A Shafarevich Conjecture for Hypersurfaces in Abelian Varieties

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(joint work with Will Sawin)

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Outline

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The Shafarevich conjecture for curves

Theorem (Faltings)

Let:

- K be a number field
- S a finite set of primes of $\mathcal{O}_K$
- $g \geq 0$ an integer.

Then there are at most finitely many curves of genus g over K, having good reduction outside S.

- Proved as part of Faltings's proof of Mordell's conjecture.

Good reduction

Definition

Let R be a DVR, K its field of fractions.
A smooth variety Y/K has good reduction if there exists a smooth $\mathcal{Y}/R$, of finite type over R, whose generic fiber is isomorphic to Y.

- Example:
$$y^2 = x(x-9)(x-18)$$
has good reduction at all $p \neq 2, 3$.
- In fact, it also has good reduction at 3: taking $y' = 3^3 y$ and $x' = 3^2 x$, we get
$$y'^2 = x'(x'-1)(x'-2),$$
which is smooth over $\mathbb{Z}_3$.

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More Shafarevich conjectures

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Also true for:

- abelian varieties of dimension g (Faltings)
- K3 surfaces (André, She)
- and many more... (Scholl, Javanpeykar, Loughran, etc.)

Our result

Theorem (L-Sawin)

Let:

- K be a number field
- A an abelian variety defined over K, of dimension not equal to 3
- S a finite set of primes of $\mathcal{O}_K$, including all places of bad reduction for A
- ϕ an ample class in the Neron–Severi group of A.

Then there are at most finitely many hypersurfaces in A belonging to the class ϕ, defined over K and having good reduction outside S.

Outline

1. Shafarevich conjectures
   - Shafarevich conjectures
   - Why should you believe Shafarevich conjectures?
2. Shafarevich for hypersurfaces in an abelian variety
   - Proof
   - Krämer–Weissauer generic vanishing

Interlude I: de Rham cohomology and Hodge structures

- Let Y be a variety over a number field K.
- (Complex) Hodge theory makes $H^*_{dR}(Y, \mathbb{C})$ into a *Hodge structure*. This is...:
  - A complex vector space $V = H^*_{dR}(Y, \mathbb{C})$.
  - An integral lattice $H^*_{dR}(Y, \mathbb{Z}) \subseteq V$.
  - A filtration of V by subspaces, coming from the Hodge–de Rham spectral sequence.
- For example, if Y is an elliptic curve, then these structures are (almost) the same as the lattice $\Lambda \subseteq \mathbb{C}$ giving the complex-analytic uniformization $Y \cong \mathbb{C}/\Lambda$.
- Exercise: Let Y be an elliptic curve. Figure out how the Hodge structure $H^1_{dR}(Y, \mathbb{C})$ and the lattice Λ determine each other.

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Interlude II: étale cohomology and Galois representations

- Let Y be a smooth variety over a number field K.
- Étale cohomology makes $H^*_{et}(Y, \mathbb{Q}_p)$ into a *Galois representation*. This is...:
  - A continuous representation

$$\rho\colon \operatorname{Gal}(\overline{K}/K) \to GL_n(\mathbb{Q}_p).$$

  - Loosely speaking, ρ keeps track of fields of definition of étale covers of Y.

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Interlude III: cohomology and motives

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- By the Hodge conjecture, the Hodge structure $H^*_{dR}(Y, \mathbb{C})$ should determine Y "as a motive".
- By the Tate conjecture, $H^*_{et}(Y, \mathbb{Q}_p)$ should determine Y "as a motive".
- Somewhat more precisely: any isomorphism $H^*(Y_1) \cong H^*(Y_2)$ should be "explained by geometry".

