (X^6 - 3X^5 - X^4 + 5X^3 - X^2 - 3X + 1)(X^4 - X^2 + 1)$.

$(r, s) = (9, 1)$ or $(5, 5)$ or $(1, 9)$.

local conditions hold.

no even, unimodular lattice of signature $(9, 1)$ or $(1, 9)$ has an isometry with characteristic polynomial F.

Hasse principle does not hold for $(9, 1)$ and $(1, 9)$.

REDUCIBLE POLYNOMIALS

$$F = \prod_{f \in J} f^{n_f} = F_0 F_1 F_2$$

$$F_1 = \prod_{f \in I_1} f^{n_f}, \quad F_0 = \prod_{f \in I_0} f^{n_f}$$

I_1 : set of irreducible, symmetric factors of F of even degree.

$I_0 = \{X-1, X+1\}, \quad I = I_0 \cup I_1.$

$F_0(X) = (X-1)^{n_+}(X+1)^{n_-}$ with n_-, n_+ even.

isometry = semi-simple isometry.

(C 1) and (C 2)

(C 1) $|F(1)|$, $|F(-1)|$ and $(-1)^n F(1) F(-1)$ are all squares.

$$m(F) = \text{number of roots } z \text{ of } F \text{ with } |z| > 1.$$

(C 2)

$$m(F) \leqslant r,\ m(F) \leqslant s,.$$

If moreover $F(1)F(-1) \neq 0$, then

$$m(F) \equiv r \equiv s \pmod 2.$$

LOCAL RESULTS

$F \in \mathbf{Z}[X]$ monic, symmetric $\deg(F) = 2n$.

(r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$.

Assume that conditions (C 1) and (C 2) hold.

• p odd prime : There exists a unimodular $\mathbf{Z}_p$-lattice having an isometry with characteristic polynomial F.

• $p = 2$: There exists an even, unimodular $\mathbf{Z}_2$-lattice having an isometry with characteristic polynomial F.

• There exists a non-degenerate quadratic form over $\mathbf{R}$ of signature (r, s) having an isometry with characteristic polynomial F.

HASSE PRINCIPLE

Hasse principle in terms of an

obstruction group

$$Ш_F$$

$$Ш_F = 0 \implies \text{conditions (C 1) and (C 2) suffice.}$$

OBSTRUCTION GROUP

$F \in \mathbf{Z}[X]$ symmetric, monic, $C(I)$: maps $I \to \mathbf{Z}/2\mathbf{Z}$.

Let $f, g \in I$. Let $V_{f,g}$ be the set of prime numbers p such that

$$f \pmod{p} \text{ and } g \pmod{p}$$

have a common symmetric irreducible factor in $\mathbf{F}_p[X]$.

$C_0(I)$: $c \in C(I)$ such that

$$c(f) = c(g) \quad \text{if} \quad V_{f,g} \neq \varnothing.$$

Ш_F : quotient of $C_0(I)$ by the constant maps.

Example.

