On classicality of overconvergent p-adic automorphic forms

§1 Motivation and Classicality results

Coleman: let f be an overconvergent p-adic modular eigenform of weight $k \geq 2$ and of finite slope (i.e. $U_p f = \lambda f$, $\lambda \neq 0$)

then $\quad \mathrm{val}_p(\lambda) < k-1 \Rightarrow f$ classical modular form

many generalizations of "small slope $\Rightarrow$ classical" to p-adic autom forms on more general grps than GL_2.

$\to$ in this talk: what happens beyond the "small slope condition".

Kisin: let f be an overconvergent p-adic eigenform of weight $k \geq 2$, of finite slope

then $\quad \rho_f$ semi-stable $\Rightarrow f$ is classical

$\to$ want to study this question for o.c. p-adic automorphic forms on definite unitary grps.

Setup: $G_{/F^+}$ unitary grp in n-var. / tot real field

G compact at ∞, split at p, quasi-split at fin places

(+ some technical conditions)

let f be an overconvergent p-adic autom. form of fin slope on $G(\mathbb{A}^\infty_{F^+})$, of tame level $K^p \subseteq G(\mathbb{A}^{\infty,p}_{F^+})$ compact open and dominant alg weight $\underline{k} \in \prod_{F^+ \hookrightarrow \overline{\mathbb{Q}}_p} \mathbb{Z}^n_+$

Assume: f is an eigenform for

- Hecke action away from p
- eigenform for Hecke operators $U_{v,i}$ at places $v|p$, $i = 1, \dots, n$

double coset operators of

$$\mathrm{diag}(1, \dots, 1, \varpi_v, 1, \dots, 1) \in G(F_v^+)$$

$(\varpi_F = \text{unif of } F_v^+)$

Question: Assume ρ_f is semi-stable, is f classical?

assume for simplicity: $F^+ = \mathbb{Q}$ (to simplify notations $\rightsquigarrow \underline{k} = (k_1 \geq \dots \geq k_n) \in \mathbb{Z}_+^n$)

given f write $U_i f = \varphi_i p^{k_i} f$

if ρ_f is semistable then $\varphi_1, \dots, \varphi_n$ are the Frob evals on $D_{st}(\rho_f)$.

assume $\varphi_i/\varphi_j \neq 1$, then the ordering $\varphi_1, \dots, \varphi_n$ defines a complete flag in $D_{st}(\rho_f) = D_{dR}(\rho_f)$

let $w \in S_n$ be the relative position of the Hodge filt w.r.t. this flag

Thm: (i) (Chenevier) $w = w_0$ is the longest elt $\Rightarrow f$ is classical

(ii) (Breuil - H. - Schraen) · tech on $\bar{\rho}_f$ + ρ_f semi-stb

· $\varphi_i/\varphi_j \notin \{1, p\}$ ($\Rightarrow \rho_f$ crystalline)

$w w_0$ product of pw distinct simple reflections

$\Rightarrow f$ classical.

(iii) (BAS, H-Weirich) · tech ass. $\bar\rho_f$ + ρ_f semi-stab.

· $\varphi_i/\varphi_j \neq 1$

$\exists$ classical automorphic form f' with

$\begin{cases} \text{same weight} \\ \text{same Hecke eigensystem} \\ \quad \text{(at } p + \text{away from } p) \end{cases}$ as f.

Expl / Thm (H.-Hernandez-Schraen)

$n = 3$, f' classical eigenform, as in Thm (iii)

but with $w = 1$, i.e. $w w_0 = w_0$ not a product

of pw simple reflections. ($\Rightarrow \rho_f|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is a sum of char's)

$\Rightarrow \exists f$ <u>non-classical</u> eigenform

$\begin{cases} \text{same weight} \\ \text{same Hecke eigensystem} \end{cases}$ as f

<u>Rem</u> (i) well known: can $\exists$ "companion forms" of f'

i.e. non-classical p-adic forms

same Hecke eval's, but not with

dom alg wt.

(ii) Seems reasonable (?) to expect that assuming conditions

on φ_i in (iii) the condition on w is optimal.

Aim for the rest of the talk: explain some ideas / constructions that enter the proof; conjectural "explanation" for failure of classicality in general.