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Interlude IV: cohomology in families

- Let $f\colon Y \to X$ be a family of varieties over a number field K.
- Consider cohomology of the fibers.
- (Complex) Hodge theory gives a *variation of Hodge structure* on X. We get a map $X \to D$, where D is a period domain ("moduli space of Hodge structures").
- Étale cohomology gives an *étale local system* on X. This is a "family of Galois representations".
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First reason: hyperbolicity

Conjecture (Lang–Vojta)

Let X be a variety of log-general type over a field K, with a model over some $\mathcal{O}_{K,S}$. Then $X(\mathcal{O}_{K,S})$ is not Zariski dense.

- If X is a proper curve, this recovers Mordell's conjecture.
- If X is a non-proper curve of genus 0 or 1, this recovers the S-unit theorem and Siegel's theorem, respectively.
- For higher-dimensional X, Lang–Vojta implies the following: If all subvarieties of X are of log-general type, then $X(\mathcal{O}_{K,S})$ is finite.

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- Suppose we have a family of varieties $Y \to X$, with X of finite type.
- Considering cohomology of the fibers, we get a variation of Hodge structure, and a period map $X \to D$ to some period domain.
- Period domains are hyperbolic.
- If $X \to D$ is finite, Lang–Vojta implies that $X(\mathcal{O}_{K,S})$ is finite.
- (This argument is in Javanpeykar–Loughran.)

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Second reason: finiteness of Galois representations

Lemma (Faltings):

Fix K, S, and positive integers n and k.
Then there are (up to isomorphism) only finitely many semisimple Galois representations

$$\mathrm{Gal}(\overline{K}/K) \to GL_n(\mathbf{Q}_p),$$

unramified at all primes outside S, and having all Frobenius eigenvalues Weil integers of weight k.

- Given a family $Y \to X$ as above, there are only finitely many possibilities for (the semisimplification of) $H^k_{et}(Y_x, \mathbb{Q}_p)$, as x ranges over $X(\mathcal{O}_{K,S})$.

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- Assuming the Tate conjecture (and semisimplicity of étale cohomology), there are only finitely many possibilities for the *motive* Y_x, up to isogeny.
- This means only finitely many possibilities for $H^k_{dR}(Y_x, \mathbb{C})$, up to isogeny.
- If the period map is finite, each Hodge structure $H^k_{dR}(Y_x, \mathbb{C})$ arises for at most finitely many $x \in X(\mathcal{O}_{K,S})$.
- (Note this is not a complete argument, because isogeny classes might be infinite.)

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Conclusion: heuristic Shafarevich

Thing that might be true

Let X be a variety over a number field K. Suppose there is a family of varieties $Y \to X$ over K, whose k-th cohomology gives rise to a finite period map $X \to D$. Then $X(\mathcal{O}_{K,S})$ is finite.

Notes:

- "Finite" means scheme-theoretically finite, i.e. finite-to-one.
- I haven't thought seriously about this statement; let me know if you see a reason it's not true.

Conclusion: heuristic Shafarevich

Thing that might be true

Fix K, S, and nonnegative integers n and k. Consider all projective varieties Y, over K with good reduction outside S, such that $\dim H^k_{dR}(Y) = n$.
As Y ranges over all such varieties, only finitely many Hodge structures appear as $H^k_{dR}(Y)$.

Notes:

- Presumably one could replace projective varieties by pure motives.
- I haven't thought seriously about this statement; let me know if you see a reason it's not true.

Outline

Vague principle (L.-Venkatesh)

Suppose a variety X over K admits a variation of Hodge structure and étale local system coming from geometry and satisfying the following conditions:

- The Frobenius centralizers are large.
- The Hodge numbers satisfy a certain numerical condition.
- The variation of Hodge structure has big monodromy.

Then $X(\mathcal{O}_{K,S})$ is not Zariski dense in X.

- We have results of this form for the following X:
  - $\mathbb{P}^1$ minus three points (L-Venkatesh, Lemma 4.2)
  - A curve (L-Venkatesh, Prop. 5.3)
  - Moduli of hypersurfaces in $\mathbb{P}^n$ (for large n and large degree) (L-Venkatesh, Thm. 10.1)
  - Moduli of hypersurfaces in an abelian variety (L-Sawin, Thm. 8.21)

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- “Large Frobenius centralizers” is a condition on crystalline cohomology (which I have not discussed here).
- See the papers for the Hodge number condition.

