

On classicality of overconvergent p -adic automorphic forms

§1 Motivation and Classicality results

Coleman: let f be an overconvergent p -adic modular eigenform of weight $k \geq 2$ and of finite slope (i.e. $U_p f = \lambda f$, $\lambda \neq 0$)

then $\text{val}_p(\lambda) < k - 1 \Rightarrow f$ classical modular form

many generalizations of "small slope \Rightarrow classical" to p -adic autom forms on more general groups than GL_2 .

\rightarrow in this talk: what happens beyond the "small slope condition".

Kisin let f be an overconvergent p -adic eigenform of weight $k \geq 2$, of finite slope

then ρ_f semi-stable $\Rightarrow f$ is classical

\rightarrow want to study this question for o.c. p -adic automorphic forms on definite unitary groups.

Setup: G/F^+ unitary group in n -var. / tot real field

G compact at ∞ , split at p , quasi-split at fin places
(+ some technical conditions)

let f be an overconvergent p -adic autom. form of fin slope on $G(\mathbb{A}_{F^+}^\infty)$, of tame level $K^P = G(\mathbb{A}_{F^+}^{\infty, P})$ compact open

and dominant alg weight $\underline{k} \in \prod_{F^+ \hookrightarrow \bar{\mathbb{Q}}_p} \mathbb{Z}_+^n$

Assume: f is an eigenform for

- Hecke action away from p

- eigenform for Hecke operators $U_{v,i}$ at places $v|p$, $i = 1, \dots, n$

double coset operators of

$$\text{diag}(1, \dots, 1, \varpi_v, 1, \dots, 1) \in G(F_v^+)$$

$$(\varpi_v = \text{unif of } F_v^+)$$

Question: Assume ρ_f is semi-stable, is f classical?

assume for simplicity: $F^+ = \mathbb{Q}$ (to simplify notation)

$$\Rightarrow \underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$$

eigen f write $U_i f = \varphi_i p^{k_i} f$

if ρ_f is semi-stable then $\varphi_1, \dots, \varphi_n$ are the Frobenius eigenvalues on $D_{\text{st}}(\rho_f)$.

assume $\varphi_i/\varphi_j \neq 1$, then the ordering $\varphi_1, \dots, \varphi_n$ defines a complete flag in $D_{\text{st}}(\rho_f) = D_{\text{cr}}(\rho_f)$

let $w \in \mathcal{J}_n$ be the relative position of the Hodge field w.r.t. this flag

Thm: (i) (Chenevier) $w = w_0$ is the largest elt $\Rightarrow f$ is classical

(ii) (Breuil - H. - Schraen) \cdot tech on $\bar{\rho}_f + \rho_f$ semi-stab

\cdot $\varphi_i/\varphi_j \notin |1, p^L| \Rightarrow \rho_f$ crystalline

$w w_0$ product of p distinct simple reflections
 $\Rightarrow f$ classical.

(iii) (BAS, H-Weirich) · tech. ass. $\bar{\rho}_f + \rho_f$ semi-stab.
 · $\varphi_i / \varphi_j \neq 1$

\exists classical automorphic form f' with

$\left\{ \begin{array}{l} \text{Same weight} \\ \text{Same Hecke eigen system} \end{array} \right.$ as f .
 (at $p + \text{away from } p$)

Expl / Thm (H. - Hernandez - Schraen)

$n = 3$, f' classical eigen form, as in Thm (ii)

but with $w = 1$, i.e. $w w_0 = w_0$ not a product
 of p simple reflections. ($\Rightarrow \rho_f|_{\text{Gal}} \text{ is a sum of char's}$)

$\Rightarrow \exists f$ non-classical eigen form

$\left\{ \begin{array}{l} \text{Same weight} \\ \text{Same Hecke eigen system} \end{array} \right.$ as f

Rem (i) well known: can \exists "companion forms" of f'

i.e. non-classical p -adic forms

same Hecke eval's, but not with
 char alg. int.

(ii) seems reasonable (?) to expect that assuming conditions

on φ_i in (ii) the condition on w is optimal.