$$F(X) = (X^6 - 3X^5 - X^4 + 5X^3 - X^2 - 3X + 1)(X^4 - X^2 + 1).$$

$$f(X) = X^6 - 3X^5 - X^4 + 5X^3 - X^2 - 3X + 1,$$

$$g(X) = X^4 - X^2 + 1.$$

$$V_{f,g} = \varnothing.$$

$$Ш_F \simeq \mathbf{Z}/2\mathbf{Z}.$$

EXAMPLES

$$f_1(X) = X^{12} - X^{11} + X^{10} - X^9 - X^6 - X^3 + X^2 - X + 1,$$

$$f_2(X) = X^6 - X^5 + X^4 - X^3 + X^2 - X + 1 = \Phi_{14}(X)$$

$$f_3(X) = X^4 - X^2 + 1 = \Phi_{12}(X).$$

$$F = f_1 f_2 f_3.$$

$$V_{f_1,f_2} = \{7\},\ V_{f_1,f_3} = \{13\},\ V_{f_2,f_3} = \varnothing.$$

$$\text{Ш}_F = 0.$$

EASY EXAMPLE

$$f \in \mathbf{Z}[X] \text{ symmetric, irreducible,}$$

$$g(X) = X - 1.$$

$$V_{f,g} : \text{ set of prime divisors of } f(1).$$

If

$$|f(1)| \neq 1,$$

then

$$V_{f,g} \neq \varnothing.$$

ANOTHER EASY EXAMPLE

$$f \in \mathbf{Z}[X] \text{ symmetric, irreducible,}$$

$$g(X) = X + 1.$$

$$V_{f,g} : \text{ set of prime divisors of } f(-1).$$

If

$$|f(-1)| \neq 1,$$

then

$$V_{f,g} \neq \varnothing.$$

THEOREM

$F \in \mathbf{Z}[X]$ monic, symmetric $\deg(F) = 2n$.

(r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$.

Theorem. Assume that conditions (C 1) and (C 2) hold, and that

$$Ш_F = 0.$$

Then there exists an even, unimodular lattice of signature (r, s) having an isometry with characteristic polynomial F.

QUESTION

$F \in \mathbf{Z}[X]$ monic, symmetric $\deg(F) = 2n$.

(r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$.

Question. Let $t \in \mathrm{SO}_{r,s}(\mathbf{R})$ be a semi-simple isometry with characteristic polynomial F.

Does t preserve an even, unimodular lattice ?

THEOREM

$F \in \mathbf{Z}[X]$ monic, symmetric $\deg(F) = 2n$.

(r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$.

Theorem. Assume that conditions (C 1) and (C 2) hold, and that

$$Ш_F = 0.$$

Let $t \in \mathrm{SO}_{r,s}(\mathbf{R})$ be a semi-simple isometry with characteristic polynomial F. Then t preserves an even, unimodular lattice.

MILNOR INDICES

Let V be a finite dimensional $\mathbf{R}$-vector space, and let

$$q : V \times V \to \mathbf{R}$$

be a quadratic form of signature (r, s).

$$\text{index of } q = r - s$$

Let $t : V \to V$ be an isometry of (V, q).

$P \in \mathbf{R}[X]$: irreducible, symmetric polynomial.

V_P : $P(t)$-primary subspace of V.

($x \in V$ such that $P(t)^N x = 0$ for large N.)

MILNOR INDICES

$P \in \mathbf{R}[X]$: irreducible, symmetric polynomial.

V_P : $P(t)$-primary subspace of V.

Milnor index of $t \in \mathrm{SO}(q)$ at P :

Index of q restricted to V_P.

Index of q = sum of Milnor indices.

MILNOR INDICES

$\mathrm{Irr}_{\mathbf{R}}(F)$: irreducible, symmetric factors of $F \in \mathbf{R}[X]$.

$n_P > 0$: integer such that P^{n_P} is the power of P dividing F.

$\mathrm{Mil}(F)$: maps

$$\tau : \mathrm{Irr}_{\mathbf{R}}(F) \to \mathbf{Z}$$

Such that

$$\tau(P) \in \{-n_P \deg(P), \ldots, n_P \deg(P)\}.$$

MILNOR INDICES

$\mathrm{Mil}_{\mathrm{m}}(F)$:

$$\tau \in \mathrm{Mil}(F)$$

such that

$$\sum_{\mathcal{P}} \tau(\mathcal{P}) = m.$$

THEOREM

$F \in \mathbf{Z}[X]$ monic, symmetric $\deg(F) = 2n$.

(r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$.

Theorem. Assume that conditions (C 1) and (C 2) hold, and that

$$Ш_F = 0.$$

Let $\tau \in \mathrm{Mil}_{r-s}(F)$. Then there exists an even, unimodular lattice having an isometry with

- characteristic polynomial F, and
- Milnor index τ.

THEOREM

$F \in \mathbf{Z}[X]$ monic, symmetric $\deg(F) = 2n$.

(r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$.

Theorem. Assume that conditions (C 1) and (C 2) hold, and that

$$Ш_F = 0.$$

Let $\tau \in \mathrm{Mil}_{r-s}(F)$. Then there exists an even, unimodular lattice having an isometry with

- characteristic polynomial F, and
- Milnor index τ.

