Vanishing theorems for Shimura varieties

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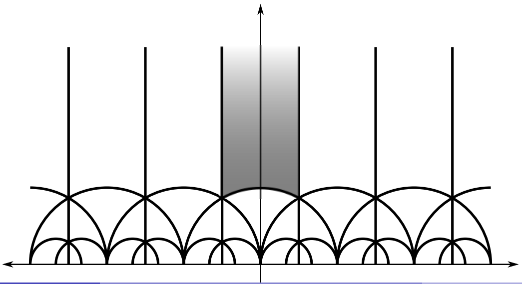

- If F is an imaginary quadratic field and $G := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_2$, then

$$X = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{R}) = \mathbb{H}^3$$

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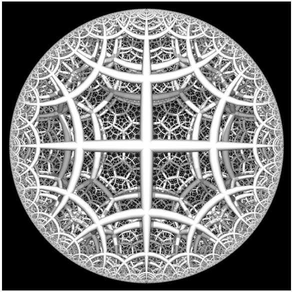

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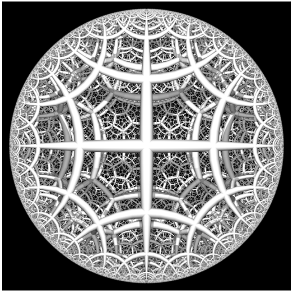

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- The condition for G to admit Shimura varieties is that X be a Hermitian symmetric domain. This happens, for example, if $G = \mathrm{GSp}_{2n}/\mathbb{Q}$ or a unitary group $\mathrm{U}(n,n)/\mathbb{Q}$.
- Some locally symmetric spaces that are not algebraic varieties can be related to higher-dimensional Shimura varieties using algebraic topology: Bianchi manifolds can be seen in the boundary of $\mathrm{U}(2,2)$-Shimura varieties.

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- Assume that $X_\Gamma(\mathbb{C})$ is compact. Matsushima's formula relates the systems of $\mathbb{T}$-eigenvalues occurring in $H^*(X_\Gamma(\mathbb{C}), \mathbb{C})$ to automorphic representations $\pi = \otimes'_v \pi_v$ of $G(\mathbb{A})$.

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- Borel–Wallach show that, if π is an automorphic representation of $G(\mathbb{A})$ with π_∞ tempered, then it contributes to $H^i(X_\Gamma(\mathbb{C}), \mathbb{C})$ only in the middle degree $i = \dim_{\mathbb{C}} X_\Gamma$.

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- For example, $G = \mathrm{SL}_2/\mathbb{Q}$, the Eichler–Shimura isomorphism expresses $H^1(X_\Gamma(\mathbb{C}), \mathbb{C})$ in terms of modular forms of level Γ and weight 2.

- Can we prove a similar result for $H^*(X_\Gamma(\mathbb{C}), \mathbb{F}_p)$?

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- However, the Calegari–Geraghty method is conjectural on a very precise understanding of torsion in the cohomology of locally symmetric spaces, such as Bianchi manifolds.

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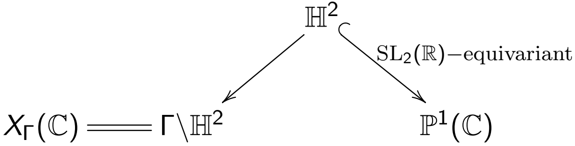

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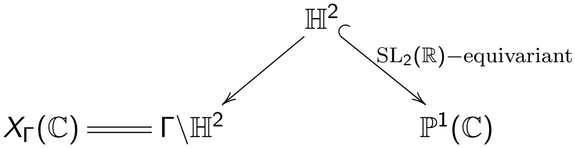

- The upper-half plane $\mathbb{H}^2$ parametrises Hodge structures of elliptic curves:

$$\mathbb{C}^2 = H^1(E(\mathbb{C}), \mathbb{C}) \simeq H^0(E, \Omega^1_E) \oplus H^1(E, \mathcal{O}_E)$$

The map on the RHS sends this Hodge decomposition to the Hodge filtration $H^0(E, \Omega^1_E) \subset H^1(E(\mathbb{C}), \mathbb{C}) = \mathbb{C}^2$.