Aim for the rest of the talk: explain some ideas / constructions that enter the proof; conjectural "explanation" for failure of classicality in general.

§2 stacks of Galois reps / (φ, Γ) -modules

general philosophy: Spaces of $(p$ -adic) automorphic forms can be interpreted as coh. sheaves on deformation spaces / stacks of Galois reps.

local counterpart:

try to construct / understand a functor

$$\left(\begin{array}{c} \text{(derived) cat. of} \\ \text{reps of } GL_n(\mathbb{Q}_p) \end{array} \right) \longrightarrow \left(\begin{array}{c} \text{(derived) cat of coh.} \\ \text{sheaves on stack of Galois} \\ \text{reps} \end{array} \right)$$

for smooth reps / stack of L -parameters:

Fargues-Scholze, Xue, Ben-Zvi-Clavin-Helm-Nalae, H.

for study of o.c. p -adic autom forms:

- cat. of admissible locally analytic reps of $G = GL_n(\mathbb{Q}_p)$

resp.: its dual: p -adic distribution alg of G .

cat of coadmissible $\mathcal{D}(G)$ -modules

- rigid analytic stack of (φ, Γ) -modules over the Robba ring \mathcal{R}

(= equivariant vb on the Fargues-Fontaine curve)

let H/\mathbb{G}_p lin alg grp. consider the fppf stack on rigid spaces

$$A \text{ aff'd } \mathbb{G}_p\text{-alg} \longmapsto \left\{ \begin{array}{l} \text{gp of } (\varphi, \pi)\text{-modules} \\ \text{with } H\text{-structure } (R_A = R \otimes_{\mathbb{G}_p} A) \end{array} \right\}$$

(if $H = GL_n$: the n (φ, π) -modules

for general H : use some Tannakian construction

e.g. Mess, of V. de Pauw.)

expl: - $X_{G_m} = \mathcal{T}/G_m$

$$G_m \curvearrowright \mathcal{T} = \text{Hom}_{\text{cont}}(\mathbb{G}_p^\times, G_m(-1)) \text{ space of cont char's of } \mathbb{G}_p^\times$$

trivial

- caution: $X_{GL_n} \neq$ generic fib. of Emerton-Gee stack

expectation: X_{GL_n} is an "Artin-stack" of dim n^2

would like to have: exact functor (fully faithful?)

$$F_{GL_n}: D_{\text{cohom}}^b(D(GL_n(\mathbb{G}_p))) \longrightarrow D_{\text{coh}}^b(X_{GL_n})$$

s.th. - $n=1$ F_{G_m} given by LFT + Emerton

(similar for split tor)

- "local global compatibility"

$X_{\bar{\rho}}$ gen fib. of univ. deform ring of ρ

Global, adic, adic (p-adic) Galois rep's $\bar{\rho}$

$X_p \longrightarrow X_{GL_n}$ by restricting to local Galois.

$$\sigma = z^{\frac{k}{2}} \cdot \sigma_{sm} \in X_T$$

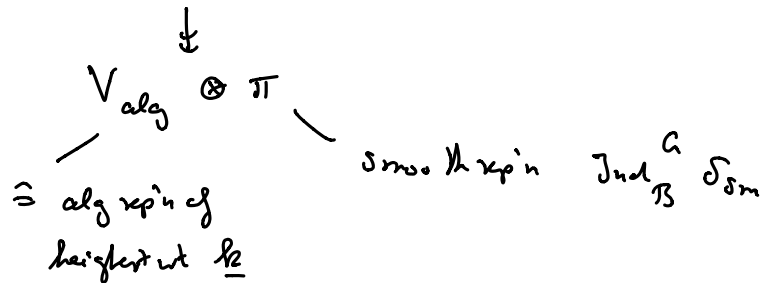
$T = B \subseteq G(\mathbb{Q}_p) = GL_n(\mathbb{Q}_p)$
 Split max Borel
 form

Then (local of space of o.c. p-adic autom forms
 of wt $\frac{k}{2}$ + Hecke eigen system at $p \hat{=} \sigma_{sm}$)

$$= \text{pullback of } F_{GL_n} \left(\mathbb{D}(G) \otimes_{\mathbb{D}(B)} h(\sigma) \right)$$

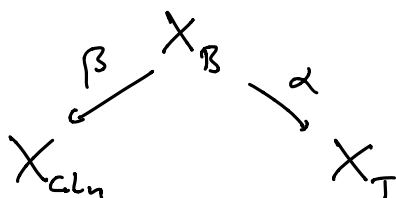
along $X_p \longrightarrow X_{GL_n}$.

