L-functions from nothing

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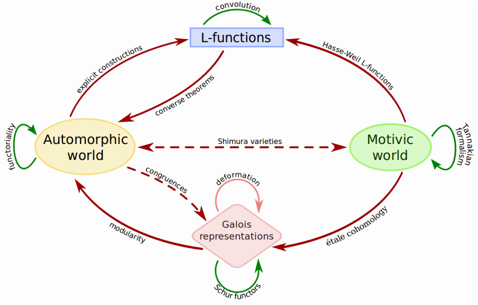
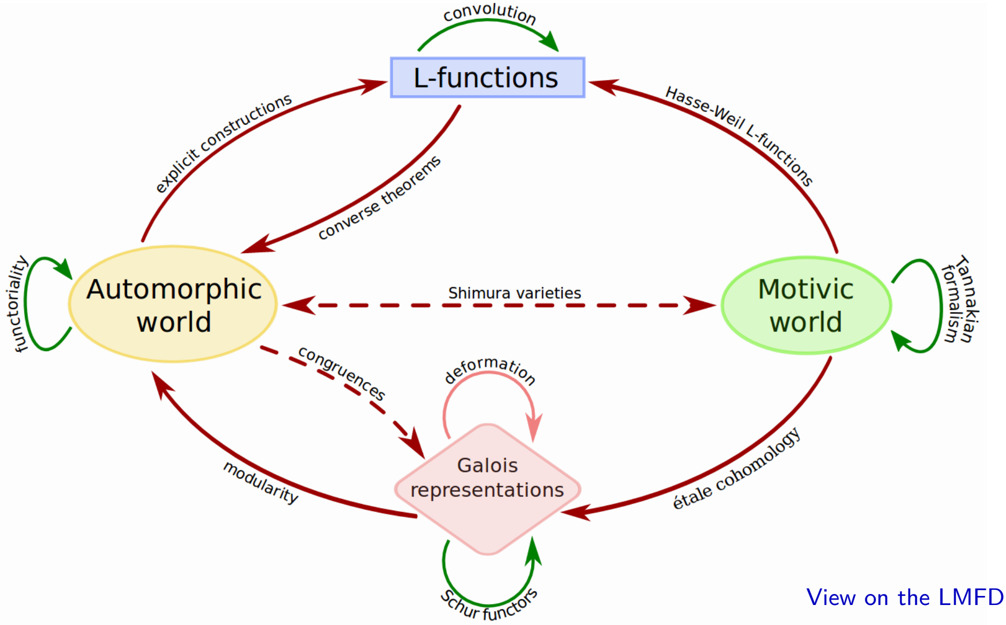


The Simons Collaboration in Arithmetic Geometry, Number Theory, and Computation

(joint work with Andrew R. Booker)



View on the LMFDB

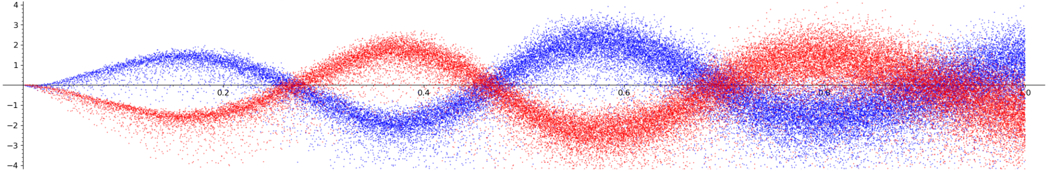
Why care about L-functions?

Many of the most important questions in number theory involve L-functions:

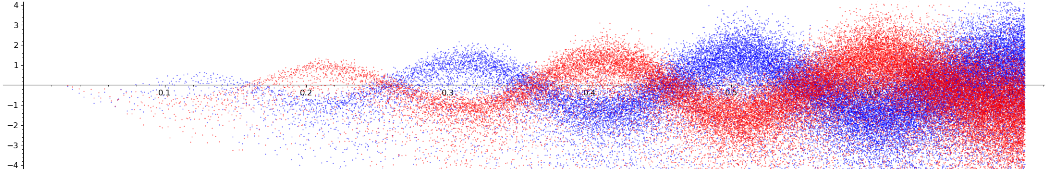
- Riemann hypothesis
- Artin conjecture
- Birch and Swinnerton-Dyer conjecture
- Sato–Tate conjecture
- Lang–Trotter conjecture
- Brumer–Stark conjecture
- Hodge conjecture
- Tate conjecture
- Modularity
- The Langlands program
- Murmurations!