Hence signature (r, s).

BIJECTION

$$t \in \mathrm{SO}_{r,s}(\mathbf{R}) \mapsto \tau_t \in \mathrm{Mil}_{r-s}(F).$$

Bijection between

• Conjugacy classes of elements of $\mathrm{SO}_{r,s}(\mathbf{R})$ with characteristic polynomial F.

and

• $\mathrm{Mil}_{r-s}(F)$.

THEOREM

$F \in \mathbf{Z}[X]$ monic, symmetric $\deg(F) = 2n$, and (r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$. Assume that conditions (C 1) and (C 2) hold.

$$\text{Let } \tau \in \mathrm{Mil}_m(F).$$

Define a homomorphism

$$\epsilon_\tau : \text{Ш}_F \to \mathbf{Z}/2\mathbf{Z}.$$

Theorem. There exists an even, unimodular lattice of signature (r, s) having an isometry with

- characteristic polynomial F, and
- Milnor index τ

if and only if

$$\epsilon_\tau = 0.$$

THEOREM

$F \in \mathbf{Z}[X]$ monic, symmetric $\deg(F) = 2n$.

(r, s) such that $r \equiv s \pmod 8$, and $r + s = 2n$.

Assume that conditions (C 1) and (C 2) hold.

Let $t \in \mathrm{SO}_{r,s}(\mathbf{R})$ be a semi-simple isometry with characteristic polynomial F. Define a homomorphism

$$\epsilon_t : \text{Ш}_F \to \mathbf{Z}/2\mathbf{Z}.$$

Theorem. The isometry t preserves an even, unimodular lattice if and only if

$$\epsilon_\tau = 0.$$

K3 SURFACES

Let d be an integer, $4 \leqslant d \leqslant 20$. Let S be a Salem polynomial of degree d, and assume that $|S(1)| \neq 1$ and that $|F(-1)|$ is a square.

Let $\delta \in S^1$ be a root of S.

Let
$$F(X) = S(X)(X-1)^{22-d}.$$

Theorem. There exists a complex analytic $K3$ surface X and an automorphism T of X such that

- The characteristic polynomial of T^* on $H^2(X, \mathbf{Z})$ is F.
- T^* acts on $H^{2,0}(X)$ by multiplication by δ.

The proof uses results of Brandhorst and McMullen.

Let $\lambda \in \mathbf{R}$ be the root of S with $\lambda > 1$.

Corollary. λ is realized as the dynamical degree of an automorphism of a complex holomorphic $K3$ surface.

Let d be an integer, $4 \leqslant d \leqslant 18$. Let S be a Salem polynomial of degree d, and assume that $|S(1)S(-1)| \neq 1$.

Theorem. λ^2 is realized as the dynamical degree of an automorphism of a complex holomorphic $K3$ surface.

Let $\Delta \in \mathbf{Z}[X]$ be a monic, symmetric polynomial such that $\Delta(1) = 1$. Suppose that Δ is a product of distinct irreducible polynomials.

Set $\deg(\Delta) = 2n$, and assume that $\Delta(-1) = (-1)^n$.

Let (r, s) such that $r \equiv s \pmod 8$, $r + s = \deg(\Delta) = 2n$.

Assume that condition (C 2) holds.

KNOTS

Let $\tau \in \mathrm{Mil}_{r-s}(\Delta)$. We have a homomorphism

$$\epsilon_\tau : Ш_\Delta \to \mathbf{Z}/2\mathbf{Z}$$

Theorem. There exists a knot with Alexander polynomial Δ and Milnor index τ if and only if

$$\epsilon_\tau = 0.$$

EXAMPLE

$$\Delta_{u,v} = \frac{(X^{uv}-1)(X-1)}{(X^u-1)(X^v-1)},$$

$u, v > 1$ odd integers with u and v prime to each other.

