The *L*-Functions and Modular Forms Database

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Calculation, tabulation, and experiment have always played a significant role in number theory. Here we describe the *L-functions and modular forms database* (LMFDB) [3], `https://lmfdb.org`, a resource for researchers in number theory and related areas.

1   What is the LMFDB?

The LMFDB is a database of mathematical objects and the connections between them. A web interface allows visitors to browse and query the database in a flexible and powerful way. At the core of the LMFDB is the *Langlands program* (see Figure 1), a set of conjectures relating objects from arithmetic algebraic geometry (such as number fields, elliptic curves, and abelian varieties) with modular and automorphic forms via Galois representations and *L*-functions (the most basic of which is the Riemann zeta function). The LMFDB catalogs these objects and the ways in which they link to one another, as well as other closely related objects like permutation groups and *p*-adic fields. Currently, the LMFDB includes tens of millions of objects totalling over 4.8 terabytes of data. Hosted on Google Cloud Platform, in 2020, it was used by more than 30 000 users in 146

different countries and all 50 states. For further information about the LMFDB (both mathematical and technical), see [1, 2].
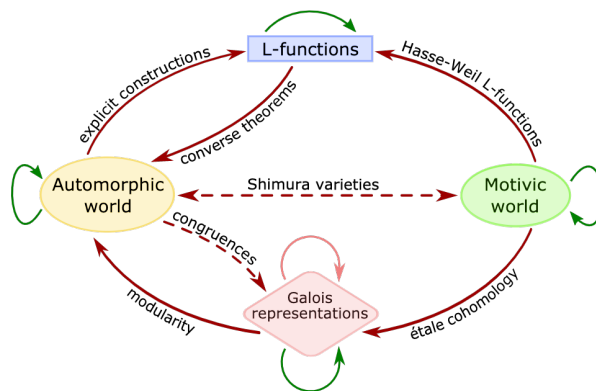

Figure 1: The LMFDB universe.

2   What is the LMFDB for?

The LMFDB is designed to be of useful to a wide range of mathematicians, from novice to expert.

A novice can easily browse the site by just "clicking around". Even if you have no idea what parameters one might want to use when searching for, say, Hilbert modular forms, the site offers several options for browsing, a list of interesting examples, and a link to produce a random example.

Every object in the LMFDB (whether a number field, an elliptic curve over $\mathbb{Q}$, or a Dirichlet *L*-function) has a homepage. This summarizes all of the data available for the object, and is generally organized so that the most well-known and commonly

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used information is at the top of the page, with more technical details appearing below. Terms that merit further explanation have a dotted underline. This signals that the word is a *knowl*[1], and clicking on the word opens a box on the page with a short explanation or definition. When no longer needed, the knowl can be closed by clicking on the term again. This makes explanations of unfamiliar terms immediately available to those who want them, without getting in the way of an expert looking at the same page.

On the other hand, an expert can use the site to search for objects based on a selection of invariants and properties relevant to their research. Once found, data is displayed on the homepage of the object and can be downloaded using a link in the right sidebar of the page. In many cases, there are buttons one can click to get a version of the information which can be directly copied and pasted into a computer algebra system such as Magma, SageMath, or PARI/GP.

Searching the LMFDB can assist the researcher by producing a large variety of examples. One can then use these examples to test conjectures, to look for patterns that may lead to new conjectures, or to learn how objects are connected by conjectures (and theorems) in the Langlands program. Searches can also be used to show that the set of objects satisfying the hypotheses of a theorem you just proved is actually non-empty—or to check "corner cases".

In some situations, theoretical considerations may show that an object of interest satisfies enough properties that there are only finitely many possibilities, in which case having a complete list moves the proof forward. To aid this sort of application, the homepage of every object in the LMFDB has links in the right sidebar with information on the source of the data including links to relevant papers, the level of rigor used to compute the data (are conjectures or heuristics being used?), and the extent to which the data is complete.

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[1]The term *knowl* was coined by the LMFDB to represent a "little bit of knowledge".