Murmurations

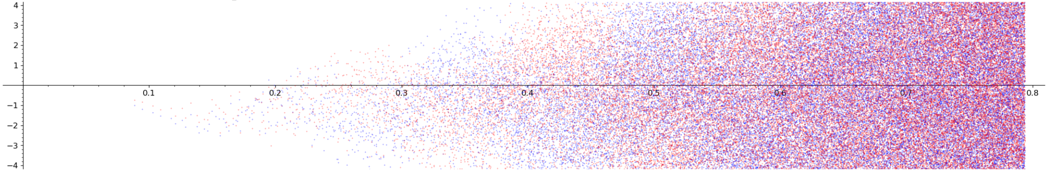
a_p averages of 530887/537808 root number +1/-1 elliptic curves E/Q of conductor 250000 < N <= 500000 for p < 500000



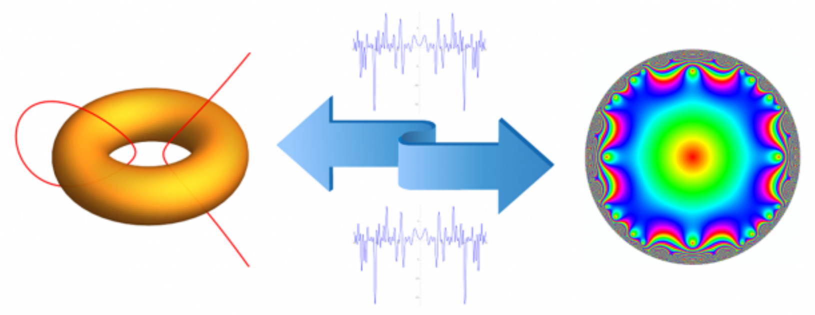
a_p averages of 356315/361597 root number +1/-1 genus 2 curves X/Q of conductor 2^19 < N <= 2^20 for p < 2^19



a_p averages of 9946/9381 root number +1/-1 genus 3 hyperelliptic curves X/Q with ST(X) = USp(6) of conductor 2^22 < N <= 2^23 for p < 2^20



Elliptic curves and their L-functions



Theorem (Eichler-Shimura, Langlands-Tunnell, Serre, Ribet, Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor)

For each positive integer N, the set of L-functions $L(E,s)$ of elliptic curves $E/\mathbb{Q}$ of conductor N is equal to the set of L-functions $L(f,s)$ of newforms $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ of weight 2 and level N with rational q-expansions.

Enumerating elliptic curves by conductor

To enumerate abelian varieties of dimension $g = 1$ over $\mathbb{Q}$ we may proceed as follows:

1. Prove the modularity theorem.
2. Enumerate rational modular forms $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ for $N = 1, 2, 3, \ldots$
3. Use Eichler-Shimura to get an isogeny class representative E_f for each f.
4. Fill out isogeny classes by finding all the elliptic curves $E/\mathbb{Q}$ isogenous to E_f.

For $N \leq 500000$ this yields 3064705 elliptic curves with 2164260 distinct L-functions that have been computed to high precision.

Each one of these steps is substantially more difficult for $g > 1$, even for $g = 2$.

There is recent progress on step 1 [Boxer-Calegari-Gee-Pilloni 2021], and on step 4 [van Bommel-Chidambaram-Costa-Kieffer 2023].

But step 2 may never be practical, and step 3 is not possible, not even in principle.

Challenges in dimension two

We currently have nothing close to the abelian surface equivalent of even the 1972 Antwerp tables of elliptic curves. We know only the first 36 modular abelian surface L-functions unconditionally, of which 5 are typical (the 1972 Antwerp tables had 749).

- Enumerating paramodular forms of a given level is very difficult; even counting them is hard, due to the absence of dimension formulas. We have provably complete lists of paramodular forms only up to level 353 (all five of them).
- Computing the L-function of a given paramodular form is very difficult; it is usually only feasible to compute a handful of Euler factors.
- There is no analog of the Eichler-Shimura construction for paramodular forms.
- Not all abelian surfaces over $\mathbb{Q}$ are Jacobians of genus 2 curves over $\mathbb{Q}$. One can generically represent an abelian surface as a projective variety in $\mathbb{P}^{15}$ defined by 72 quadratic forms, but this is not a very pleasant thing to do.
- There is no algorithm known to enumerate all genus 2 curves over $\mathbb{Q}$ of a given conductor. Even computing the conductor of a single curve can be very hard.