[$\mathbb{D}(G) \otimes_{\mathbb{D}(B)} h(\sigma)$ has a locally algebraic quotient



local of space of classical forms = pullback of $F_{GL_n}(V_{\text{alg}} \otimes \pi)$]

- compatibility with parabolic induction



$$F_n \left(\mathbb{D}(G) \otimes_{\mathbb{D}(B)} h(\sigma) \right) \cong \beta_{\pi} \alpha^! F_T(h(\sigma))$$

Rem: (i) using Taylor-Wiles patching (\sim C.F.G.P.S)

can produce a candidate for F_{GL_n} (in a slightly different setup)

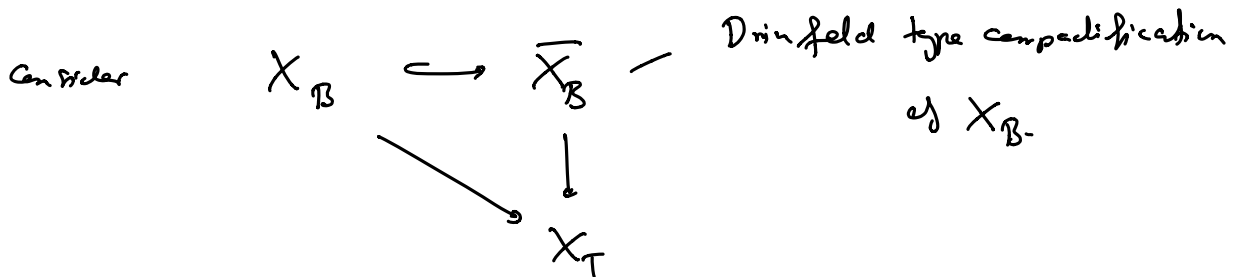
studying this + geometry of stacks introduced \leadsto results in §1.

(ii) comp. w. periodic induction can't work like this:

β is not proper \leadsto need to replace X_B by a compactification \overline{X}_B

§3

Drinfeld-type compactifications.



let $X_T^{\text{reg}} = \{ (\sigma_1, \dots, \sigma_n) \mid \sigma_i / \sigma_j \neq z^u, u \in \mathbb{Z} \} \subseteq X_T$
open

$\leadsto X_B^{\text{reg}}, \overline{X}_B^{\text{reg}}$.

Thm (i) $\overline{X}_B^{\text{reg}}$ is an Artin stack, $X_B \hookrightarrow \overline{X}_B^{\text{reg}}$ Zar open.

don't know whether $X_B^{\text{reg}} \subseteq \overline{X}_B^{\text{reg}}$ scheme-theoretically dense.

\leadsto replace $\overline{X}_B^{\text{reg}}$ by closure of X_B in the following

