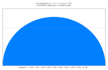
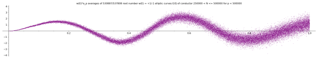
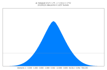
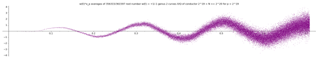
The power of pictures

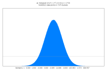
Andrew V. Sutherland, MIT

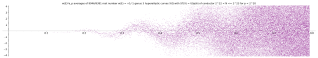












Sato

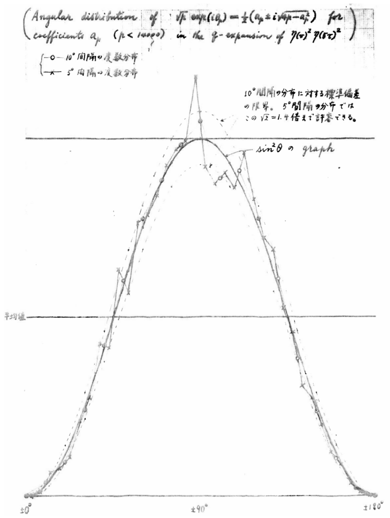
In 1962 Mikio Sato returned to the University of Tokyo after visiting IAS with an interest in the following question about the sequence of Frobenius eigenangles ϑ_p associated to an elliptic curve $E/\mathbb{Q}$ at primes p of good reduction.

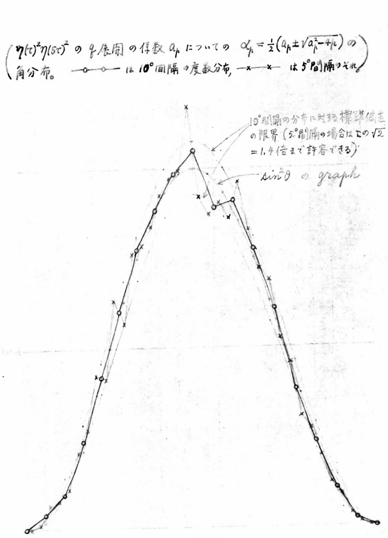
How is ϑ_p distributed on $[0, \pi]$ as p varies over primes?

The university had recently installed a HIPAC 103 computer, and several young researchers and students had started "playing" with it, including Kanji Namba.

> *. . . on a nice summer evening, Sato and some of his colleagues, including Namba, instead of parting at the Ikebukuro suburban train terminal were drawn to a roof beer garden on a department store. Then Sato, explaining the beauty of arithmetic of elliptic curves and modular forms, said to Namba something like "why not use the new computer for something more worthwhile than examining the Goldbach conjecture; for example, for collecting data for this question".*

The $\sin^2\vartheta$-law (May 15, 1963 letter from Sato to Namba)





The $\sin^2\vartheta$-law

In the letter Sato wrote to Namba in May, 1963 (as translated by Namba in a March 2007 letter to Ralf Schmidt)

> *. . . according to the figure and table, it is estimated that the angular distribution of α_p is proportional to $\sin^2\vartheta$. It could be said that the above hypothesis is very plausible.*

Here $\alpha_p = \sqrt{p}e^{i\vartheta_p}$ and $\bar{\alpha}_p = \sqrt{p}e^{-i\vartheta_p}$ are p-Weil numbers with $a_p = \alpha_p + \bar{\alpha}_p$. Sato continued

> *This fact is, I think, probably, if we spend sufficiently long time and deep conversation, even under our present knowledge, it would be possible to explain theoretically, but now, I would like to postpone such heavy brain work, and instead, collect experimental muscular obtainable data.*

The Sato-Tate distribution

Fix an elliptic curve $E/\mathbb{Q}$. For each good prime p the trace of Frobenius

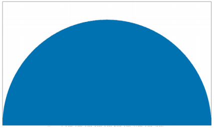
$$a_p := p + 1 - \#E_p(\mathbb{F}_p)$$

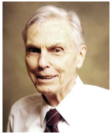
satisfies $|a_p| \leq 2\sqrt{p}$. Let $x_p := -a_p/\sqrt{p} \in [-2, 2]$. If E does not have CM then $(x_2, x_3, x_5, x_7, x_{11}, \ldots)$ should be equidistributed with respect to the measure

$$\frac{2}{\pi}\sqrt{4 - x^2}dx$$

If we construct a histogram of x_p-values for $p \leq B$ and rescale by $\frac{\pi}{2}$, as B tends to infinity our histogram should converge to a semicircle of radius 2.

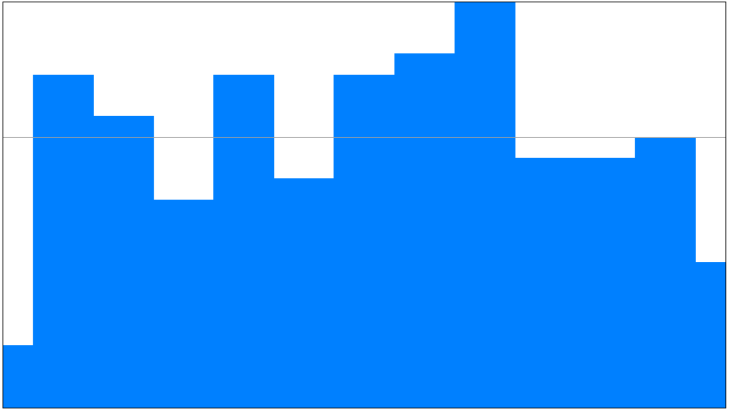
Mikio Sato





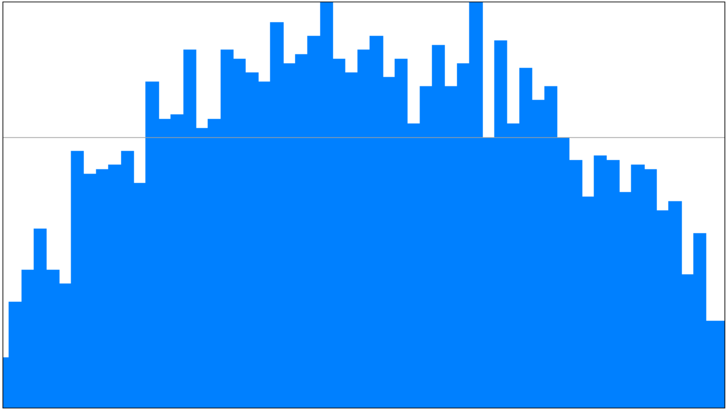
John Tate

a1 histogram of y^2 = x^3 + x + 1 for p <= 2^10
170 data points in 13 buckets, z1 = 0.029, out of range data has area 0.018



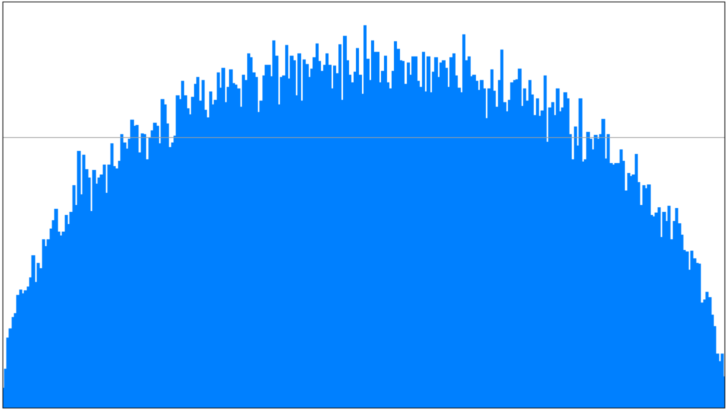
Moments: 1 0.051 1.039 0.081 2.060 0.294 4.971 1.134 13.278 4.308 37.954

a1 histogram of y^2 = x^3 + x + 1 for p <= 2^15
3510 data points in 59 buckets, out of range data has area 0.001



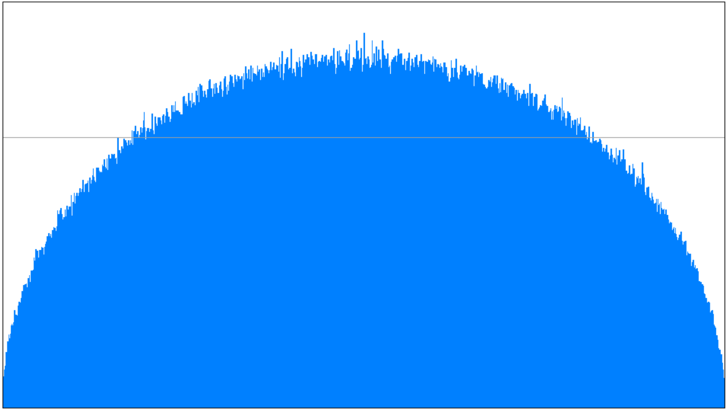
Moments: 1 0.003 0.981 0.030 1.957 0.114 4.914 0.398 13.838 1.388 41.768

a1 histogram of y^2 = x^3 + x + 1 for p <= 2^20
82023 data points in 286 buckets

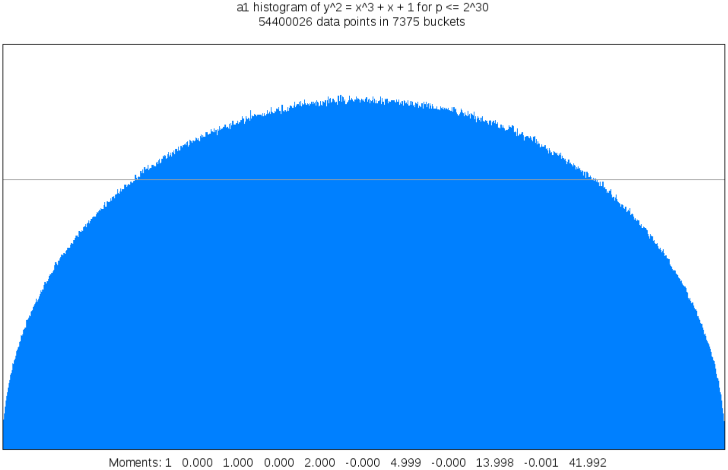


Moments: 1   0.002   0.999   0.003   1.992   0.004   4.969   0.005   13.885   -0.006   41.567

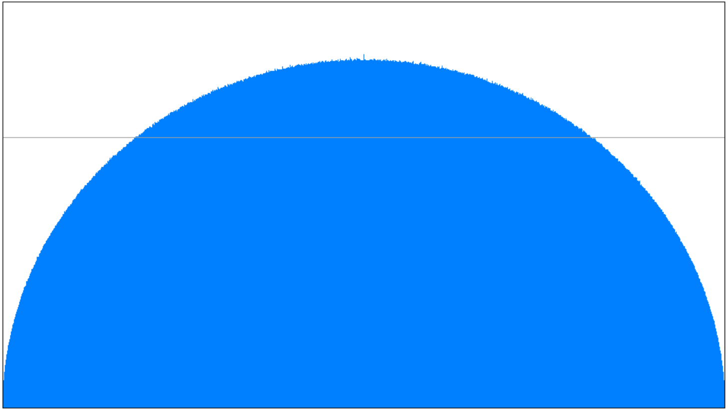
a1 histogram of y^2 = x^3 + x + 1 for p <= 2^25
2063687 data points in 1436 buckets