Automorphic forms associated to abelian surfaces

Type	Conductor	Curve Equation	Motive	Modular form
$\mathbf{A}[C_1]_{(s)}$	$277 = 277^1$	$y^2+(x^3+x^2+x+1)y = -x^2-x$	typical surface	paramodular form
$\mathbf{B}[C_1]_s$	$529 = 23^2$	$y^2+(x^3+x+1)y = -x^5$	surface with RM by $\mathbb{Q}(\sqrt{5})$ over $\mathbb{Q}$	CMF 23.2.1.a
$\mathbf{B}[C_1]_{ns}$	$294 = 2^1 3^1 7^2$	$y^2+(x^3+1) = x^4+x^2$	product of ECs 14a4 and 21a4 over $\mathbb{Q}$	CMFs 14.2.1.a and 21.2.1.a
$\mathbf{B}[C_2]_s$	$10368 = 2^7 3^4$	$y^2+x^2y = 3x^5-4x^4+6x^3-3x^2+1$	surface with RM by $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}(\sqrt{2})$	HMF 162.1-a over $\mathbb{Q}(\sqrt{2})$
$\mathbf{B}[C_2]_{ngs}$	$1088 = 2^6 17^1$	$y^2+(x^3+x^2+x+1)y = x^4+x^3+2x^2+x+1$	Weil restriction of 17.1-a1 over $\mathbb{Q}(\sqrt{2})$	HMF 17.1-a over $\mathbb{Q}(\sqrt{2})$
$\mathbf{C}[C_2]_{(ns)}$	$448 = 2^6 7^1$	$y^2+(x^3+x)y = x^4-7$	product of PCM EC 32a3 and EC 14a6 over $\mathbb{Q}$	CMFs 32.2.1.a and 14.2.1.a
$\mathbf{D}[C_4]_{(s)}$	$3125 = 5^5$	$y^2+y = x^5$	surface with CM by $\mathbb{Q}(\zeta_5)$ over $\mathbb{Q}(\zeta_5)$	CM HMF 125.1-a over $\mathbb{Q}(\sqrt{5})$
$\mathbf{D}[D_2]_{(ns)}$	$8192 = 2^{13}$	$y^2 = x^6-9x^4+16x^2-8$	product of PCM ECs 32a3 and 256d1 over $\mathbb{Q}$	CMFs 32.2.1.a and 256.2.1.d
$\mathbf{E}[C_1]_{(ns)}$	$196 = 2^2 7^2$	$y^2+(x^2+x)y = x^6+3x^5+6x^4+7x^3+6x^2+3x+1$	square of EC 14a1 over $\mathbb{Q}$	CMF 14.2.1.a
$\mathbf{E}[C_2, \mathbb{C}]_{(ngs)}$	$576 = 2^6 3^2$	$y^2+(x^3+x^2+x+1)y = -x^3-x$	square of EC 9.1-a3 over $\mathbb{Q}(\sqrt{2})$	CMF 24.2.13.a
$\mathbf{E}[C_3]_{(ngs)}$	$324 = 2^2 3^4$	$y^2+(x^3+x+1)y = x^5+2x^4+2x^3+x^2$	square of EC 8.1-a1 over 3.3.81.1	CMF 18.2.13.a
$\mathbf{E}[C_4]_{(ngs)}$	$256 = 2^8$	$y^2+y = 2x^5-3x^4+x^3+x^2-x$	square of EC 1.1-a5 over 4.4.2048.1	CMF 16.2.5.a
$\mathbf{E}[C_6]_{(ngs)}$	$169 = 13^2$	$y^2+(x^3+x+1)y = x^5+x^4$	square of EC 1.1-a3 over 6.6.371293.1	CMF 13.2.4.a
$\mathbf{E}[C_2, \mathbb{R}\times\mathbb{R}]_s$	$455625 = 3^6 5^4$	$y^2+(x^3+x^2+x+1)y = x^5-3x^4-2x-1$	surface with QM $(D=6)$ over 2.0.3.1	BMF over 2.0.3.1 of level 50625
$\mathbf{E}[C_2, \mathbb{R}\times\mathbb{R}]_{ngs}$	$3969 = 3^4 7^2$	$y^2+(x^2+x+1)y = -3x^5+5x^4-4x^3+x$	Weil restriction of 441.2-a over 2.0.3.1	BMF 2.0.3.1-441.2-a
$\mathbf{E}[C_2, \mathbb{R}\times\mathbb{R}]_{ns}$	$675 = 3^3 5^2$	$y^2 = -x^6-14x^5-44x^4+28x^3-44x^2-14x-1$	product of ECs 15a2 and 45a2 over $\mathbb{Q}$	CMFs 15.2.1.a and 45.2.1.a
$\mathbf{E}[D_2]_s$	$20736 = 2^8 3^4$	$y^2 = -27x^6-54x^5-27x^4+18x^3+18x^2-2$	surface with QM $(D=6)$ over 4.0.576.2	HMF 324.1-b over $\mathbb{Q}(\sqrt{2})$
$\mathbf{E}[D_3]_s$	$34992 = 2^4 3^7$	$y^2 = -2x^6-6x^5+10x^3+9x^2-18x+6$	surface with QM $(D=6)$ over 6.0.2834352.2	BMF over 2.0.3.1 of level 3888
$\mathbf{E}[D_4]_s$	$20736 = 2^8 3^4$	$y^2+y = 6x^5+9x^4-x^3-3x^2$	surface with QM $(D=6)$ over 8.0.339738624.10	BMF over 2.0.3.1 of level 2304
$\mathbf{E}[D_6]_s$	$8100 = 2^2 3^4 5^2$	$y^2+x^3y = x^6+3x^5-42x^4+43x^3+21x^2-60x-28$	surface with QM $(D=6)$ over degree 12 field	BMF over 2.0.3.1 of level 900
$\mathbf{E}[D_2]_{ngs}$	$6400 = 2^8 5^2$	$y^2 = 2x^5+5x^4+8x^3+7x^2+6x+2$	square of EC 256.1-a1 over $\mathbb{Q}(\sqrt{5})$	HMF 2.2.5.1-256.1-a
$\mathbf{E}[D_3]_{ngs}$	$2187 = 3^7$	$y^2+(x^3+1)y = -1$	square of EC over 6.0.177147.2	BMF over 2..0.3.1 of level 243
$\mathbf{E}[D_4]_{ngs}$	$3600 = 2^4 3^2 5^2$	$y^2+x^2y = x^5-3x^4+11x^2-16x$	square of EC over 4.0.13500.2	BMF over $\mathbb{Q}(i)$ of level 225
$\mathbf{E}[D_6]_{ngs}$	$3600 = 2^4 3^2 5^2$	$y^2+x^3y = 14x^3-20$	square of EC over 6.0.7200000.1	BMF over 2.0.3.1 of level 400
$\mathbf{F}[D_2, C_2, \mathcal{H}]_{ngs}$	$576 = 2^6 3^2$	$y^2+x^3y = 5x^3-2$	square of PCM EC 1.1-a2 over $\mathbb{Q}(\sqrt{6})$	CM HMF 1.1-a over $\mathbb{Q}(\sqrt{6})$
$\mathbf{F}[C_2, C_1, \mathrm{M}_2(\mathbb{R})]_{ns}$	$729 = 3^6$	$y^2+y = -48x^6+15x^3-1$	square of PCM EC 27.a4 over $\mathbb{Q}$	CM CMF 27.2.1.a