§2 Stacks of Galois rep'ns / (φ, Γ)-modules

general philosophy: Spaces of (p-adic) automorphic (eigen-) forms can be interpreted as coh. sheaves on deformation spaces / stacks of Galois rep'ns.

local counterpart:

try to construct / understand a functor

$$\left(\begin{array}{c}\text{(derived) cat. of}\\ \text{rep'ns of } GL_n(\mathbb{Q}_p)\end{array}\right) \longrightarrow \left(\begin{array}{c}\text{(derived) cat of coh.}\\ \text{sheaves on stack of Galois}\\ \text{rep'ns}\end{array}\right)$$

for smooth rep'ns / stack of L-parameters:

Fargues - Scholze, Xu, Ben-Zvi - Chen - Helm - Nadler, H.

for study of o.c. p-adic autom forms:

- cat. of admissible locally analytic rep'ns of $G = GL_n(\mathbb{Q}_p)$
  resp.: its dual: cat of coadmissible $\mathcal{D}(G)$-modules ($\mathcal{D}(G)$ = p-adic distribution alg of G.)

- rigid analytic stack of $\underbrace{(\varphi, \Gamma)\text{-modules over the Robba ring } \mathcal{R}}$

  (= equivariant vb on the Fargues-Fontaine curve) let $H/\mathbb{Q}_p$ lin alg grp. consider the fpqc stack on rigid spaces

$$\underset{\text{affid } \mathbb{Q}_p\text{-alg}}{A} \longmapsto \left\{ \begin{array}{l} \text{grpd of } (\varphi,\Gamma)\text{-modules} \\ \text{with } H\text{-structure } / \mathcal{R}_A = \mathcal{R} \hat{\otimes}_{\mathbb{Q}_p} A \end{array} \right\}$$

(if $H = GL_n$: rk n (φ,Γ)-modules

for general H: use some Tannakian construction

e.f. thesis of V. de Daruvar.)

expl: - $X_{\mathbb{G}_m} = \mathcal{J}/\mathbb{G}_m$

$\mathbb{G}_m \underset{\text{trivial}}{\curvearrowright} \mathcal{J} = \mathrm{Hom}_{cont}(\mathbb{Q}_p^\times, \mathbb{G}_m(-1))$ space of cont char's of $\mathbb{Q}_p^\times$

- caution: $X_{GL_n} \neq$ generic fib. of Emerton-Gee stack

expectation: X_{GL_n} is an "Artin-stack" of dim n^2

would like to have: exact functor (fully faithful?)

$$F_{GL_n} : D^b_{coadm}(D(GL_n(\mathbb{Q}_p))) \longrightarrow D^b_{coh}(X_{GL_n})$$

s.th. - $n=1$ $F_{\mathbb{G}_m}$ given by LCFT + Emerton

(similar for split tori)

- "local global compatibility"

$\mathfrak{X}_{\bar\rho}$ gen fib of univ. deform ring of a

Global, automorphic (polarizable, ...) Galois rep'n $\bar\rho$

$\mathfrak{X}_{\bar{p}} \longrightarrow X_{GL_n}$ by restricting to local Galgp.

$\delta = z^{\underline{k}} \cdot \delta_{sm} \in X_T \qquad \underset{\text{split max torus}}{T} \subseteq \underset{\text{Borel}}{B} \subseteq G(\mathbb{Q}_p) = GL_n(\mathbb{Q}_p)$

Thm $\left(\begin{array}{l} \text{dual of space of o.c. p-adic autom forms} \\ \text{of wt } \underline{k} \quad + \quad \text{Hecke eigensystem at } p \ \hat{=}\ \delta_{sm} \end{array} \right)$

$=$ pullback of $\quad F_{GL_n}(\mathcal{D}(G) \otimes_{\mathcal{D}(B)} k(\delta))$

along $\quad \mathfrak{X}_{\bar{p}} \longrightarrow X_{GL_n}$.