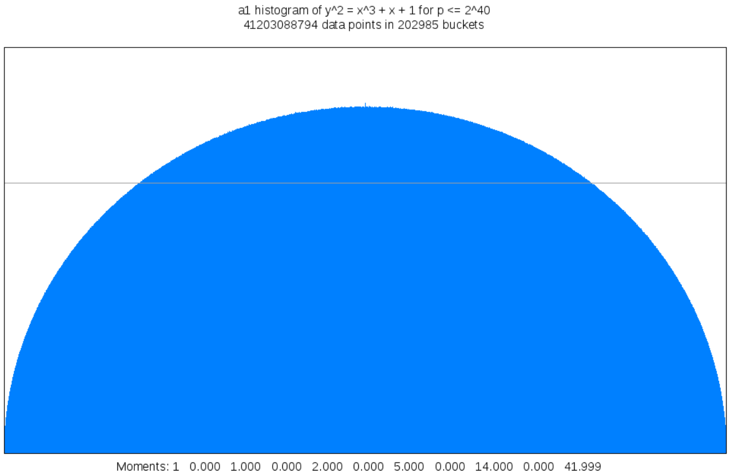
Moments: 1 0.000 0.999 0.001 1.997 0.003 4.991 0.003 13.972 -0.005 41.914


a1 histogram of y^2 = x^3 + x + 1 for p <= 2^30
54400026 data points in 7375 buckets
Moments: 1 0.000 1.000 0.000 2.000 -0.000 4.999 -0.000 13.998 -0.001 41.992

a1 histogram of y^2 = x^3 + x + 1 for p <= 2^35
1480206277 data points in 38473 buckets

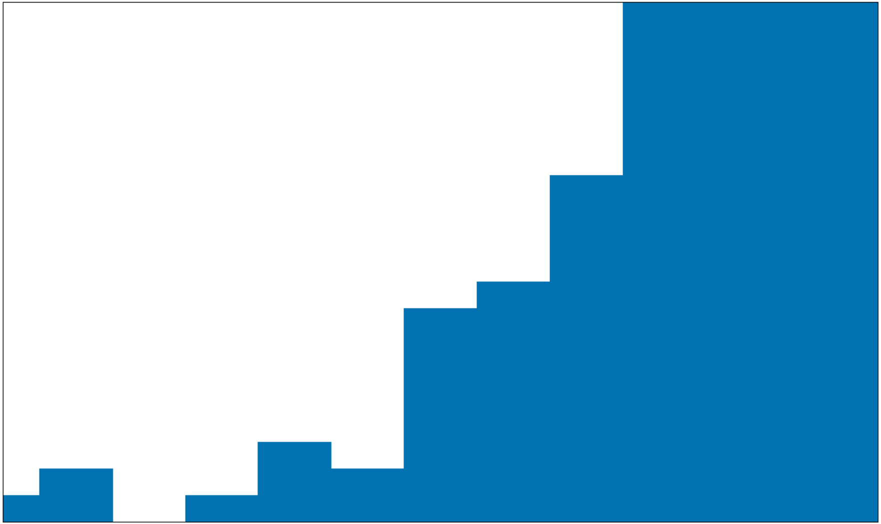


Moments: 1 0.000 1.000 0.000 2.000 0.000 5.000 0.000 13.999 0.001 41.998


a1 histogram of y^2 = x^3 + x + 1 for p <= 2^40
41203088794 data points in 202985 buckets
Moments: 1 0.000 1.000 0.000 2.000 0.000 5.000 0.000 14.000 0.000 41.999

a1 histogram of $y^2 + xy = x^3 - 270061832416309222184346521452974537847680546218363579547373 85x$
$+ 55258058551342376475736699591118191821521067032535079608372404779149413277716173425636721497$ for primes $p < 2^{10}$

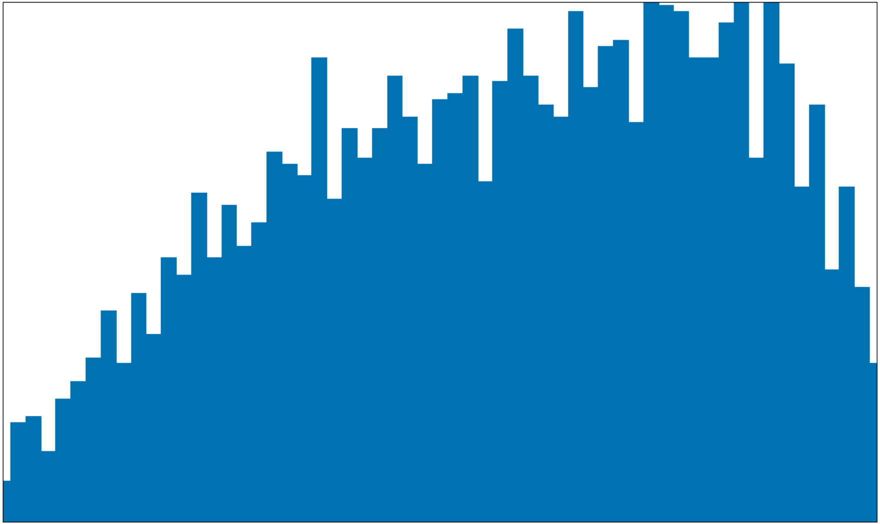
159 data points in 13 buckets, z1 = 0.025, out of range data has area 0.252



Moments: 1 1.114 1.775 2.579 4.523 7.055 12.986 20.973 39.725 65.587 126.589

a1 histogram of $y^2 + xy = x^3 - 27006183241630922218434652145297453784768054621836357954737385x + 552580585513423764757366995911181918215210670325350796083724047791494132777161734256367214 97$ for primes $p < 2^{15}$

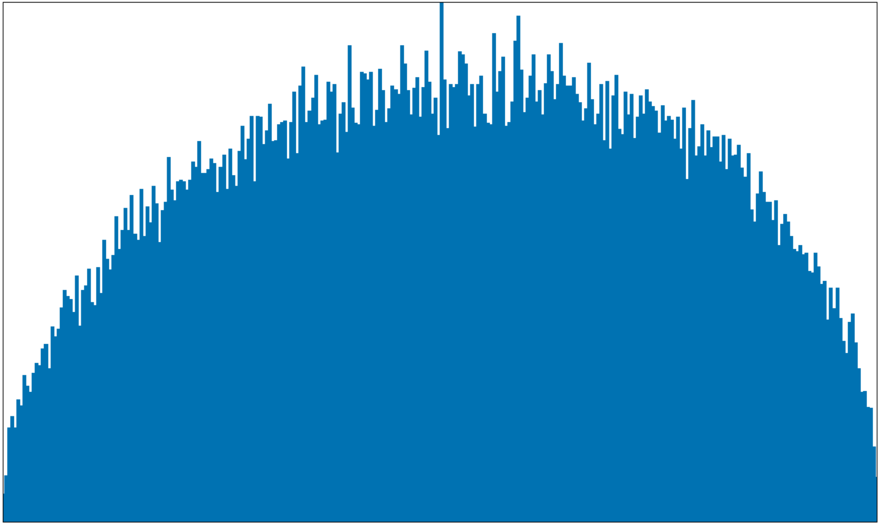
3498 data points in 59 buckets, out of range data has area 0.003



Moments: 1 0.285 1.054 0.604 2.154 1.541 5.465 4.355 15.506 13.144 47.133

a1 histogram of $y^2 + xy = x^3 - 270061832416309222184346521452974537847680546218363579547373 85x + 55258058551342376475736699591118191821521067032535079608372404779149413277716173425636721497$ for primes $p < 2^{20}$

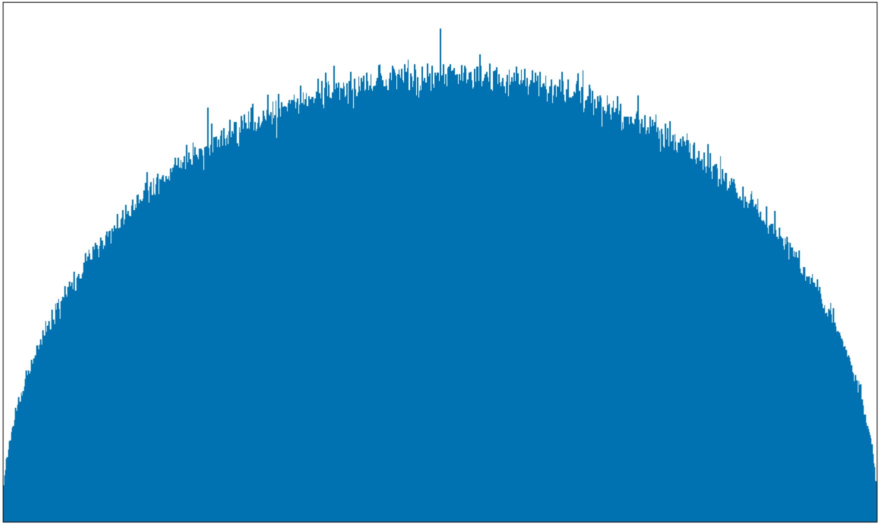
82011 data points in 286 buckets



Moments: 1   0.057   1.010   0.119   2.032   0.305   5.106   0.874   14.367   2.675   43.313

a1 histogram of $y^2 + xy = x^3 - 27006183241630922218434652145297453784768054621836357954737385x$
$+ 55258058551342376475736699591118191821521067032535079608372404779149413277716173425636721497$ for primes $p < 2^{25}$

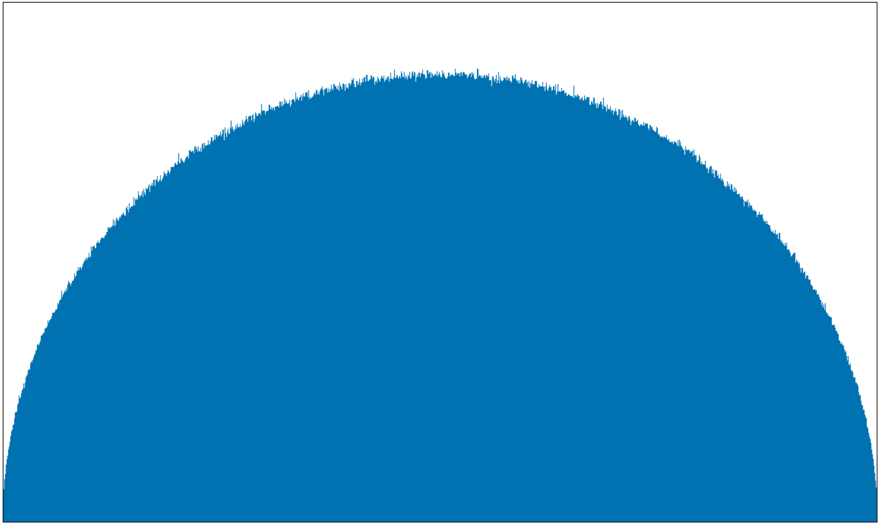
2063675 data points in 1436 buckets



Moments: 1   0.010   1.000   0.020   2.001   0.050   5.006   0.143   14.024   0.431   42.097

a1 histogram of $y^2 + xy = x^3 - 27006183241630922218434652145297453784768054621836357954737385x + 552580585513423764757366995911181918215210670325350796083724047791494132777161734256367214 97$ for primes $p < 2^{30}$