One page of the "giant table" [Booker-Sijsling-S-Voight-Yasaki 2024?]

An axiomatic approach to arithmetic L-functions [FPRS]

An arithmetic L-function of motivic weight $w \in \mathbb{Z}_{\geq 0}$ with field of coefficients K is a Dirichlet series $L(s) = \sum_{n \geq 1} a_n n^{-s}$ with $a_1 = 1$, $a_n \in \mathcal{O}_K$, $\mathbb{Q}(a_n) = K$ such that:

- Analytic continuation: $L_{\mathrm{an}}(s) := L(s + w/2)$ converges absolutely on $\mathrm{Re}(s) > 1$ and has a meromorphic continuation with finitely many poles, all on $\mathrm{Re}(s) = 1$.
- Functional equation: For some $N \in \mathbb{Z}_{<0}$, $\varepsilon \in \mathbb{C}$ and $\mu_i, \nu_j \in \mathbb{Z}$ or $\mu_i, \nu_j \in \frac{1}{2} + \mathbb{Z}$,

  $$\Lambda_{\mathrm{an}}(s) := \Gamma_{\mathbb{R}}(s + \mu_1) \cdots \Gamma_{\mathbb{R}}(s + \mu_{d_1}) \Gamma_{\mathbb{C}}(s + \nu_1) \cdots \Gamma_{\mathbb{C}}(s + \nu_{d_2}) L_{\mathrm{an}}(s)$$

  is bounded in vertical strips away from $\mathrm{Re}(s) = 1$ with $\Lambda(s) = \varepsilon N^{1-s} \bar{\Lambda}(1 - s)$. Here ε is the root number, N is the conductor, and $d = d_1 + 2d_2$ is the degree.
- Euler product: $L_{\mathrm{an}}(s) = \prod_p F_p(p^{-s})^{-1}$ where $F_p(z) = (1 - \alpha_{1,p} z) \cdots (1 - \alpha_{d_p,p} z)$ with $d_p \leq d$ ($d_p = p$ if $p \nmid N$) and $|\alpha_{j,p}| = p^{-m_j/2}$ with $m_j \in \mathbb{Z}_{\geq 0}$, $\sum m_j \leq d - d_p$.
- Central character: There is a Dirichlet character χ of modulus N for which $F_p(z) = 1 - a_p Z + \cdots + (-1)^d \chi(p) z^d$ and $\chi(-1) = (-1)^{\sum \mu_j + \sum (2\nu_k + 1)}$.