- Set $\mathcal{X}_\Gamma := (X_\Gamma \times_{\mathbb{Q}} \mathbb{Q}_p)^{\mathrm{ad}}$. There exists a diagram

$$\begin{array}{ccc} & \mathcal{X}_{\Gamma(p^\infty)} & \\ \swarrow & & \searrow^{\pi_{\mathrm{HT}},\ \mathrm{SL}_2(\mathbb{Q}_p)-\text{equivariant}} \\ \mathcal{X}_\Gamma & & \mathbb{P}^{1,\mathrm{ad}}, \end{array}$$

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- Set $C := \widehat{\overline{\mathbb{Q}_p}}$. The map π_{HT} measures the relative position of the Hodge–Tate filtration

$$0 \to H^1(E, \mathcal{O}_E) \to H^1_{\mathrm{et}}(E, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to H^0(E, \Omega^1_E) \to 0.$$

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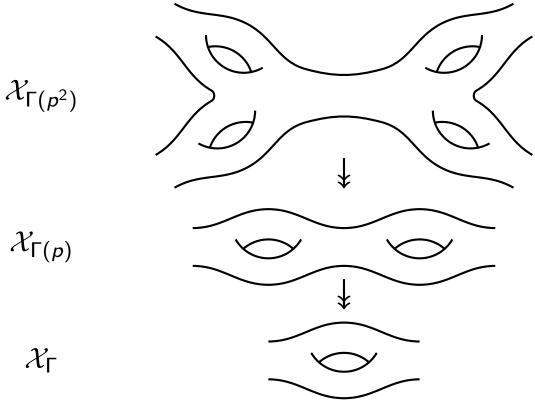
  of the family of elliptic curves over $\mathcal{X}_{\Gamma(p^\infty)}$. An elliptic curve E/C together with a trivialisation $H^1_{\mathrm{et}}(E, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \simeq C^2$ gets sent to

$$H^1(E, \mathcal{O}_E) \subset H^1_{\mathrm{et}}(E, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \simeq C^2.$$



A perfectoid space is a fractal-like object, invented by Scholze in 2011, that is particularly well-suited to doing p-adic Hodge theory in families.

Theorem 1 (Scholze 2013, C–Scholze 2015, 2019, Hamacher 2016)

Let (G, X) be a Shimura datum of Hodge type. There exists a perfectoid space $\mathcal{X}_\Gamma(p^\infty) \sim \varprojlim_n \mathcal{X}_{\Gamma(p^n)}$ and a morphism of adic spaces

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that measures the relative position of the Hodge–Tate filtration. Moreover, there exists a decomposition into Newton strata

$$\mathscr{F}\ell_{G,\mu} = \sqcup \mathscr{F}\ell^b_{G,\mu}$$

and the fibers of π_{HT} over each $\mathscr{F}\ell^b_{G,\mu}$ can be identified with perfectoid Igusa varieties Ig^b.

Recall that our goal is to understand

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Upshot: we can compute $H^*_{\mathrm{et}}(\mathcal{X}_\Gamma, \mathbb{F}_\ell)$ using the diagram:

$$\begin{array}{ccc} & \mathcal{X}_{\Gamma(p^\infty)} & \\ \swarrow {\scriptstyle G(\mathbb{Z}_p)-\mathrm{torsor}} & & \searrow {\scriptstyle \pi_{\mathrm{HT}}} \\ \mathcal{X}_\Gamma & & \mathscr{F}\ell_{G,\mu}. \end{array}$$

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Theorem 2 (C-Scholze 2015, 2019)

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- The complex $R\Gamma(\mathrm{Ig}^b, \mathbb{Q}_\ell)_{\mathfrak{m}}$ is also supported in one degree when $\dim \mathscr{F}\ell^b_{G,\mu}$ is maximal. We compute the alternating sum of its cohomology groups using the trace formula, building on work of Shin.

For the modular curve, we have $B(G,\mu) = \{\mathrm{ord}, \mathrm{ss}\}$:

$$\begin{array}{ccccccc} \mathcal{X}^*_{K^p} & = & \mathcal{X}^{*,\mathrm{ord}}_{K^p} & \sqcup & \mathcal{X}^{\mathrm{ss}}_{K^p} \\ \downarrow \pi_{\mathrm{HT}} & & \downarrow & & \downarrow \\ \mathbb{P}^{1,\mathrm{ad}} & = & \mathbb{P}^{1,\mathrm{ad}}(\mathbb{Q}_p) & \sqcup & \Omega \end{array}$$

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- If π is an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that π_f^p contributes to $R\Gamma(\mathrm{Ig}^{\mathrm{ss}}, \mathbb{Q}_\ell)$, then π is in the image of the global Jacquet–Langlands map and so π_p cannot be a principal series representation.

Theorem 2 is a crucial ingredient in the following results, obtained by Allen–Calegari–C–Gee–Helm–Le Hung –Newton–Scholze–Taylor–Thorne:

Theorem 3 (2018)

Let F be a CM field and E/F a non-CM elliptic curve. Then E satisfies the Sato–Tate conjecture.

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Theorem 4 (2018)

Let F be a CM field and π a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of parallel weight 2. Then π satisfies the generalised Ramanujan–Petersson conjecture.

- The key point for proving both Theorems 3 and 4 is to establish the potential modularity of the symmetric powers of the associated Galois representations. This is achieved by implementing the Calegari–Geraghty method unconditionally.