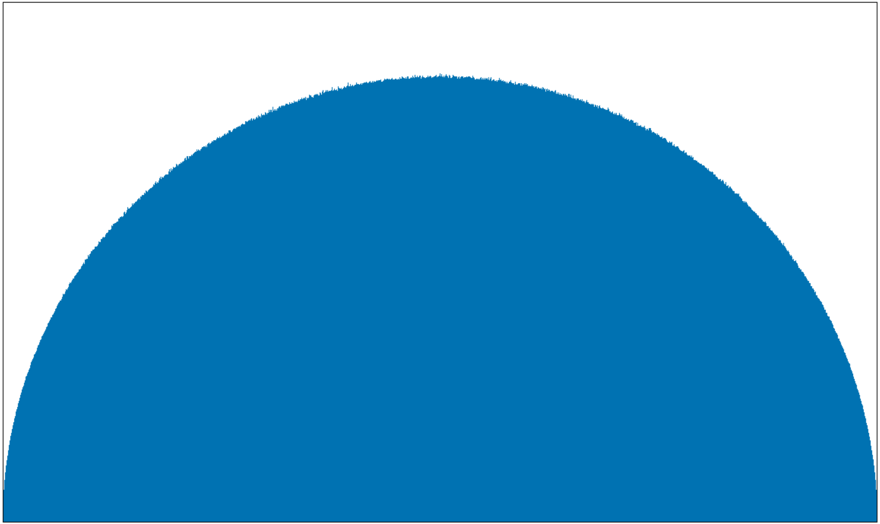
54400014 data points in 7375 buckets



Moments: 1 0.002 1.000 0.003 1.999 0.009 4.996 0.025 13.988 0.078 41.958

a1 histogram of $y^2 + xy = x^3 - 27006183241630922218434652145297453784768054621836357954737385x + 55258058551342376475736699591118191821521067032535079608372404779149413277716173425636721497$ for primes $p < 2^{35}$

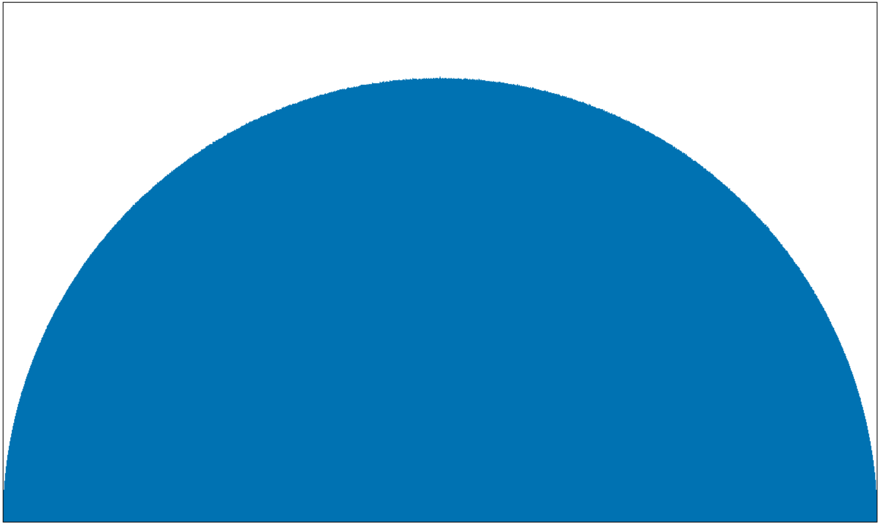
1480206265 data points in 38473 buckets



Moments: 1 0.000 1.000 0.001 2.000 0.002 5.000 0.005 14.000 0.014 42.001

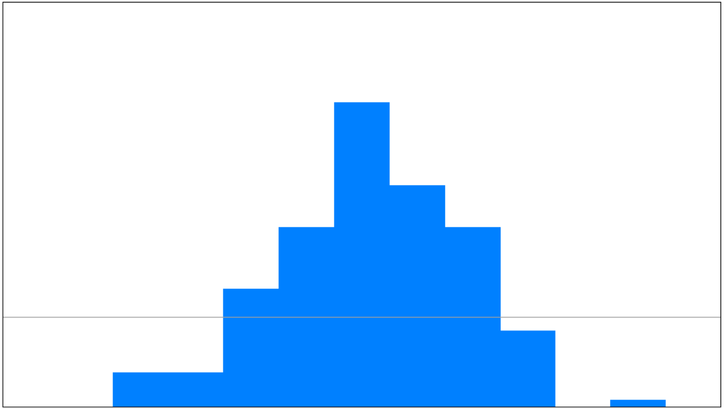
a1 histogram of $y^2 + xy = x^3 - 27006183241630922218434652145297453784768054621836357954737385x + 5525805855134237647573669959111819182152106703253507960837240477914941327771617342563672 1497$ for primes $p < 2^{40}$

41203088782 data points in 202985 buckets

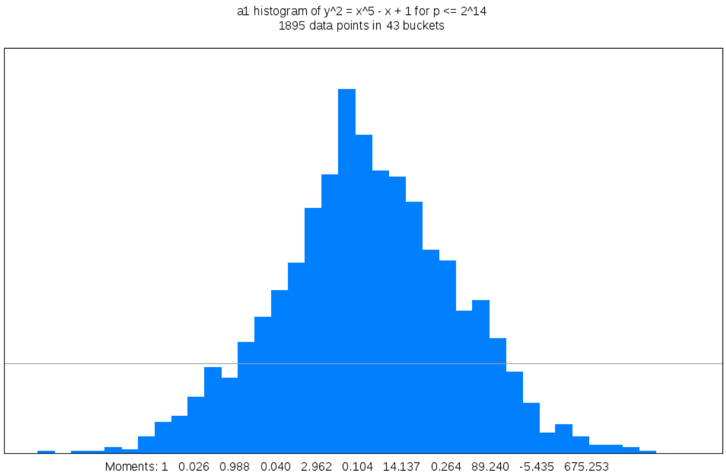


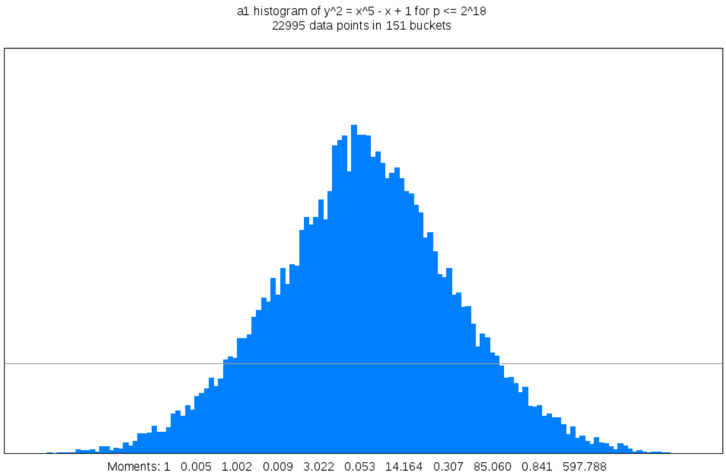
Moments: 1   0.000   1.000   0.000   2.000   0.000   5.000   0.001   14.000   0.002   42.001

a1 histogram of y^2 = x^5 - x + 1 for p <= 2^10
167 data points in 13 buckets, z1 = 0.030

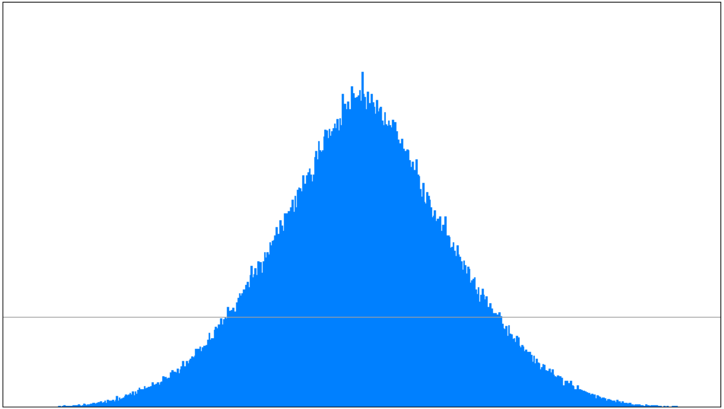


Moments: 1 0.098 1.031 -0.011 3.041 -0.725 13.944 -3.026 81.644 4.428 547.633


a1 histogram of y^2 = x^5 - x + 1 for p <= 2^14
1895 data points in 43 buckets
Moments: 1 0.026 0.988 0.040 2.962 0.104 14.137 0.264 89.240 -5.435 675.253

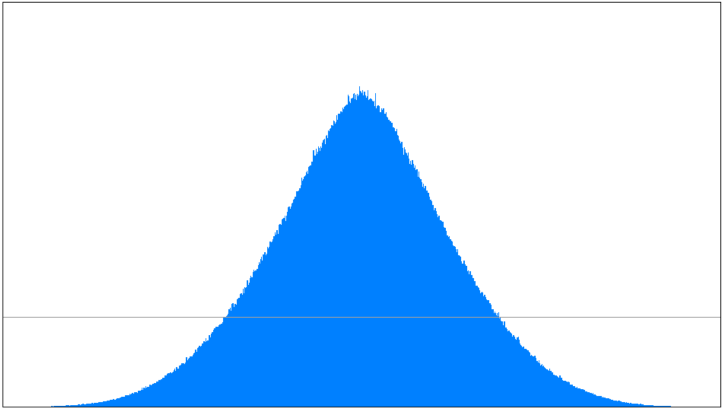

a1 histogram of y^2 = x^5 - x + 1 for p <= 2^18
22995 data points in 151 buckets
Moments: 1 0.005 1.002 0.009 3.022 0.053 14.164 0.307 85.060 0.841 597.788

a1 histogram of y^2 = x^5 - x + 1 for p <= 2^22
295942 data points in 544 buckets

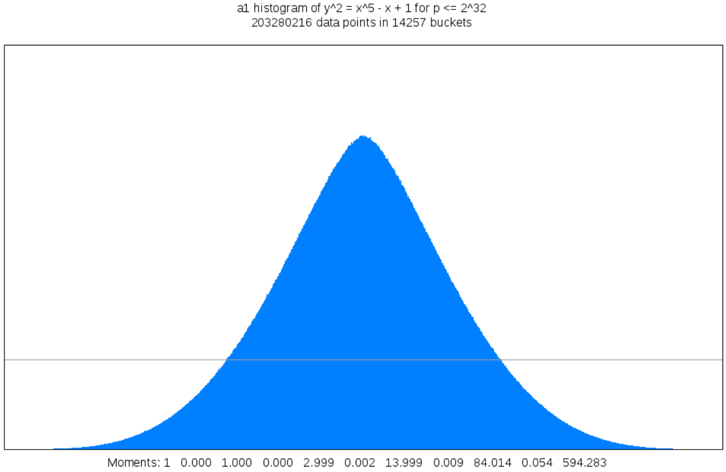


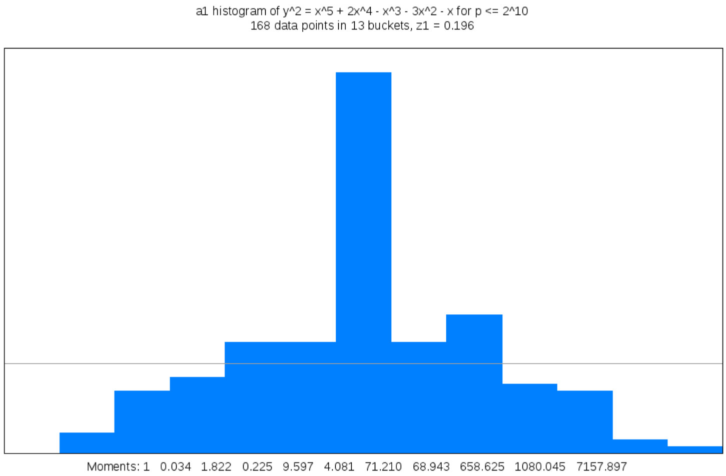
Moments: 1 0.002 0.999 0.006 2.998 0.019 13.999 0.091 84.022 0.763 594.374