3 Example

As mentioned above, every object in the LMFDB database has a homepage. The object has a unique label assigned to it that provides a permanent URL for its homepage. This label can be used in papers as a shorthand for referring to specific objects. For example, the first elliptic curve over $\mathbb{Q}$ of rank 3 (when ordered by conductor) has label `5077.a1` and URL `https://www.lmfdb.org/EllipticCurve/Q/5077/a/1`. Those reading this article online may wish to try clicking on the label `5077.a1`. You can include such links in your papers.

The homepage of each object contains many invariants and data related to the object that has been computed. For the elliptic curve `5077.a1`, you would see that its Mordell–Weil group of rational points is isomorphic to $\mathbb{Z}^3$, with explicit generators. There is a wealth of additional information that we will not try to describe here, but it ranges from basics like the j-invariant to terms appearing in the Birch and Swinnerton-Dyer conjecture or arising from Iwasawa theory.

The homepage of every object includes four boxes that appear on the right of the page. The Properties box provides a succinct recap of key invariants of the object. Next, a box containing a list of links to Related Objects appears: for example, an elliptic curve's homepage includes links to the homepages of the associated modular form and the curve's L-function. The Downloads box provides links for downloading data (which may include information too large to display on the page). Finally, the Learn more box provides links to pages with more information about how the data was computed, its level of precision and rigor, and credits including citations to the literature and those who contributed to this page.

4 Conclusion

The LMFDB is undergoing continual development: its databases are being expanded and new search features are being added. In the near future, we hope to add sections on hypergeometric motives, lattices, and finite groups. Changes large and small are over-
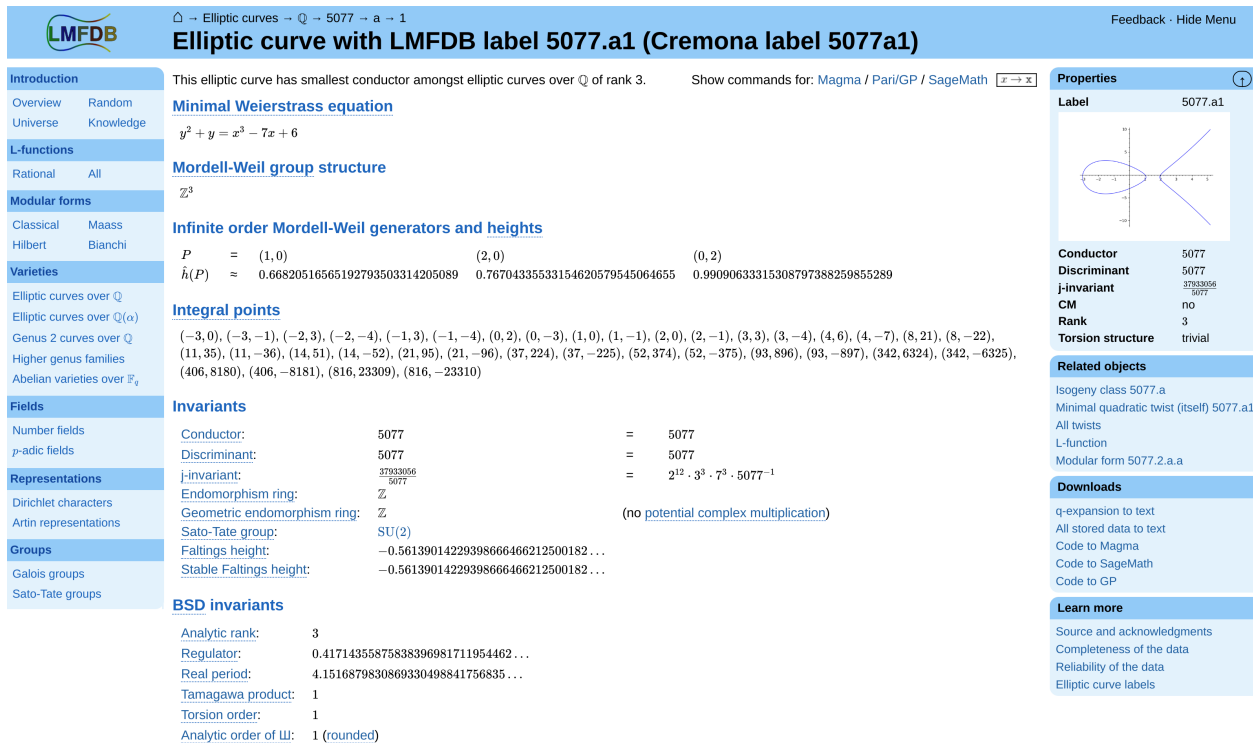
Elliptic curve with LMFDB label 5077.a1 (Cremona label 5077a1)