An axiomatic approach to L-functions of abelian varieties over $\mathbb{Q}$

Fix a positive integer g. We shall consider arithmetic L-functions of degree $2g$, motivic weight 1, field of coefficients $\mathbb{Q}$, defined by an Euler product

$$L(s) := \sum_n a_n n^{-s} = \prod_p L_p(p^{-s})^{-1},$$

with $L_p \in \mathbb{Z}[T]$. We further assume that

- $\Lambda(s) := \Gamma_{\mathbb{C}}(s)^g L(s)$ is holomorphic on $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s) = \varepsilon N^{1-s} \Lambda(2-s)$$

  with root number $\varepsilon = \pm 1$ and conductor N.

- the a_n are integers that satisfy $|a_n| \leq d_{2g}(n)\sqrt{n}$, where $d_r(n) = \sum_{n_1 \cdots n_r = n} 1$.

Under the Hasse–Weil conjecture, every $A/\mathbb{Q}$ of dimension g has such an L-function.

Conductor bounds for abelian varieties over $\mathbb{Q}$

The formula of [Brumer–Kramer 94] gives explicit bounds on the p-adic valuation of the conductor N of an abelian variety $A/\mathbb{Q}$ of dimension g:

$$v_p(N) \leq 2g + pd + (p-1)\lambda_p(d),$$

where $d = \lfloor \frac{2g}{p-1} \rfloor$ and $\lambda_p(d) = \sum i d_i p^i$, with $d = \sum d_i p^i$ with $0 \leq d_i < p$.

g	$p=2$	$p=3$	$p=5$	$p=7$	$p>7$
1	8	5	2	2	2
2	20	10	9	4	4
3	28	21	11	13	6

For $g \leq 2$ these bounds are tight (see `www.lmfdb.org` for examples).

An integral converse theorem for GL_2

Theorem (Dimitrov 2023)

Let K be a number field, $k, q \in \mathbb{Z}_{>0}$, $L(s) = \sum_{n \geq 1} a_n n^{-s}$ be an L-function with $a_1 = 1$, $qa_n \in \mathbb{Z}$ for $n \geq 1$, $a_n = O(n^{k-1})$, and $\widetilde{L}(s)$ any L-function. Suppose $L(s)$ and $\widetilde{L}(s)$ admit a holomorphic continuation to $\mathbb{C}$ that is bounded on vertical strips such that

$$\Lambda(s) = i^k N^{k/2-s} \widetilde{\Lambda}(k-s)$$

for some $N \in \mathbb{Z}_{>0}$, with $\Lambda(s) := \Gamma_{\mathbb{C}}(s) L(s)$ and $\widetilde{\Lambda}(s) := \Gamma_{\mathbb{C}}(s) \widetilde{L}(s)$.
Then $L(s) = L(f, s)$ and $\widetilde{L}(s) = L(f|_{W_N}, s)$ for some $f \in S_k(\Gamma_0(N))$.

Corollary

Every rational L-function of degree 2, conductor N, and motivic weight w with $L_\infty(s) = \Gamma_{\mathbb{C}}(s)$ is the L-function of a newform in $S_k^{\mathrm{new}}(\Gamma_0(N))$ with $k = w + 1$.
If $w = 1$, it is also the L-function of an elliptic curve of conductor N.

Builds on Calegari-Dimitrov-Tang proof of the unbounded denominators conjecture.

A finite problem

Let $\mathcal{S}(g, N, \varepsilon)$ denote the set of L-functions $L(s)$ that satisfy our axioms for a particular choice of $g, N \in \mathbb{Z}_{>0}$ and $\varepsilon = \pm 1$.

We expect every $L \in \mathcal{S}(g, N, \varepsilon)$ to be the L-function of a g-dimensional $A/\mathbb{Q}$ (this is far beyond anything we can currently hope to prove, but we don't need to).

Shafarevich's conjecture (proved by Faltings), then implies that $\mathcal{S}(g, N, \varepsilon)$ is finite. Moreover there is an effectively computable $n_0 = O(\sqrt{N})$ for which the coefficients $a_1, \ldots, a_{n_0}$ uniquely determine each $L \in \mathcal{S}(g, N, \varepsilon)$ (and $n_0 = O(\log^2 N)$ under GRH).

We seek an algorithm that takes inputs g, N, ε, determines a suitable n_0, and then outputs a list of distinct tuples $(a_1, \ldots, a_{n_0})$, one for each $L \in \mathcal{S}(g, N, \varepsilon)$.
See Booker and Farmer–Koutsoliotas–Lemurell for prior work in this direction.