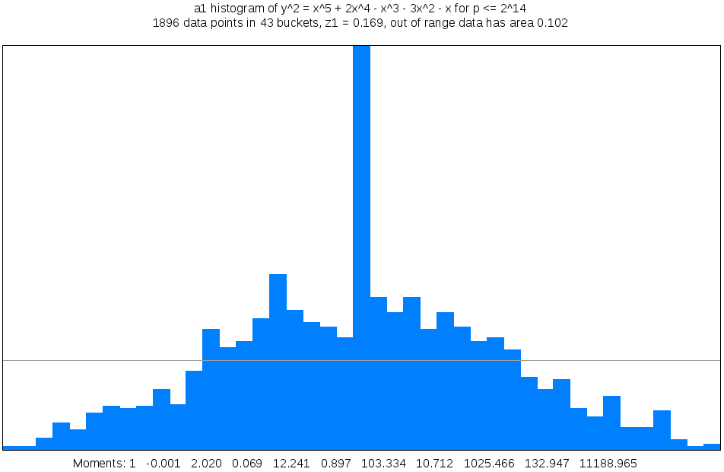
a1 histogram of y^2 = x^5 - x + 1 for p <= 2^26
3957804 data points in 1989 buckets

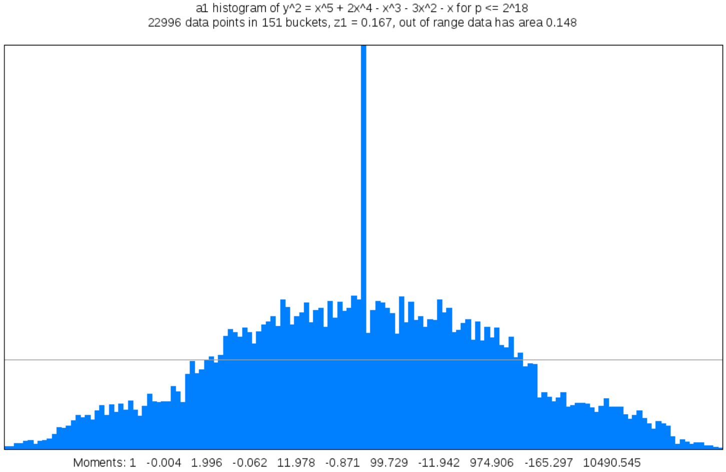


Moments: 1 0.000 1.000 0.001 2.998 0.008 13.977 0.081 83.788 0.775 592.260


a1 histogram of y^2 = x^5 - x + 1 for p <= 2^32
203280216 data points in 14257 buckets
Moments: 1 0.000 1.000 0.000 2.999 0.002 13.999 0.009 84.014 0.054 594.283

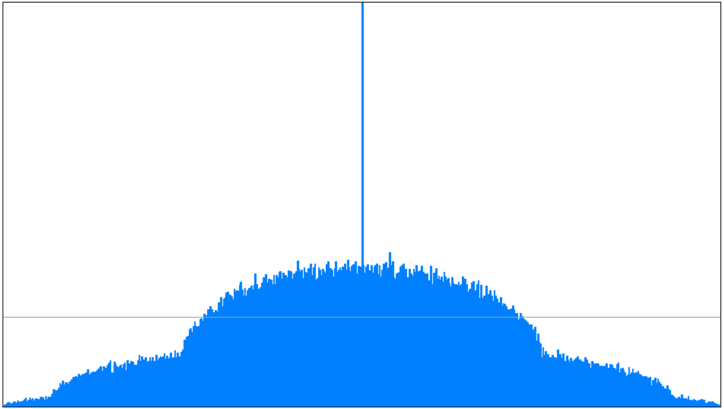

a1 histogram of y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x for p <= 2^10
168 data points in 13 buckets, z1 = 0.196
Moments: 1 0.034 1.822 0.225 9.597 4.081 71.210 68.943 658.625 1080.045 7157.897


a1 histogram of y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x for p <= 2^14
1896 data points in 43 buckets, z1 = 0.169, out of range data has area 0.102
Moments: 1 -0.001 2.020 0.069 12.241 0.897 103.334 10.712 1025.466 132.947 11188.965


a1 histogram of y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x for p <= 2^18
22996 data points in 151 buckets, z1 = 0.167, out of range data has area 0.148
Moments: 1 -0.004 1.996 -0.062 11.978 -0.871 99.729 -11.942 974.906 -165.297 10490.545

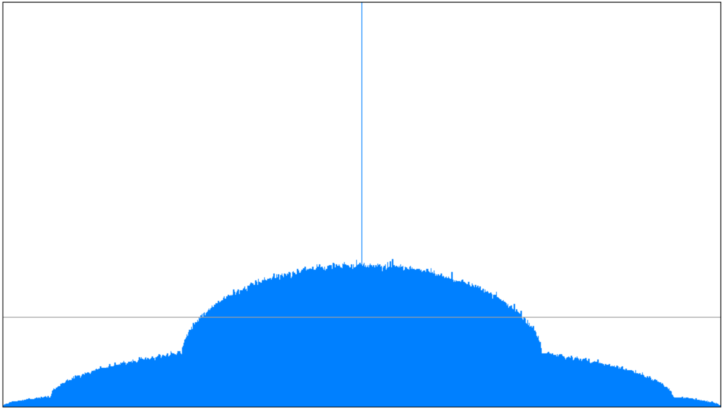
a1 histogram of y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x for p <= 2^22

295943 data points in 544 buckets, z1 = 0.167, out of range data has area 0.161



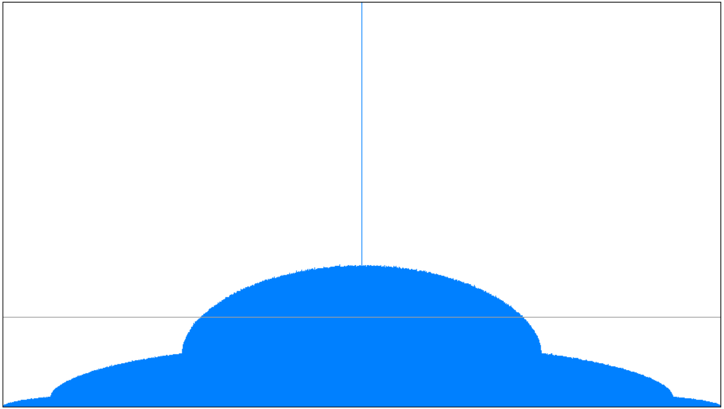
Moments: 1 0.000 2.001 0.005 12.018 0.062 100.219 1.003 982.339 16.061 10606.396

a1 histogram of y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x for p <= 2^26
3957805 data points in 1989 buckets, z1 = 0.167, out of range data has area 0.165



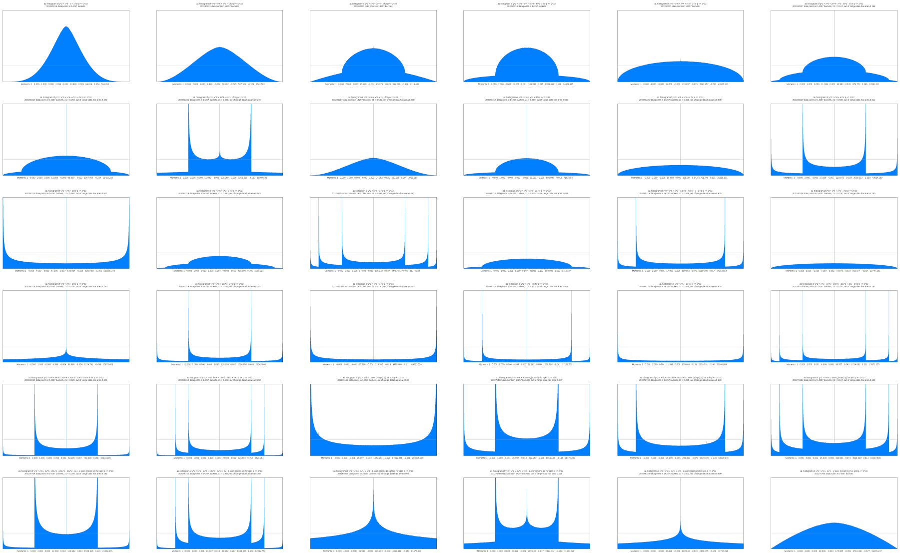
Moments: 1 -0.000 1.999 -0.001 11.992 -0.011 99.894 -0.191 978.446 -3.099 10559.868

a1 histogram of y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x for p <= 2^32
203280217 data points in 14257 buckets, z1 = 0.167, out of range data has area 0.166



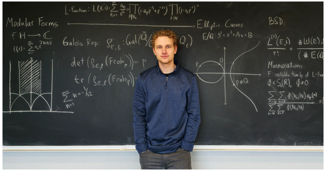
Moments: 1 0.000 2.000 0.000 11.999 0.003 99.983 0.030 979.773 0.286 10581.031

Sato-Tate trace distributions of genus 2 curves



Murmurations of elliptic curves

In 2022, He, Lee, Oliver, and Pozdnyakov ran a series of machine learning experiments in an attempt to predict ranks of elliptic curves over $\mathbb{Q}$ using Frobenius traces.

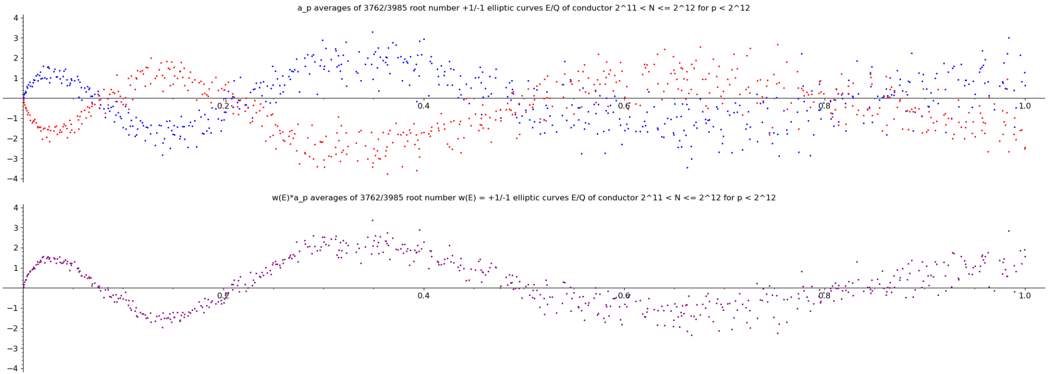


Their efforts to predict ranks worked for curves of small conductor, but not in general. However, they noticed a previously unobserved oscillation in average Frobenius traces in families of elliptic curves ordered by conductor when separated by rank.

You can read more about their discovery in this 2024 Quanta article.