[$\quad \mathcal{D}(G) \otimes_{\mathcal{D}(B)} k(\delta)$ has a locally algebraic quotient

$$\begin{array}{c} \mathcal{D}(G) \otimes_{\mathcal{D}(B)} k(\delta) \\ \downarrow \\ V_{alg} \otimes \pi \end{array}$$

V_{alg} $\hat{=}$ alg rep'n of highest wt $\underline{k}$

π smooth rep'n $\operatorname{Ind}_B^G \delta_{sm}$

dual of space of classical forms $=$ pullback of $F_{GL_n}(V_{alg} \otimes \pi)$]

- compatibility with parabolic induction

$$\begin{tikzcd} & X_B \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ X_{GL_n} & & X_T \end{tikzcd}$$

$$F_n(\mathcal{D}(G) \otimes_{\mathcal{D}(B)} k(\delta)) \cong \beta_* \alpha^! F_T(k(\delta))$$

<u>Rem</u>: (i) using Taylor-Wiles patching ($\sim$ CEGGPS)

can produce a candidate for F_{GL_n} (in a slightly different setup)

studying this + geometry of stacks introduced $\rightsquigarrow$ results in §1.

(ii) comp. w. parabolic induction can't work like this:

β is not proper $\rightsquigarrow$ need to replace X_B by a compactification $\overline{X_B}$

§3 Drinfeld-type compactifications.

Consider

$$\begin{tikzcd} X_B \arrow[r, hook] \arrow[dr] & \overline{X_B} \arrow[d] \\ & X_T \end{tikzcd}$$

— Drinfeld type compactification of X_B.

let $X_T^{reg} = \{ (\delta_1, \dots, \delta_n) \mid \delta_i/\delta_j \neq z^n, n \in \mathbb{Z} \} \underset{\text{open}}{\subseteq} X_T$

$\rightsquigarrow X_B^{reg}, \overline{X_B}^{reg}$.

Thm (i) $\overline{X_B}^{reg}$ is an Artin stack, $X_B \hookrightarrow \overline{X_B}^{reg}$ for open.

don't know whether $X_B^{reg} \subseteq \overline{X_B}^{reg}$ scheme-theoretically dense.

$\rightsquigarrow$ replace $\overline{X_B}^{reg}$ by closure of X_B in the following

(ii) $\overline{X_B}^{reg}$ equidim of dim = dim B

if $x \in \overline{X_B}^{reg}$ defines a semi-stab (φ, Γ)-module with regular HT weights + pw. distinct Frob evals

$\Rightarrow$ $\overline{X_B}^{reg}$ is normal + CM at x.

Closing Remarks

let $D \in X_{GL_n}$ be a semi-stab (φ,Γ)-module, $\varphi_1, \ldots, \varphi_n$ = evals of Frob on $D_{st}(D)$

If $\varphi_i/\varphi_j \notin \{1, p\}$: in nbhd of $D \in X_{GL_n}$

$$\begin{tikzcd} & \overline{X_B}^{reg} \arrow[dl] \arrow[dr] & \\ X_{GL_n} & & X_T^{reg} \end{tikzcd}$$

up to formally smooth morphisms looks like

$$\begin{tikzcd} & \coprod_{w \in S_n} X_w \arrow[dl] \arrow[dr, "\amalg \kappa|_{X_w}"] & \\ \tilde{\mathfrak{g}} \supseteq q^{-1}(\mathfrak{b}) = \bigcup_{w \in W} X_w & & \coprod \mathfrak{t} \end{tikzcd}$$

decomp in irred compon

$\tilde{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} = \mathfrak{gl}_n$, $\tilde{\mathfrak{g}} \xrightarrow{\pi} G/B$ proj to flag var

Grothendieck's simult. resolution

X_w = closure of $\pi^{-1}(BwB/B) \cap q^{-1}(\mathfrak{b})$

$\kappa : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{t}$ graded pieces of stable flag

in this case $F_{GL_n}(D(G) \otimes_{D(B)} h(\delta))$ should come by pull back from a functor constructed by Bezrukavnikov

block of BGG-cat $\mathcal{O}$ — $\mathcal{O}_\lambda \longrightarrow \mathrm{Coh}^B(q^{-1}(\mathfrak{b}))$

λ dot dom alg

$M(ww_0 \cdot \lambda) \longmapsto \omega_{X_w}$ dualizing sheaf

$L(\lambda) \longmapsto$ line bdl on $\{0\} \times G/B \subseteq q^{-1}(\mathfrak{b})$

alg rep'n

X_w CM (Bezrukavnikov - Riche) but can be very singular

even <u>non - Gorenstein</u> !