Our plan: Compute $\mathcal{S}(g, N, \varepsilon)$ via linear algebra, then search for corresponding $A/\mathbb{Q}$.

Our plan depends crucially on being able to compute $\mathcal{S}(g, N, \varepsilon)$ exactly.
This not only tells us when to stop searching, knowing $a_1, \ldots, a_{n_0}$ helps us search.

The approximate functional equation

Fix g, N, ε. For each nonnegative integer k we define $S_k(x) := \sum_n f_k(n/x) a_n/n$, where

$$f_k(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (s-1)^k \Gamma_{\mathbb{C}}(s)^g x^{1-s}\, ds.$$

The functional equation then implies the identity

$$S_k(x) = \varepsilon(-1)^k S_k(N/x),$$

valid for all $x > 0$; this is the approximate functional equation.If we choose k so that $(-1)^k = -\varepsilon$ and put $x = \sqrt{N}$ we obtain a nontrivial linear constraint on the a_n:

$$\sum_n \frac{a_n}{n} f_k(n/\sqrt{N}) = 0. \tag{1}$$

The $O(\sqrt{n})$ bounds on a_n and rapid decay of $f_k(x)$ allows us to compute an interval $I_{k,m}$ containing the truncated sum in (1) for $n \le m$ that does not depend on the a_n.

A system of linear constraints

For each $k \geq 0$ of the correct parity (meaning $(-1)^k = -\varepsilon$), we have linear constraints

$$\sum_{n \leq m} f_k(n/\sqrt{N}) a_n/n \in I_{k,m}.$$

These become less useful as k grows, so we restrict to $k = O(N^{1/4})$.
We also have the constraints $|a_n| \leq d_{2g}(n)\sqrt{n}$ for $n \geq 1$.

If we further assume that the $L \in \mathcal{S}(g, N, \varepsilon)$ are automorphic (which we do), we can obtain additional constraints by twisting $L(s)$ by a Dirichlet character $\chi \colon \mathbb{Z} \to \mathbb{C}$, equivalently, taking the Rankin-Selberg convolution of $L(s)$ with $L(\chi, s)$.

This generally increases the conductor and widens the corresponding interval $I_{\chi,k,m}$, but for χ of small conductor q and small k we obtain useful constraints

$$\sum_{n \leq m} f_k(n/\sqrt{q^4 N}) \chi(n) a_n/n \in I_{\chi,k,m}.$$

Solving the system rigorously using the simplex method

The Euler product for $L(s)$ implies that the a_n are determined by the a_q for prime powers $q = p^e$ with $e \leq 2g$. In order to take advantage of this, and to obtain rigorous results using off-the-shelf simplex solvers with fixed precision, we proceed as follows.

Let $q \leq n_0 < m$ be a prime power. Assume we have recursively fixed values for $a_1, \ldots, a_{q-1}$ that we cannot rule out this sequence as a prefix of a feasible solution.

We now apply the simplex method to a system of linear constraints on variables $a_{q'}$, with q' ranging over prime powers $q \leq q' \leq m$, using the objective functions $\pm a_q$.

The dual solution yields a linear combination of constraints we can compute using interval arithmetic. Plugging in bounds on $a_{q'}$ yields an interval I_q containing a_q.

If $I_q \cap \mathbb{Z}$ is empty, then $a_1, \ldots, a_{q-1}$ is not the prefix of any $L \in \mathcal{S}(g, N, \varepsilon)$. Otherwise, for each $a \in I_q$ we add the tuple $(a_1, \ldots, a_{q-1}, a)$ to our list of feasible tuples.

We continue in this fashion until we run out of feasible prefixes or reach $q = n_0$.

A toy example

A short proof that the set $\mathcal{S}(1, 13, 1)$ is empty, which implies that there are no elliptic curves $E/\mathbb{Q}$ of conductor 13 (this only requires Hasse-Weil, not modularity).