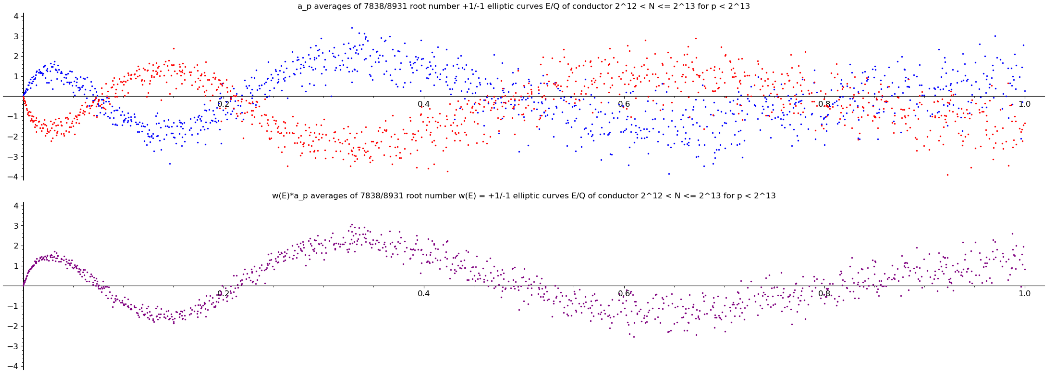
Murmurations of elliptic curves

Elliptic curves of conductor $N \in (2^n, 2^{n+1}]$ for $11 \le n \le 18$. Blue/red/purple dots at $(p, \bar{a}_p$ or $\bar{m}_p)$ are averages of a_p or $m_p := (-1)^r a_p(E)$ over even/odd/all $E/\mathbb{Q}$.



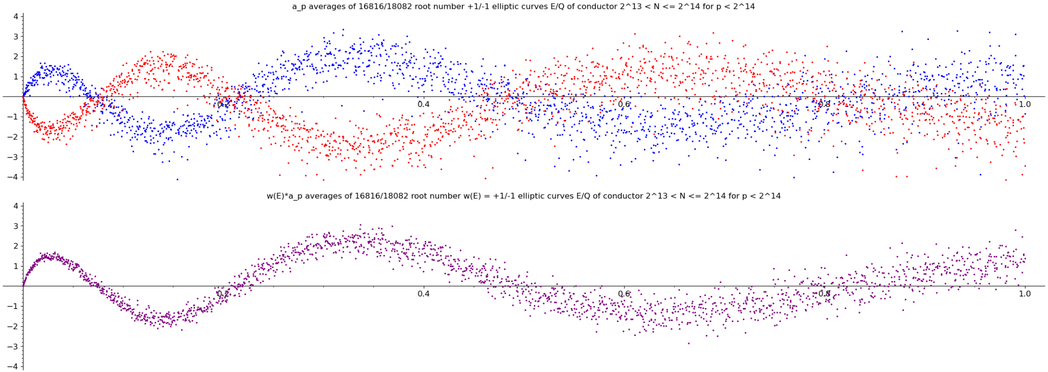
Murmurations of elliptic curves

Elliptic curves of conductor $N \in (2^n, 2^{n+1}]$ for $11 \leq n \leq 18$. Blue/red/purple dots at $(p, \bar{a}_p$ or $\bar{m}_p)$ are averages of a_p or $m_p := (-1)^r a_p(E)$ over even/odd/all $E/\mathbb{Q}$.



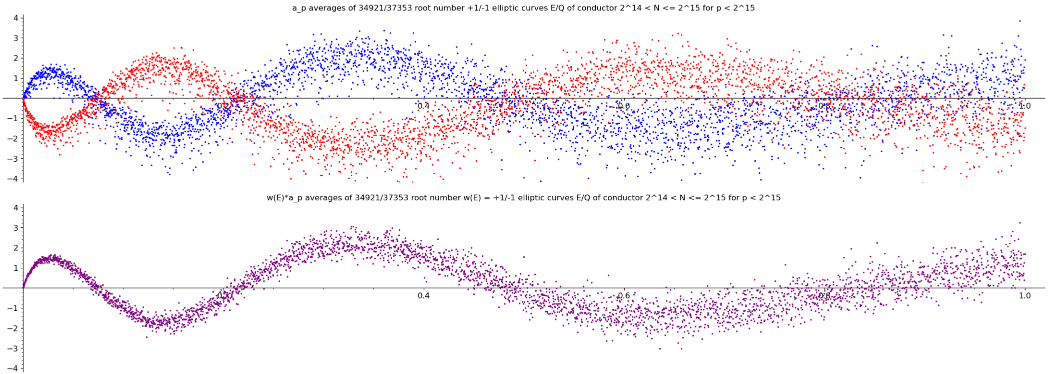
Murmurations of elliptic curves

Elliptic curves of conductor $N \in (2^n, 2^{n+1}]$ for $11 \leq n \leq 18$. Blue/red/purple dots at $(p, \bar{a}_p$ or $\bar{m}_p)$ are averages of a_p or $m_p := (-1)^r a_p(E)$ over even/odd/all $E/\mathbb{Q}$.



Murmurations of elliptic curves

Elliptic curves of conductor $N \in (2^n, 2^{n+1}]$ for $11 \leq n \leq 18$. Blue/red/purple dots at $(p, \bar{a}_p$ or $\bar{m}_p)$ are averages of a_p or $m_p := (-1)^r a_p(E)$ over even/odd/all $E/\mathbb{Q}$.



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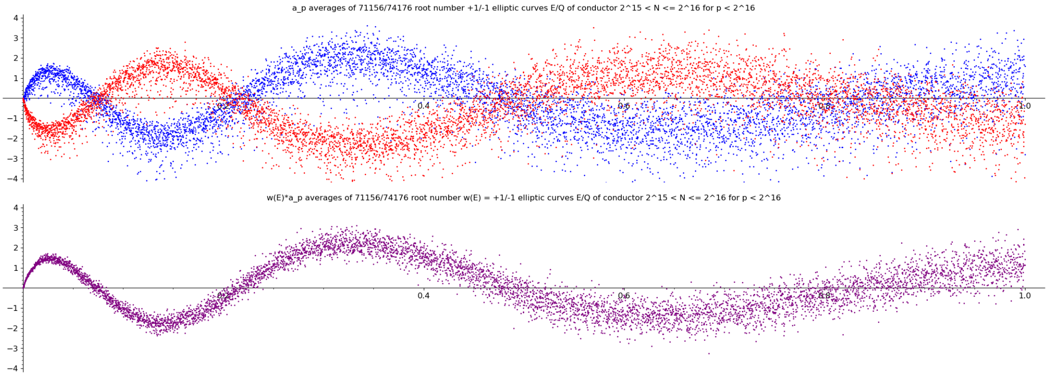

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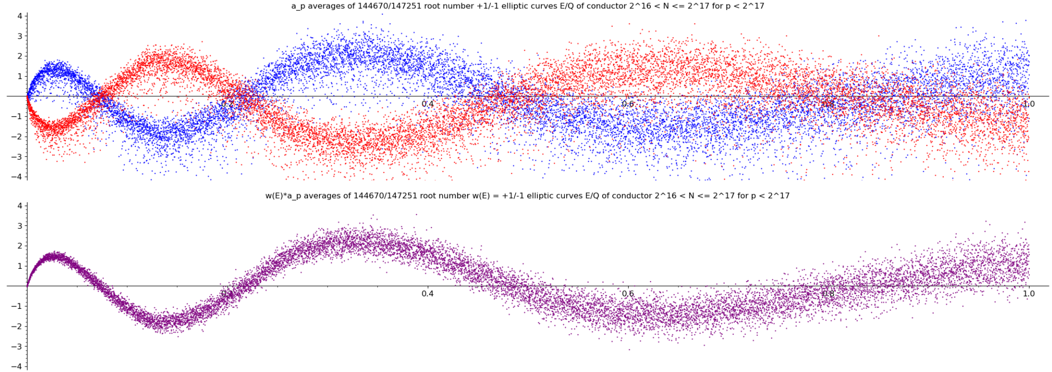

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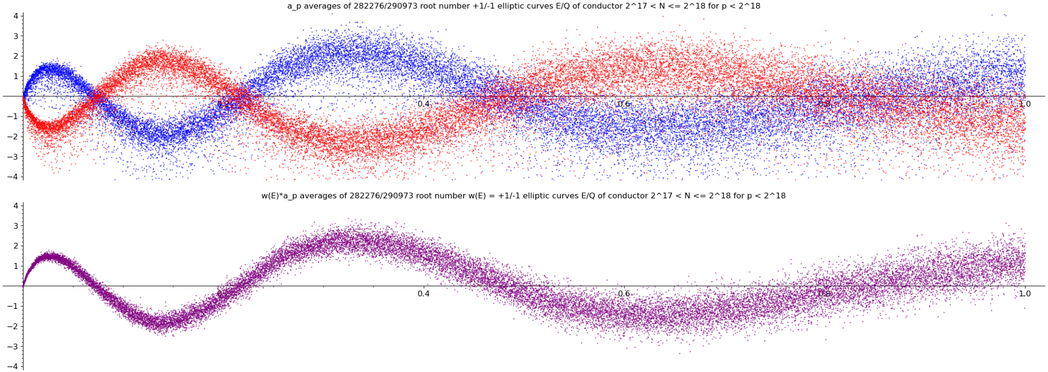

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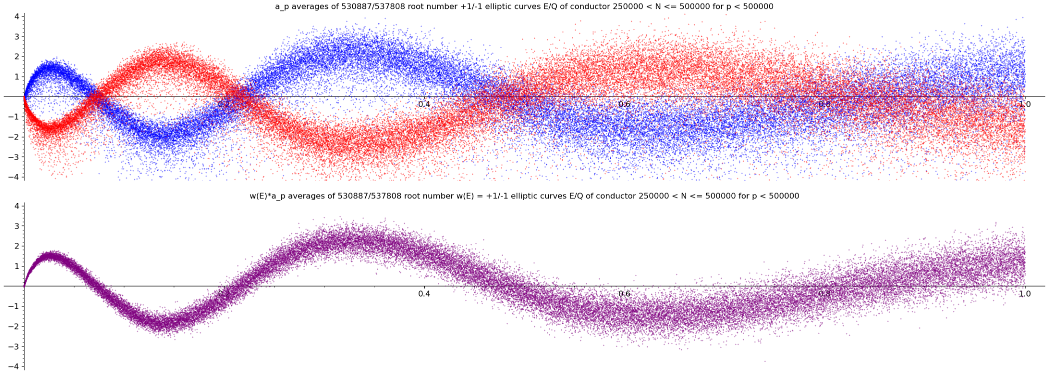
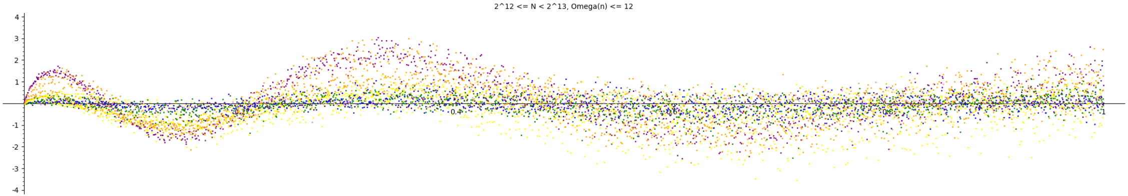


Murmurations of elliptic curves over a_n (not just a_p)

Elliptic curves of conductor $N \in (2^n, 2^{n+1}]$ for $11 \leq n \leq 18$. Dots at $(n, \bar{m}_n)$ show the average of $m_n := w(E)a_n(E)$ over all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$. The color of each dot indicates the number of prime factors of n (with multiplicity).