(ii) $\overline{X}_B^{\text{reg}}$ regular of $\dim = \dim B$

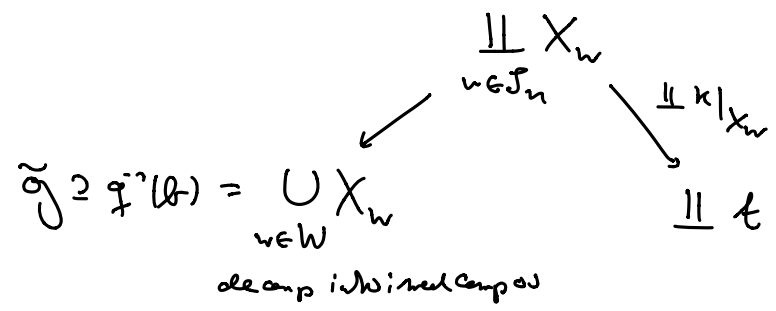
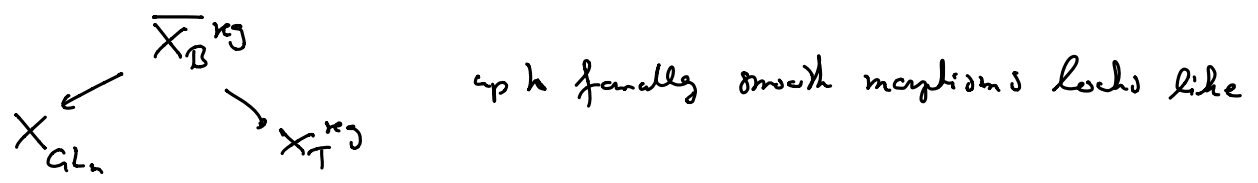
$\exists \mathfrak{x} \in \overline{X}_B^{\text{reg}}$ defines a semi-stab (φ, P) -module

with regular HT weights + pm. distinct Frobenius

$\rightarrow \overline{X}_B^{\text{reg}}$ is normal + C.M. at \mathfrak{x} .

Closing Remarks

let $D \in X_{GL_n}$ be a semi-stab (φ_i) -module, $\varphi_1, \dots, \varphi_n =$ evals of Frob on $D_{\text{st}}(D)$
 If $\varphi_i / \varphi_j \notin \mathbb{Z}, \mathbb{Q}$: inhbld of $D \in X_{GL_n}$

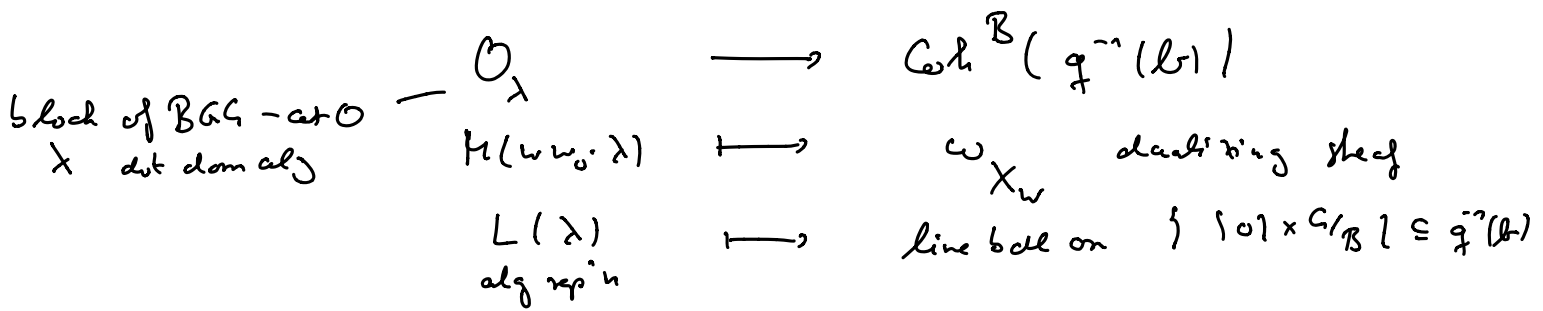


$\tilde{\mathcal{G}} \xrightarrow{\mathcal{F}} \mathcal{G} = \mathcal{O}_{GL_n}$
 Grothendieck's simult. resolution

$\tilde{\mathcal{G}} \xrightarrow{\pi} G/B$ proj to flag var
 $X_w =$ closure of $\pi^{-1}(BwB/B) \cap \mathcal{F}^{-1}(W)$

$k: \tilde{\mathcal{G}} \rightarrow t$ general pieces of stable flag

in this case $F_{GL_n} (D(G) \otimes_{D(B)} h(S))$ should come by pull back from a functor constructed by Beilinson-Karvits



X_w CM (Beilinson-Karvits - Riche) but can be very singular even non-Gorenstein!