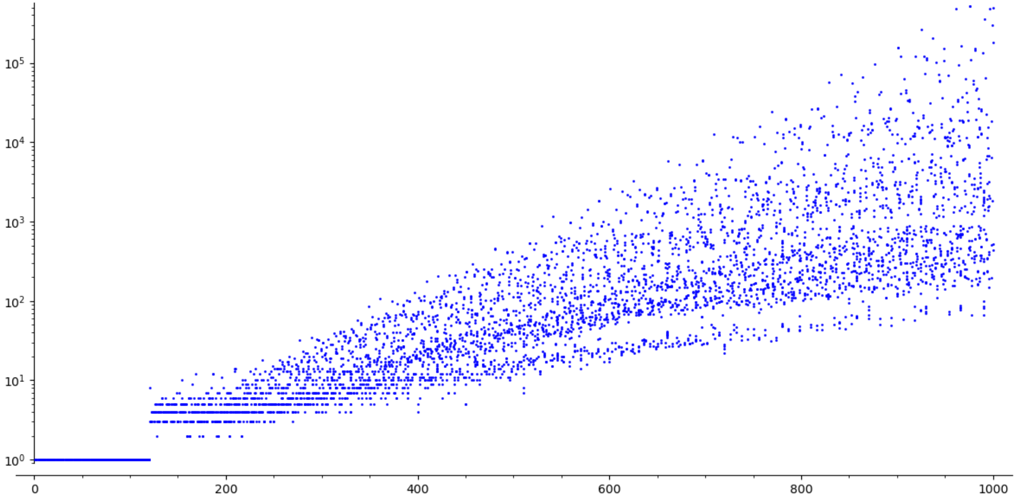
```
> LFN(1,13,-1);
[ 1 ], considering a_2 in { -2, -1, 0, 1, 2 }
[ 1 ], possible a_2: {}

There are no degree 2 motivic weight 1 rational L-functions for N=13 and eps=-1

> LFN(1,13,1);
[ 1 ], considering a_2 in { -2, -1, 0, 1, 2 }
[ 1 ], possible a_2: { -2, -1 }
    [ 1, -2 ], considering a_3 in { -3, -2, -1, 0, 1, 2, 3 }
    [ 1, -2 ], possible a_3: {}
    [ 1, -1 ], considering a_3 in { -3, -2, -1, 0, 1, 2, 3 }
    [ 1, -1 ], possible a_3: { -2 }
        [ 1, -1, -2, -1 ], considering a_5 in { -4, -3, -2, -1, 0, 1, 2, 3, 4 }
        [ 1, -1, -2, -1 ], possible a_5: {}

There are no degree 2 motivic weight 1 rational L-functions for N=13 and eps=1.
```

Timings



Proving completeness

If our algorithm outputs a nonempty list of feasible tuples $(a_1, \ldots, a_{n_0})$,
the next step is to show there is at most one L-function in $\mathcal{S}(g, N, \varepsilon)$ for each prefix.

For this step, if we suppose that $(a_1, \ldots, a_{n_0})$ is the prefix of two distinct L-functions $L(s, \pi_1)$ and $L(s, \pi_2)$ of isobaric cuspidal automorphic representations of $\mathrm{GL}_{2g}(\mathbb{A}_\mathbb{Q})$ whose L-functions lie in $\mathcal{S}(g, N, \varepsilon)$. Using the Rankin–Selberg convolution L-function $L(s, \pi_1 \boxtimes \pi_2)$ we construct an inequality which will be violated if n_0 is sufficiently large.

If it is not violated, we increase n_0, extend our tuples, and try again.

Eventually we obtain a list of distinct tuples $(a_1, \ldots, a_{n_0})$, each of which is provably the prefix of at most one automorphic L-function in $\mathcal{S}(g, N, \varepsilon)$.

This gives us an upper bound for our search that we expect to be tight.
Finding an abelian variety for each prefix proves completeness subject to modularity.

We can then attempt to use Faltings-Serre or other methods to prove modularity for each abelian variety. Either our list is complete or we find an explicit nonmodular $A/\mathbb{Q}$.

Proving completeness

Let A and B be automorphic abelian varieties of dimensions g_A, g_B, conductors N_A, N_B, with corresponding automorphic representations π_A, π_B. We define

$$L(A \boxtimes B, s) = L(s-1, \pi_A \boxtimes \pi_B),$$

where $L(s, \pi_A \boxtimes \pi_B)$ is the Rankin–Selberg L-function associated to the pair (π_A, π_B).

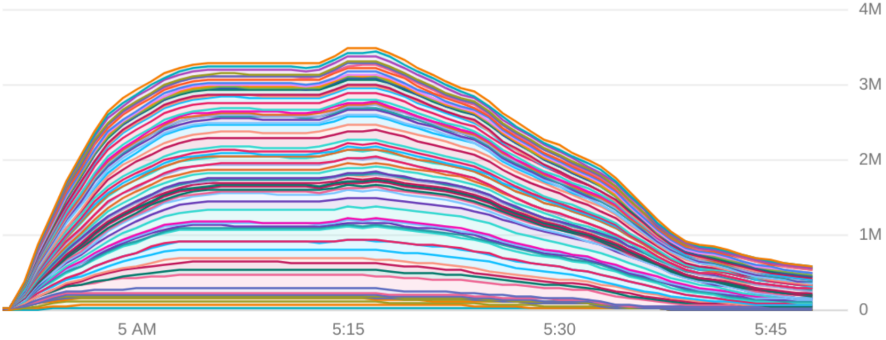
Lemma

Assume that $L(A \boxtimes B, s)$ is entire and has no zeros in the region $\{s \in \mathbb{C} : \mathrm{Re}(s) > \sigma\}$ for some $\sigma \geq \frac{3}{2}$. Let $g : [0, \infty) \to \mathbb{R}$ be continuous of compact support, with a non-negative cosine transform. Then