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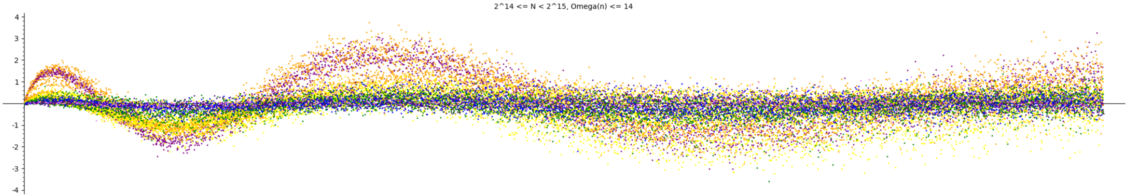

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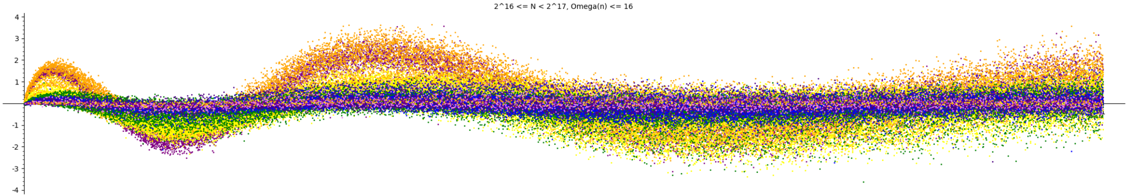

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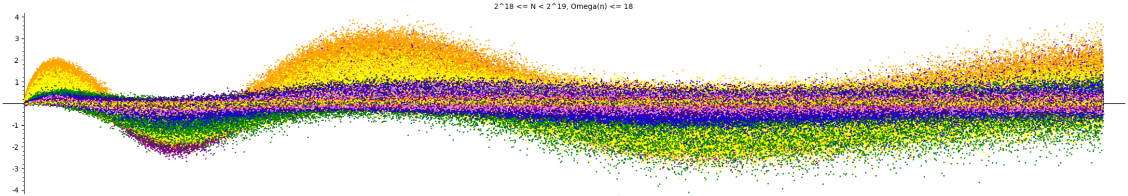
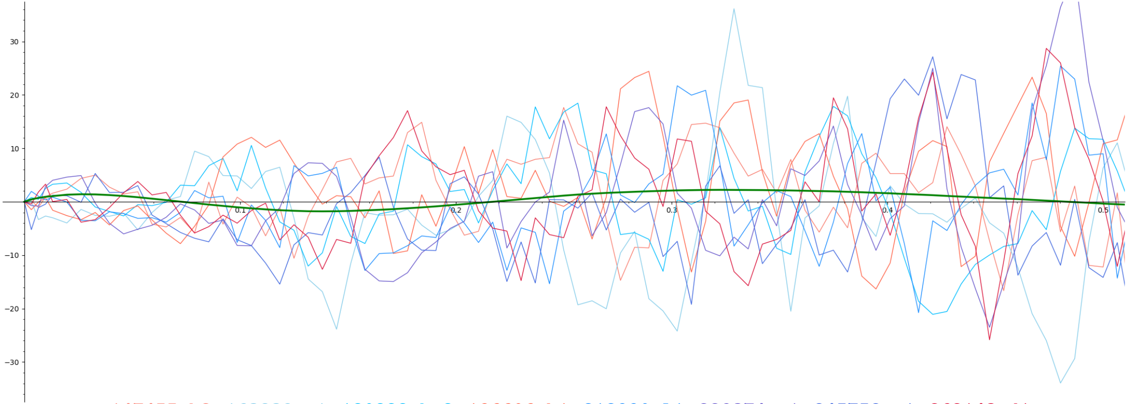


Murmurations are an aggregate phenomenon

Moving average line plots of $\bar{m}_p$ for 8 individual and all $E/\mathbb{Q}$ with $N_E \in (M, 2M]$, using subintervals of size $\sqrt{M}$ for $p \leq 2M$, with $M = 2^{17}$.



147455.b2, 163839.a1, 180222.be2, 196606.b1, 212990.l1, 229374.a1, 245758.a1, 262143.d1

Zubrilina's theorem

Definition. Let $U_n \in \mathbb{Z}[x]$ denote the Chebyshev polynomial defined by $U_n(\cos\vartheta)\sin\vartheta = \sin((n+1)\vartheta)$. The murmuration density function is

$$M_k(y) := D_k\Big(Ay \ - (-1)^{k/2} B \sum_{1\le r\le 2y} c(r)\sqrt{4y^2 - r^2}\ U_{k-2}(\tfrac{r}{2y}) - \pi y^2 \delta_{k=2}\Big),$$

$A := \prod_p \big(1 + \frac{p}{(p+1)^2(p-1)}\big)$, $B := \prod_p \frac{p^4-2p^2-p+1}{(p^2-1)^2}$, $c(r) := \prod_{p|r}\big(1 + \frac{p^2}{p^4-2p^2-p+1}\big)$, $D_k := \frac{12}{(k-1)\pi \prod_p (1-\frac{1}{p^2+p})}$.

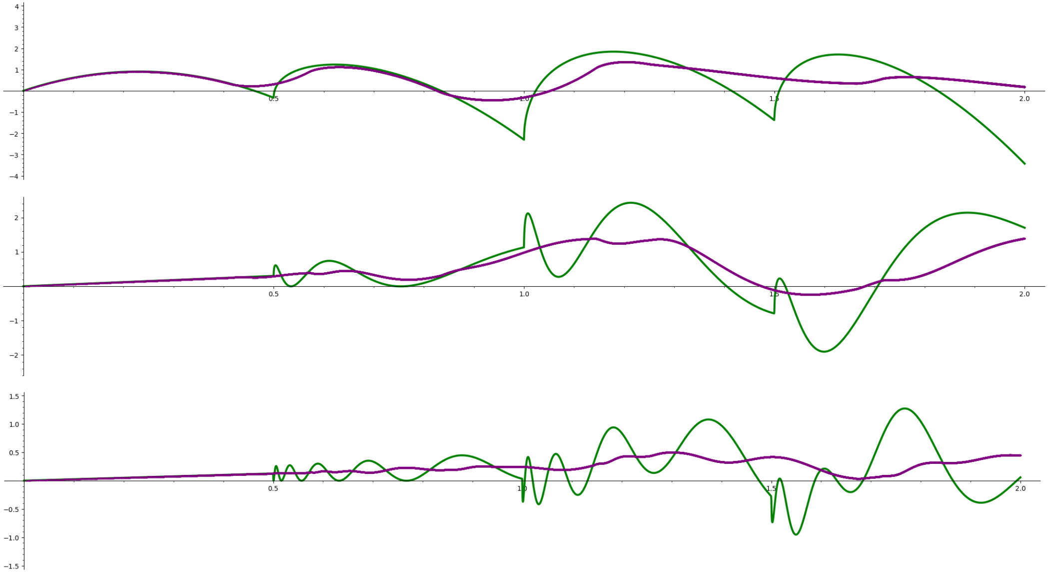
Theorem (Zubrilina 2023)

Let $\sum a_n(f)q^n$ denote a weight-k newform for $\Gamma_0(N)$ with root number $w(f)$. Let $X, Y, P \to \infty$ with P prime, $Y \sim X^{1-\delta}$, $P \ll X^{1+\delta_1}$, $\delta, \delta_1 > 0$ and $2\delta_1 < \delta < 1$, and put $y := \sqrt{P/X}$. Then for every $\varepsilon > 0$ we have

$$\frac{\sum_{N\in[X,X+Y]}^{\square\text{-free}} \sum_f w(f) a_P(f) P^{(1-k/2)}}{\sum_{N\in[X,X+Y]}^{\square\text{-free}} \sum_f 1} = M_k(y) + O_\varepsilon(X^{-\delta'+\varepsilon} + P^{-1})$$

where $\delta' := \max(\delta/2 - \delta_1, (\delta+1)/9 - \delta_1)$; for $\delta_1 < 2/9$ we can choose δ so $\delta' > 0$.

Zubrilina's theorem for $k = 2, 14, 32$ (click here for other k)



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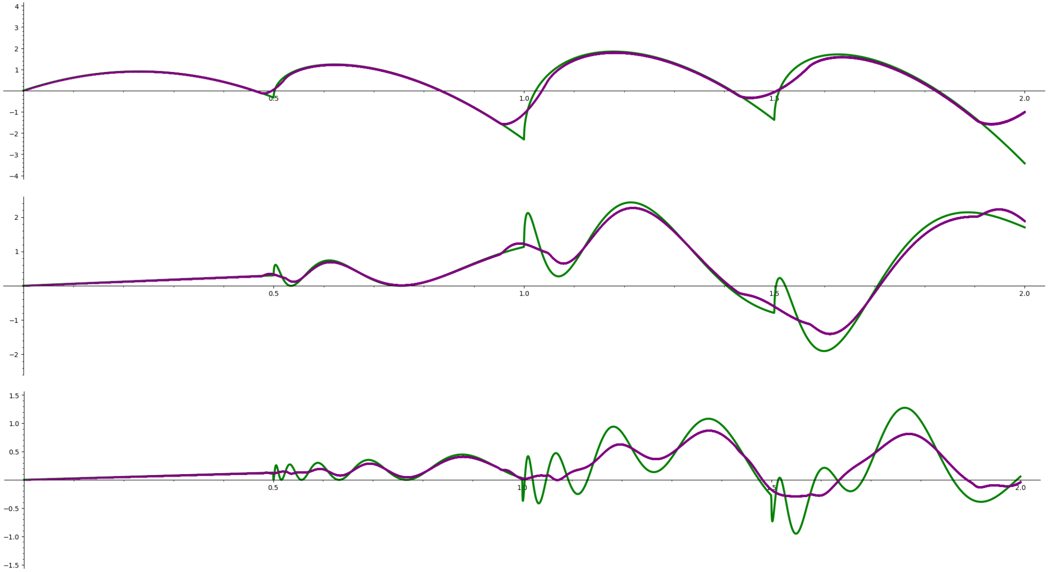

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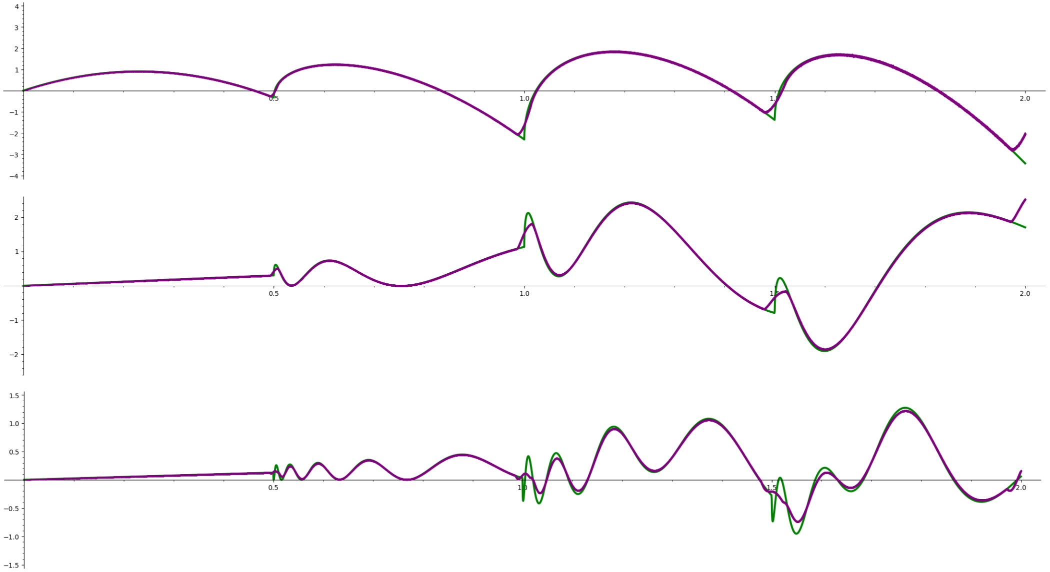