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- Theorem 2 is used to establish the first instances of local-global compatibility at $\ell = p$ for the Galois representations constructed by Scholze. This is a prerequisite for the Calegari–Geraghty method.
- In joint work in progress with Newton, we are using Theorem 2 to prove that these Galois representations satisfy local-global compatibility at $\ell = p$ in the crystalline case (under technical assumptions and with $F^+ \neq \mathbb{Q}$).

- Recall Emerton's completed cohomology groups

$$\widetilde{H}^i_{(c)}(\mathbb{F}_p) := \varinjlim_n H^i_{(c)}(X_{\Gamma(p^n)}(\mathbb{C}), \mathbb{F}_p).$$

Motivated by heuristics from the p-adic Langlands programme, Calegari–Emerton conjectured that

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- In fact the Calegari–Emerton conjecture applies to general locally symmetric spaces, and the case of tori is equivalent to the Leopoldt conjecture.

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- By a classical argument in Hida theory, this is equivalent to showing that the ordinary completed cohomology groups

$$\widetilde{H}^i_c(B_0, \mathbb{F}_p)^{\mathrm{ord}} := \varinjlim_n H^i_c(X_{\Gamma_0(p^n)}(\mathbb{C}), \mathbb{F}_p)^{\mathrm{ord}}$$

vanish for $i > d$.

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Theorem 5 (C–Gulotta–Johansson, 2019)

Assume that $G_{\mathbb{Q}_p}$ is split. Let $N_0 := N(\mathbb{Z}_p)$, where N is the unipotent radical of the Borel B of G. Then

$$\widetilde{H}^i_c(N_0, \mathbb{F}_p) = \varinjlim_n H^i_{\mathrm{et},c}(\mathcal{X}_{\Gamma_1(p^n)}, \mathbb{F}_p) = 0 \text{ whenever } i > d.$$

- p-adic Hodge theory:

$$R\Gamma_{\mathrm{et},c}\left(\mathcal{X}_{N_0}, \mathbb{F}_p\right) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \overset{a}{\simeq} R\Gamma_{\mathrm{et}}\left(\mathcal{X}^*_{N_0}, \mathcal{I}^+/p\right),$$

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- The Bruhat stratification into $B(\mathbb{Q}_p)$-orbits

$$\mathscr{F}\ell_{G,\mu} = \bigsqcup_{w \in W/W_{P_\mu}} \mathscr{F}\ell^w_{G,\mu},$$

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- By quantifying when different subsets of $(\mathcal{X}^*_\Gamma)_\Gamma$ become perfectoid, we show that the cohomological amplitude of $R\pi_{\mathrm{HT}/N_0,*}\mathcal{I}^{+a}/p$ restricted to $\mathscr{F}\ell^w_{G,\mu}/N_0$ lies in $[0, d - \dim \mathscr{F}\ell^w]$.

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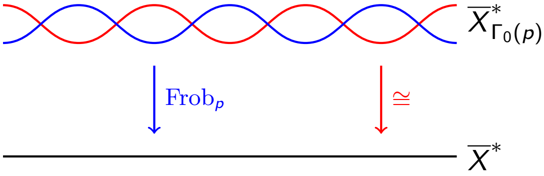

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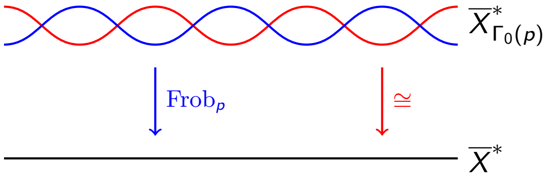

$$\overline{X}^{*,\mathrm{ord}}_{\Gamma_0(p)} = \overline{X}^{*,\mathrm{anti}}_{\Gamma_0(p)} \sqcup \overline{X}^{*,\mathrm{can}}_{\Gamma_0(p)}$$

matches the Bruhat stratification:

$$\mathbb{P}^{1,\mathrm{ad}} = \mathbb{A}^{1,\mathrm{ad}} \sqcup \{\infty\}.$$

In particular, $\mathcal{X}^{*,\mathrm{anti}}_{K^pN_0}$ is perfectoid (Ludwig).

Let $G = \mathrm{SL}_2/\mathbb{Q}$. Very recently, Boxer–Pilloni constructed two dual families of ordinary p-adic modular forms by interpolating the cohomology of automorphic line bundles on the modular curve in degrees 0 and 1.

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Theorem 6 (C–Mantovan–Newton, in progress)

There is a Hecke-equivariant short exact sequence

$$0 \to M^1_c \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \to \mathcal{M}\left(H^1_c(N_0, \mathbb{Z}_p)^{\mathrm{ord}}\right) \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \to M^0 \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \to 0.$$

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The short exact sequence is induced by the Bruhat stratification on $\mathbb{P}^{1,\mathrm{ad}}$. Theorem 6 recovers an older result of Ohta, but our proof sees Higher Hida theory intrinsically and the ingredients are very general.

Thank you!