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) c_n^{\natural}(A \boxtimes B) g(\log n)}{n^{\sigma}} \leq \frac{1}{2} g(0) \sum_p \left(s_p(A \boxtimes B) + 4 g_A g_B - d_p^{\natural}(A \boxtimes B)\right) \log p$$
$$+ g_A g_B \left((\psi(\sigma - 1) + \psi(\sigma) - 2\log(2\pi)) g(0) + \int_0^{\infty} \frac{e^{(1-\sigma)x}(g(0) - g(x))}{\tanh(x/2)}\, dx \right).$$

Searching for genus 2 curves

Over the past several years we have conducted several searches for genus 2 curves of small conductor (we expect to run one more this year). Below is CPU histogram from a computation we ran in 2022 that enumerated more than 10^{19} genus 2 curves using a large parallel computation running on Google cloud platform.



We used a total of 4,034,560 Intel/AMD vCPUs in 73 data centers across the globe.

Searching for genus 2 curves

We found millions of genus 2 curves of small conductor, including the curve

$$C_{903} : y^2 + (x^2 + 1)y = x^5 + 3x^4 - 13x^3 - 25x^2 + 61x - 28$$

of conductor 903 and whose L-function coefficients match those of the paramodular form of level 903 computed by Poor–Yuen that had not previously been matched.

We also found curves of conductor 657, 760, 775, 924 not previously known to occur, and many new genus 2 L-functions of small conductor:

conductor bound	1000	10000	100000	1000000
curves in LMFDB	159	3069	20265	66158
curves found	807	25438	447507	5151208
L-functions in LMFDB	109	2807	19775	65534
L-functions found	200	9409	212890	2426708

A provisional result

Provisional Theorem (proof in progress)

Assume the paramodular conjecture.
There are 456 L-functions of abelian surfaces $A/\mathbb{Q}$ with conductor $N \leq 1000$, of which

- *360 arise from products of elliptic curves over $\mathbb{Q}$;*
- *17 arise from weight-2 newforms with quadratic coefficients;*
- *2 arise from the Weil restriction of an elliptic curve over a quadratic field;*
- *77 arise from generic abelian surfaces, of which at least 67 are Jacobians.*

It may be feasible to remove the paramodular hypothesis, but that will depend largely on work by others and it almost certainly won't be feasible much past $N \leq 1000$.

In addition to proving this theorem, we hope to extend it well past $N \leq 1000$.
But this requires algorithmic improvements.

Exploiting Galois representations

Let $A/\mathbb{Q}$ be an abelian surface of conductor N. For each $m \in \mathbb{Z}_{>1}$ we have a mod-m Galois representation

$$\rho_{A,m}\colon \operatorname{Gal}(\mathbb{Q}(A[m])/\mathbb{Q}) \to \operatorname{GSp}_4(\mathbb{Z}/m\mathbb{Z}).$$

For $p \nmid mN$ the charpoly $\chi_p \in (\mathbb{Z}/m\mathbb{Z})[T]$ of $\rho_{A,m}(\mathrm{Frob}_p) \in \operatorname{GSp}_4(\mathbb{Z}/m\mathbb{Z})$ satisfies

$$\chi_p(T) \equiv T^{2g} L_p(T^{-1}) \bmod \ell.$$

The m-torsion field $\mathbb{Q}(A[m])$ is unramified away from $p|mN$ and of degree at most $\#\operatorname{GSp}_4(\mathbb{Z}/m\mathbb{Z})$. For small m and N it is feasible to enumerate all such fields K and their associated mod-m GSp_4-representations, especially $m = 2$ and N a prime power.

Each representation yields a mod-m congruence constraints on $L_p(T)$ for primes $p \nmid mN$. This dramatically reduces the amount of branching in our algorithm.

Other opportunities for improvement

- Modify our test functions f_k to obtain linear systems that are better conditioned by using a basis that is orthogonal with respect to dx/x.
- Exploit integrality by finding small linear combinations of constraints whose sum has coefficients close to an integer (possibly using LLL) and use this to construct a better objective function. This should help problems caused by twin primes.
- Write a custom simplex solver that is better suited to the shape of our systems and that would allow us to better exploit integrality.
- Use ML methods or other heuristic algorithms to guess linear combinations of constraints that we can then exploit rigorously.
- Consider non-linear approaches (quadratic programs).