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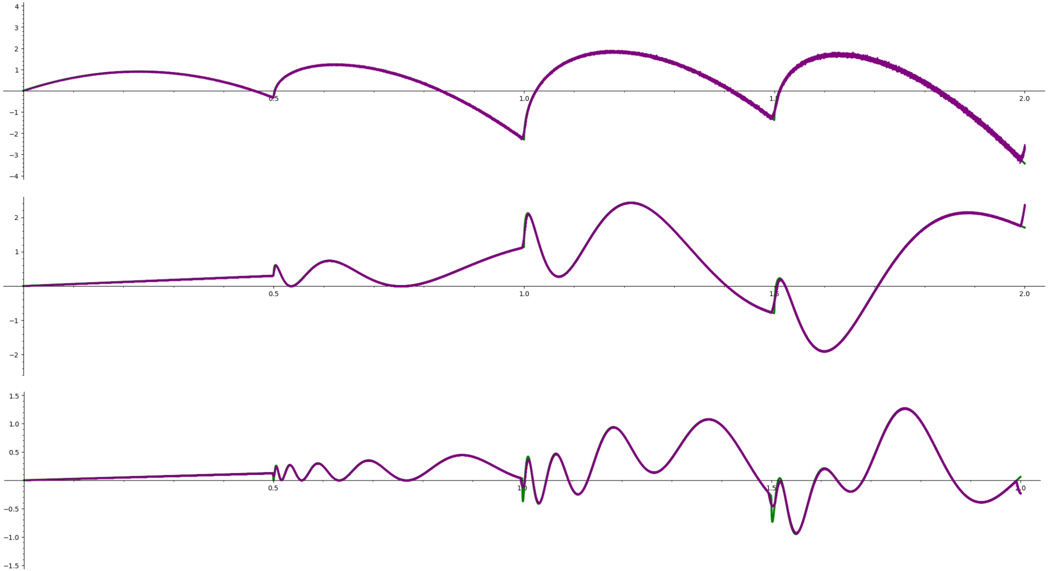

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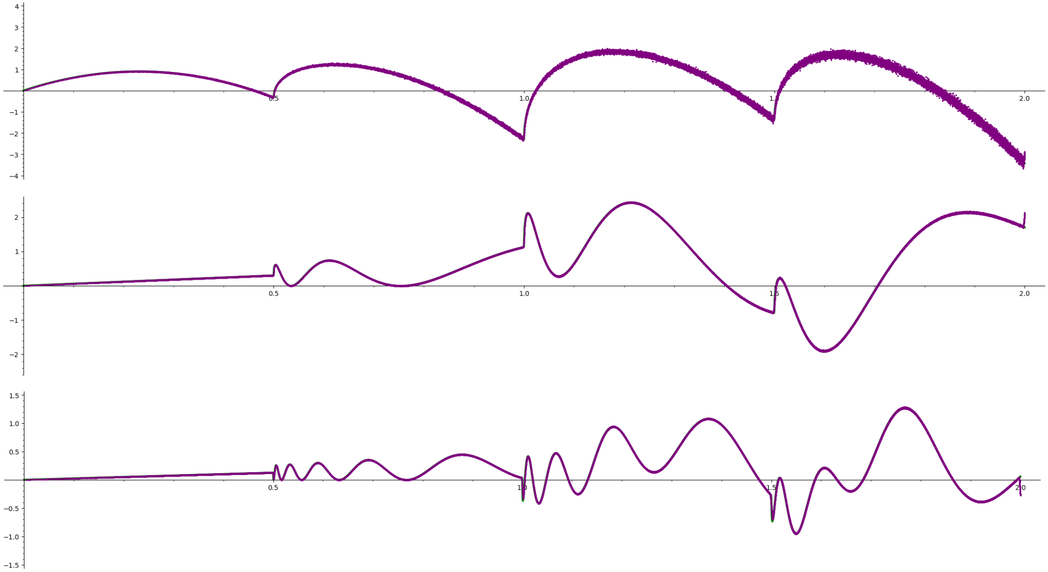


A murmuration theorem for elliptic curves

Let $\mathcal{E}(X) := \left\{ y^2 = x^3 + ax + b : a, b \in \mathbb{Z}, p^4 | a \Rightarrow p^6 \nmid b, \max(4|a|^3, 27b^2) \leq X \right\}$ be the set of isomorphism classes of elliptic curves over $\mathbb{Q}$ of naive height at most X.

Theorem (S–Sawin 2025)

For any smooth $W : \mathbb{R}_{>0} \to \mathbb{R}$ with compact support, the limit

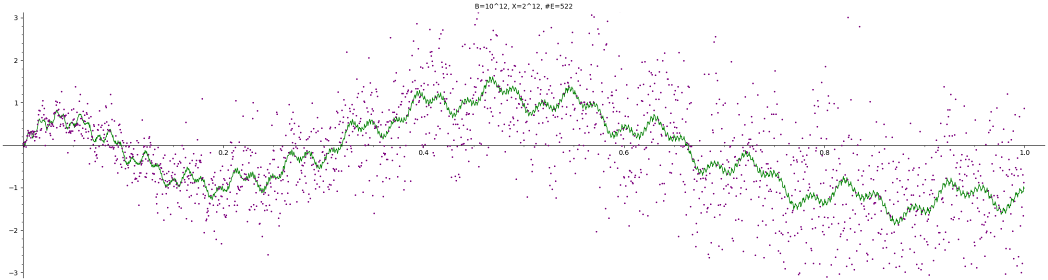
$$\lim_{X\to\infty} \frac{1}{\#\mathcal{E}(X)} \sum_{E\in\mathcal{E}(X)} \frac{\varepsilon(E)}{N_E} \sum_{n\geq 1} W(n/N_E)a_n(E)$$

exists and is equal to

$$\int_0^\infty 2\pi W(u) \sum_{n=1}^\infty \frac{\prod_{p|n} \ell_{p^{\nu_p(n)}}}{\sqrt{n}} \sqrt{u} J_1(4\pi\sqrt{un}) du,$$

with $\ell_{2^\nu} = \frac{t_2(\nu+2)}{1023}, \ell_{3^\nu} = \cdots, \ell_{p^\nu} = \frac{p^9-p^8}{p^{10}-1} t_p(\nu+2)$, *where* $t_p(k) = \mathrm{tr}(T_p)$ *on* $S_k(1)$.

A murmuration theorem for elliptic curves



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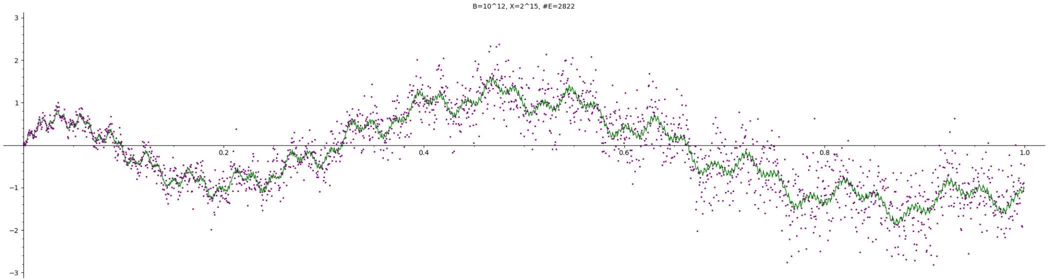

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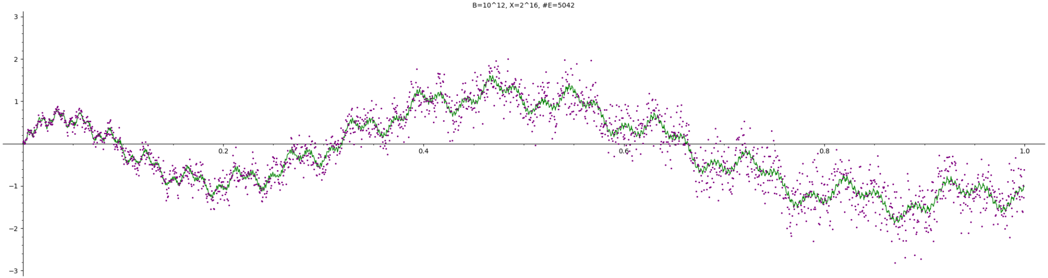

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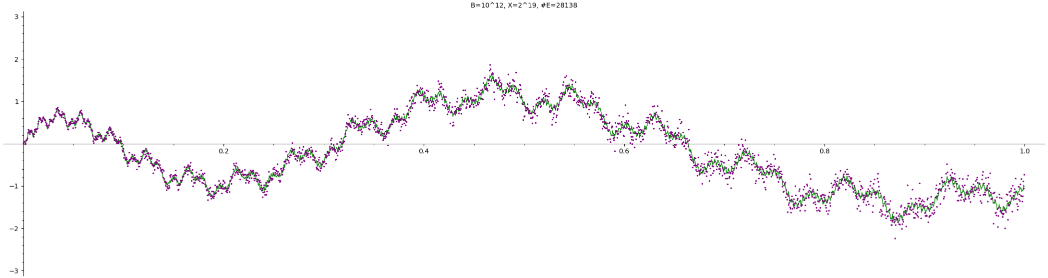

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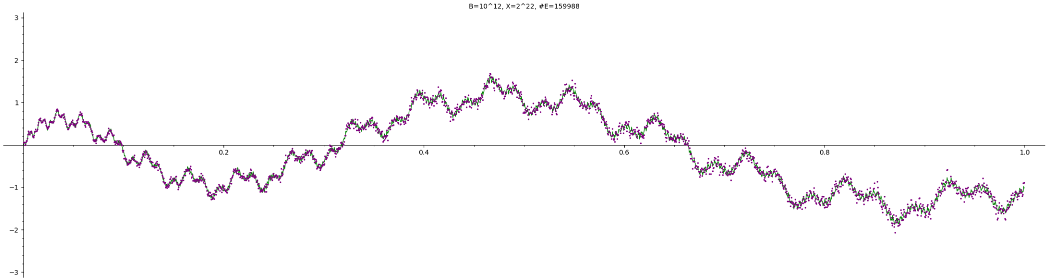

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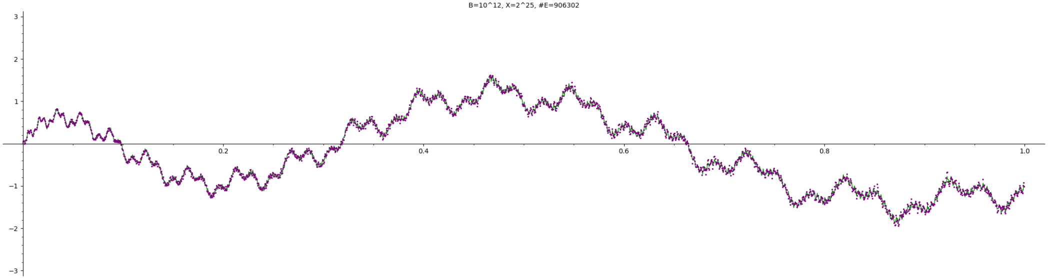
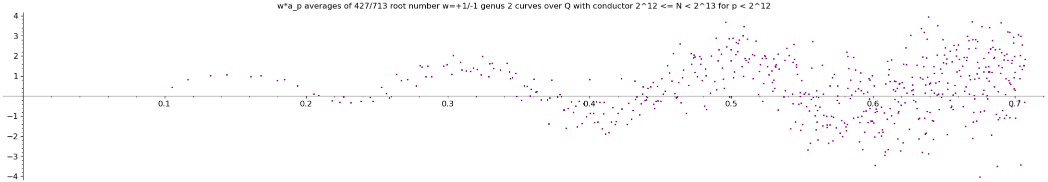
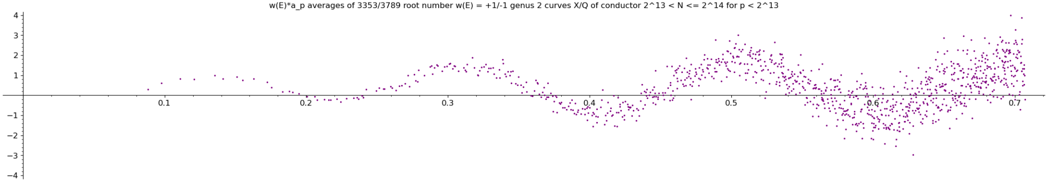


L-functions of genus 2 curves over $\mathbb{Q}$ with Sato-Tate group USp(4).