This elliptic curve has smallest conductor amongst elliptic curves over ℚ of rank 3.   Show commands for: Magma / Pari/GP / SageMath $x \to x$

Minimal Weierstrass equation

$y^2 + y = x^3 - 7x + 6$

Mordell-Weil group structure

$\mathbb{Z}^3$

Infinite order Mordell-Weil generators and heights

P	$=$	$(1,0)$	$(2,0)$	$(0,2)$
$\hat{h}(P)$	$\approx$	$0.6682051656519279350331420508$	$0.7670433553315462057954506465$	$0.9909063331530879738825985528$

Integral points

$(-3,0), (-3,-1), (-2,3), (-2,-4), (-1,3), (-1,-4), (0,2), (0,-3), (1,0), (1,-1), (2,0), (2,-1), (3,3), (3,-4), (4,6), (4,-7), (8,21), (8,-22),$
$(11,35), (11,-36), (14,51), (14,-52), (21,95), (21,-96), (37,224), (37,-225), (52,374), (52,-375), (93,896), (93,-897), (342,6324), (342,-6325),$
$(406,8180), (406,-8181), (816,23309), (816,-23310)$

Invariants

Conductor:	5077	$=$	5077	
Discriminant:	5077	$=$	5077	
j-invariant:	$\frac{37933056}{5077}$	$=$	$2^{12} \cdot 3^3 \cdot 7^3 \cdot 5077^{-1}$	
Endomorphism ring:	$\mathbb{Z}$			
Geometric endomorphism ring:	$\mathbb{Z}$		(no potential complex multiplication)	
Sato-Tate group:	$\mathrm{SU}(2)$			
Faltings height:	$-0.56139014229398666466212500182\ldots$			
Stable Faltings height:	$-0.56139014229398666466212500182\ldots$			

BSD invariants

Analytic rank:	3
Regulator:	$0.41714355875838396981711954462\ldots$
Real period:	$4.15168798308693304988417568 35\ldots$
Tamagawa product:	1
Torsion order:	1
Analytic order of Ш:	1 (rounded)

Properties

Label	5077.a1
Conductor	5077
Discriminant	5077
j-invariant	$\frac{37933056}{5077}$
CM	no
Rank	3
Torsion structure	trivial

Related objects

Isogeny class 5077.a
Minimal quadratic twist (itself) 5077.a1
All twists
L-function
Modular form 5077.2.a.a

Downloads

q-expansion to text
All stored data to text
Code to Magma
Code to SageMath
Code to GP

Learn more

Source and acknowledgments
Completeness of the data
Reliability of the data
Elliptic curve labels

Figure 2: Elliptic curve `5077.a1`.

seen by an editorial board which functions similarly to that of a journal, to ensure the correctness and consistency of its contents.

We would like to thank all who have contributed to the LMFDB. Current funding comes through a grant from the Simons Foundation; previous grants have come from the NSF and EPSRC. Just as importantly, there are several institutions and over 100 mathematicians who have contributed their time and effort. A full list is available at `https://lmfdb.org/acknowledgment`. The LMFDB has already been cited by almost 400 papers: for information on how to cite the LMFDB, see `https://www.lmfdb.org/citation`.

We hope the LMFDB will be a useful resource for you. We invite everyone to contribute by providing data, writing code, or just by giving feedback!

References

[1] Edgar Costa and David Roe, *Zen and the art of database maintenance*, accepted to Simons Symp.

[2] John Cremona, *The L-functions and modular forms database project*, Found. Comput. Math. **16** (2016), no. 6, 1541–1553.

[3] The LMFDB Collaboration, *The L-functions and modular forms database*, `http://www.lmfdb.org`, 2021, [Online; accessed 18 May 2021].