Let p and q be two distinct odd prime numbers with $q \equiv 3 \pmod 4$, let $e \geqslant 1$ be an integer, and set

$$u = p^e,\ v = q;$$

set

$$\Delta = \Delta_{p^e,q}.$$

EXAMPLE

- $(\frac{p}{q}) = -1$

There exists a knot with Alexander polynomial Δ and index m

if and only if

$$m \equiv 0 \pmod 8, \text{ and } |m| \leq \deg(\Delta).$$

- $(\frac{p}{q}) = 1$

There exists a knot with Alexander polynomial Δ and index m

if and only if

$$m \equiv 0 \pmod 8, \text{ and } |m| \leq \deg(\Delta) - 4(e-1).$$

EXAMPLE

$p = 3$, $q = 7$, $e = 2$.

$$\Delta = \Delta_{3^2,7} = \Delta_{9,7} =$$

$$\Phi_{21}\Phi_{63}.$$

$(\frac{3}{7}) = -1$

$$Ш_{\Delta} = 0.$$

EXAMPLE - INDICES

Necessary conditions

$$m \equiv 0 \pmod{8}, \text{ and } |m| \leq \deg(\Delta).$$

$\deg(\Delta) = 12 + 36 = 48.$

$$m \equiv 0 \pmod{8}, \text{ and } |m| \leq 48.$$

$$|m| = 0, 8, 16, 24, 32, 40, 48.$$

EXAMPLE - MILNOR INDICES

Let $\tau \in \mathrm{Mil}_m(\Delta)$ for

$$|m| = 0, 8, 16, 24, 32, 40, 48.$$

There exists a knot with

- Alexander polynomial Δ, and
- Milnor index τ.

EXAMPLE - MILNOR INDICES

$$|\mathrm{Mil}_{48}(\Delta)| = 1, \quad |\mathrm{Mil}_{40}(\Delta)| = \binom{24}{2}, \quad |\mathrm{Mil}_{32}(\Delta)| = \binom{24}{4},$$

$$|\mathrm{Mil}_{24}(\Delta)| = \binom{24}{6}, \quad |\mathrm{Mil}_{16}(\Delta)| = \binom{24}{8},$$

$$|\mathrm{Mil}_{8}(\Delta)| = \binom{24}{10}, \quad |\mathrm{Mil}_{0}(\Delta)| = \binom{24}{12}.$$

EXAMPLE

$p = 7$, $q = 3$, $e = 2$.

$$\Delta = \Delta_{3,7^2} = \Delta_{3,49} =$$

$$\Phi_{21}\Phi_{147}.$$

$(\frac{7}{3}) = 1$

$$Ш_{\Delta} \simeq \mathbf{Z}/2\mathbf{Z}.$$

EXAMPLE - INDICES

Necessary conditions

$$m \equiv 0 \pmod 8, \text{ and } |m| \leq \deg(\Delta).$$

$\deg(\Delta) = 12 + 84 = 96.$

$$m \equiv 0 \pmod 8, \text{ and } |m| \leq 96.$$

EXAMPLE

$\tau \in \mathrm{Mil}_m(\Delta)$

$$\epsilon_\tau : Ш_\Delta \to \mathbf{Z}/2\mathbf{Z}$$

there exists a knot with

- Alexander polynomial Δ, and
- Milnor index τ

if and only if $\epsilon_\tau = 0$.

EXAMPLE

$\tau \in \mathrm{Mil}_m(\Delta)$

$$\epsilon_\tau : Ш_\Delta \to \mathbf{Z}/2\mathbf{Z}$$

there exists a knot with

- Alexander polynomial Δ, and
- Milnor index τ

if and only if $\epsilon_\tau = 0$.

Set

$$\mathrm{Mil}_m(\Delta)^+ : \quad \tau \in \mathrm{Mil}_m(\Delta) \text{ such that } \epsilon_\tau = 0.$$

EXAMPLE

$$\mathrm{Mil}_{96}(\Delta)^{+} = \varnothing.$$

$$|\mathrm{Mil}_{88}(\Delta)| = \binom{48}{2} = 24.47 = 12.94$$

$$|\mathrm{Mil}_{88}(\Delta)^{+}| = 6.42 = 12.21$$

EXAMPLE

NO KNOT with

- Alexander polynomial Δ
- signature 96.

There exist knots with

- Alexander polynomial Δ
- signature 88,

but not all Milnor indices occur.

Thank you