Before and after genus 2 murmuration plots (top LMFDB, bottom new dataset).





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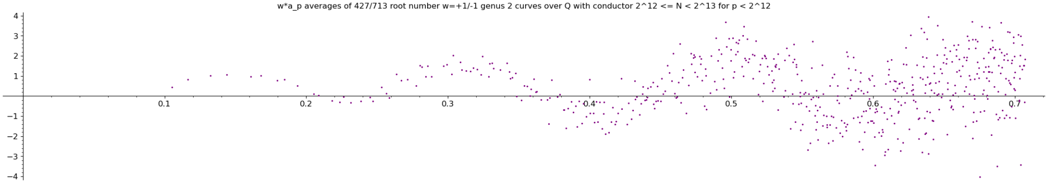

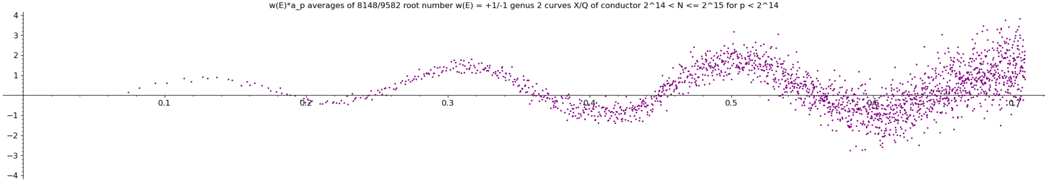

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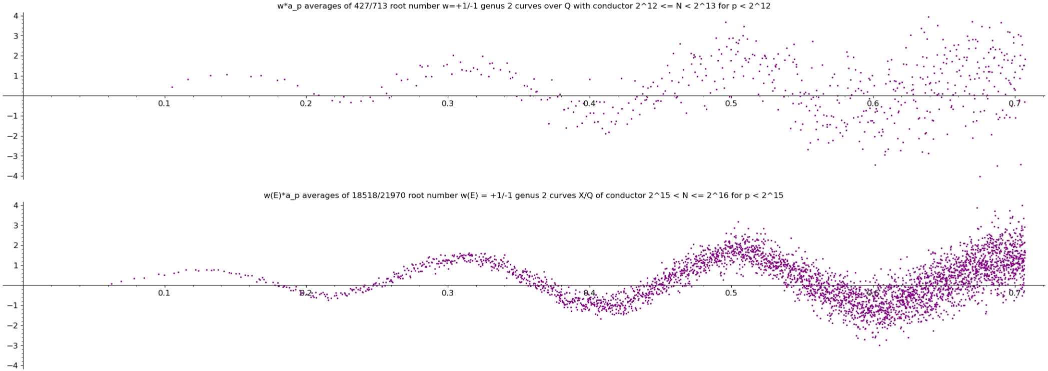

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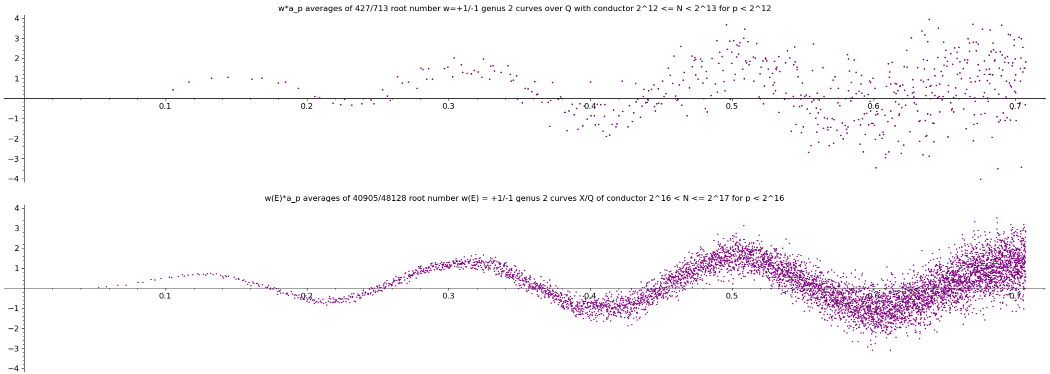

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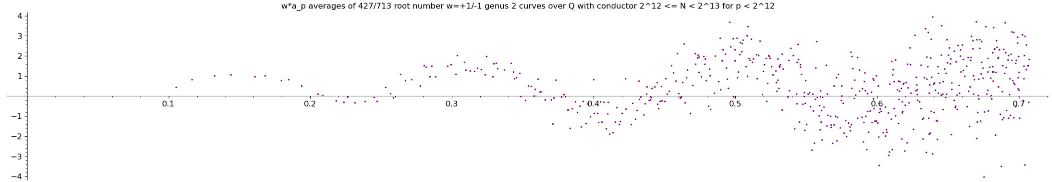

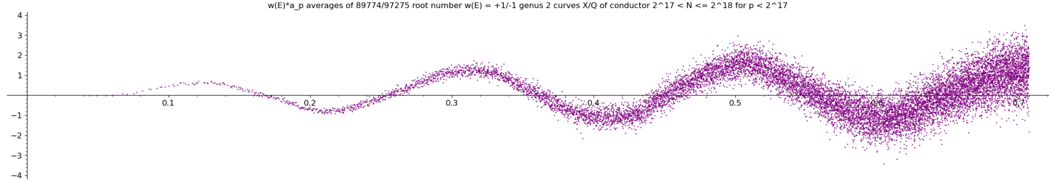

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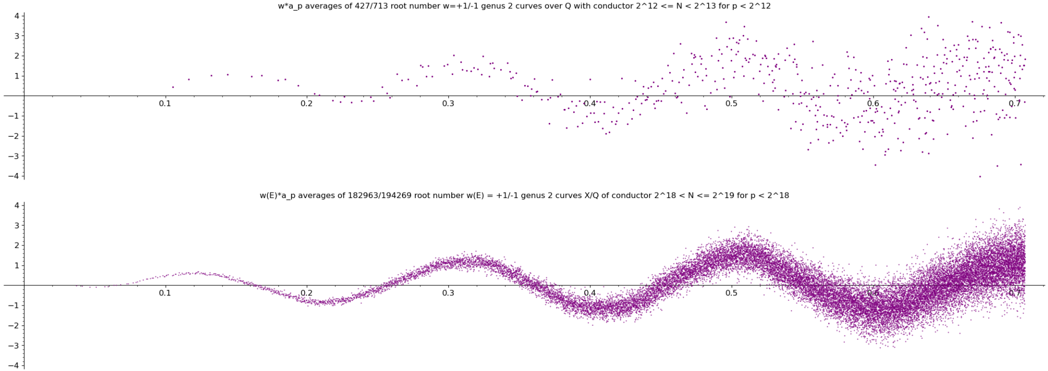

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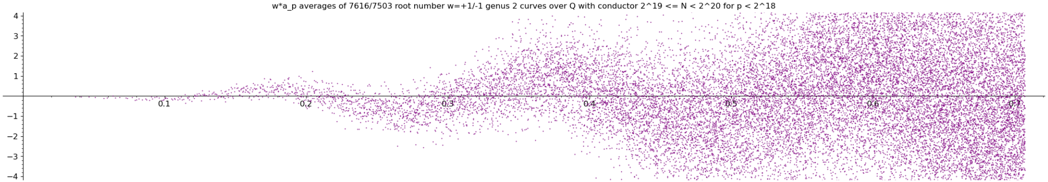
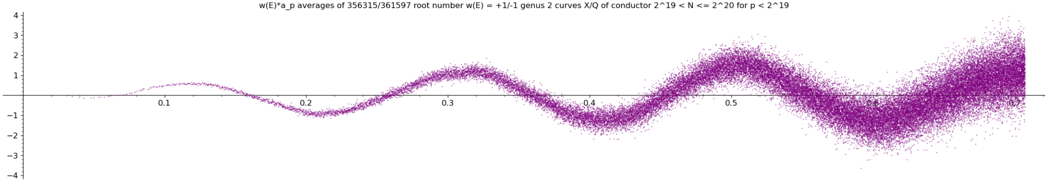
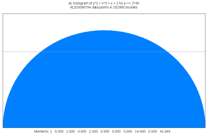
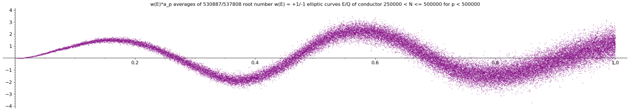
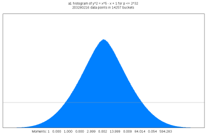


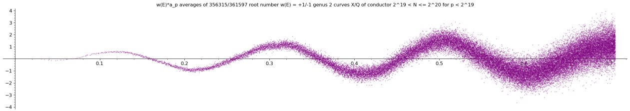


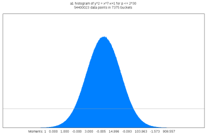
Thank you!

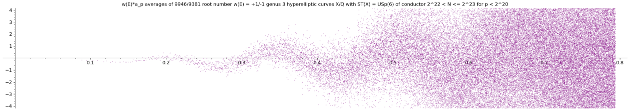

a1 histogram of y^2 = x^3 + x + 1 for p <= 2^40
41203088794 data points in 202985 buckets
Moments: 1 0.000 1.000 0.000 2.000 0.000 5.000 0.000 14.000 0.000 41.999


w(E)*a_p averages of 530887/537808 root number w(E) = +1/-1 elliptic curves E/Q of conductor 250000 < N <= 500000 for p < 500000


a1 histogram of y^2 = x^5 - x + 1 for p <= 2^32
203280216 data points in 14257 buckets
Moments: 1 0.000 1.000 0.000 2.999 0.002 13.999 0.009 84.014 0.054 594.283


w(E)*a_p averages of 356315/361597 root number w(E) = +1/-1 genus 2 curves X/Q of conductor 2^19 < N <= 2^20 for p < 2^19


a1 histogram of y^2 = x^7-x+1 for p <= 2^30
54400023 data points in 7375 buckets
Moments: 1 0.000 1.000 -0.000 3.000 -0.005 14.996 -0.093 103.963 -1.573 908.557


w(E)*a_p averages of 9946/9381 root number w(E) = +1/-1 genus 3 hyperelliptic curves X/Q with ST(X) = USp(6) of conductor 2^22 < N <= 2^23 for p < 2^20

Temporary page!

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If you rerun the document (without altering it) this surplus page will go away, becau
LaTeX now knows how many pages to expect for this document.