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- For "big monodromy", we are only concerned with the Zariski closure of the image of the monodromy map
$$\pi_1(X, x_0) \to \mathrm{Aut}\, H^k(Y_{x_0}, \mathbb{Q}).$$
- This image is an algebraic group. It's sufficient to show it's the largest possible group (GL, Sp or O), but we can sometimes make do with less.

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- To prove our finiteness result, need to apply this to all subvarieties of the moduli space of hypersurfaces of Neron–Severi class ϕ in A.
- The first two conditions hold uniformly for subvarieties. The monodromy condition is a problem.

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Toy problem (uniform big monodromy)

Let X be the moduli space of smooth hypersurfaces of degree 50 in $\mathbb{P}^{10}$. This X is a high-dimensional projective space with a discriminant locus removed. It comes with an action of PGL_{11}, the automorphism group of $\mathbb{P}^{10}$.

Every such hypersurface has interesting cohomology in the middle degree (H^9).

For any irreducible subvariety $Z \subseteq X$ not contained in a single PGL_{11}-orbit, we can consider the monodromy representation

$$\mathrm{Mon}\colon \pi_1(Z, z_0) \to \mathrm{Aut}(H^9(\text{hypersurface})).$$

Can you give a nontrivial, uniform lower bound for the dimension of the Zariski closure of the image of monodromy?

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Can you give a nontrivial, uniform lower bound for the dimension of the Zariski closure of the image of monodromy?

- This looks hard.

- Solution: Work with hypersurfaces in an abelian variety A.
- They have lots of local systems.
- For any finite-order character χ of $\pi_1(A)$, get a local system $\mathcal{L}_\chi$.
- If $H \subseteq A$ is a hypersurface, we can consider $H^{n-1}(\mathcal{L}_\chi|_H)$.
- If $f\colon Y \to X$ is the universal hypersurface over the moduli space X, we can consider $R^{n-1}f_*(\mathcal{L}_\chi)$.

Theorem (L-Sawin, imprecisely stated)

For every subvariety $Z \subseteq X$ (not contained in an orbit of A), there exists χ such that $R^{n-1}f_*(\mathcal{L}_\chi)$ has big monodromy on Z.

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- Our proof uses work of Krämer and Weissauer on sheaf convolution and generic vanishing for perverse sheaves on abelian varieties.
- Let $H \subseteq A$ be a smooth subvariety of dimension m. Krämer and Weissauer study

$$H^\bullet(\mathcal{L}_\chi, H)$$

as χ varies over characters of $\pi_1(A)$.
- They determine the "generic" behavior of this cohomology, which holds for almost all χ.

Generic vanishing theorem (Krämer–Weissauer)

Let $H \subseteq A$ be a smooth subvariety of dimension m. For all characters χ of A outside a finite union of torsion translates of proper subtori of the dual torus of A, we have

$$H^k(\mathcal{L}_\chi, H) = 0$$

for all $k \neq m$.

- In fact, Krämer and Weissauer prove a vanishing theorem for $H^k(K \otimes \mathcal{L}_\chi)$, with K an arbitrary perverse sheaf on A. (The result above comes from taking K a constant sheaf on H.)
- They also interpret the middle cohomology as a fiber functor on a certain Tannakian category.
- This lets us prove the uniform big monodromy result we need.

Arxiv links

- Javanpeykar–Loughran: `https://arxiv.org/abs/1505.02249`
- L–Venkatesh: `https://arxiv.org/abs/1807.02721`
- L–Sawin: `https://arxiv.org/abs/2004.09046`
- Krämer–Weissauer 1: `https://arxiv.org/abs/1111.4947`
- Krämer–Weissauer 2: `https://arxiv.org/abs/1309.3754`