The generalized Sato-Tate conjecture

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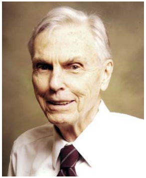

John Tate

Sato-Tate in genus 1

Let $E/\mathbb{Q}$ be an elliptic curve, which we can write in the form

$$y^2 = x^3 + ax + b$$

corresponding to the projective curve $zy^2 = x^3 + axz^2 + bz^3$.

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Let p be a prime of good reduction ($p \nmid \operatorname{disc}(E) = -16(4a^3 + 27b^2)$).
The number of $\mathbb{F}_p$-points on the reduction $\overline{E}$ of E modulo p is

$$\#\overline{E}(\mathbb{F}_p) = p + 1 - t_p.$$

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The number of $\mathbb{F}_p$-points on the reduction $\overline{E}$ of E modulo p is

$$\#\overline{E}(\mathbb{F}_p) = p + 1 - t_p.$$

The trace of Frobenius t_p is an integer in the interval $[-2\sqrt{p}, 2\sqrt{p}]$.

We are interested in the limiting distribution of $x_p = -t_p/\sqrt{p} \in [-2, 2]$, as p varies over primes of good reduction.

Example: $y^2 = x^3 + x + 1$

p	t_p	x_p	p	t_p	x_p	p	t_p	x_p
3	0	**0.000000**	71	13	**−1.542816**	157	−13	**1.037513**
5	−3	**1.341641**	73	2	**−0.234082**	163	−25	**1.958151**
7	3	**−1.133893**	79	−6	**0.675053**	167	24	**−1.857176**
11	−2	**0.603023**	83	−6	**0.658586**	173	2	**−0.152057**
13	−4	**1.109400**	89	−10	**1.059998**	179	0	**0.000000**
17	0	**0.000000**	97	1	**−0.101535**	181	−8	**0.594635**
19	−1	**0.229416**	101	−3	**0.298511**	191	−25	**1.808937**
23	−4	**0.834058**	103	17	**−1.675060**	193	−7	**0.503871**
29	−6	**1.114172**	107	3	**−0.290021**	197	−24	**1.709929**
37	−10	**1.643990**	109	−13	**1.245174**	199	−18	**1.275986**
41	7	**−1.093216**	113	−11	**1.034793**	211	−11	**0.757271**
43	10	**−1.524986**	127	2	**−0.177471**	223	−20	**1.339299**
47	−12	**1.750380**	131	4	**−0.349482**	227	0	**0.000000**
53	−4	**0.549442**	137	12	**−1.025229**	229	−2	**0.132164**
59	−3	**0.390567**	139	14	**−1.187465**	233	−3	**0.196537**
61	12	**−1.536443**	149	14	**−1.146925**	239	−22	**1.423062**
67	12	**−1.466033**	151	−2	**0.162758**	241	22	**−1.417145**

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Complex multiplication

An *endomorphism* of E is a rational map from E to itself.

The set of endomorphisms forms a ring $\mathrm{End}(E)$ in which addition corresponds to point addition and multiplication is composition.

$\mathrm{End}(E)$ always contains a subring isomorphic to $\mathbb{Z}$, arising from the "multiplication-by-n" maps defined by $nP = P + P + \cdots + P$.

In some cases $\mathrm{End}(E)$ contains additional endomorphisms. These can be viewed as "multiplication-by-α" maps for some complex number α in an imaginary quadratic number field.

When this happens E is said to have *complex multiplication* (CM).

Sato-Tate distributions in genus 1

1. Typical case (no CM)

Elliptic curves $E/\mathbb{Q}$ w/o CM have the semi-circular trace distribution. (This is also known for E/k, where k is a totally real number field).

[Clozel, Harris, Shepherd-Barron, Taylor, Barnet-Lamb, and Geraghty]

2. Exceptional cases (CM)

Elliptic curves E/k with CM have one of two distinct trace distributions, depending on whether k contains the CM field or not.

[classical]

Sato-Tate groups in genus 1

The *Sato-Tate group* G of E is a closed subgroup of $\mathrm{SU}(2) = \mathrm{USp}(2)$.
(A definition will be given in the second half of the talk).

The generalized Sato-Tate conjecture implies that the normalized trace distribution of E converges to the distribution of the traces in G given by the *Haar measure* (the unique translation-invariant measure).

Each such distribution is uniquely determined by its *moment sequence*.

G	G/G^0	example curve	k	moments of $\mathrm{tr}(A)^2$
$\mathrm{U}(1)$	C_1	$y^2 = x^3 + 1$	$\mathbb{Q}(\sqrt{-3})$	$1, 2, 6, 20, 70, 252, \ldots$
$N(\mathrm{U}(1))$	C_2	$y^2 = x^3 + 1$	$\mathbb{Q}$	$1, 1, 3, 10, 35, 126, \ldots$
$\mathrm{SU}(2)$	C_1	$y^2 = x^3 + x + 1$	$\mathbb{Q}$	$1, 1, 2, 5, 14, 42, \ldots$

In genus 1 there are three possible Sato-Tate groups.
Only two of these arise for elliptic curves defined over $\mathbb{Q}$.

Moment sequences in genus 1

Let $e^{i\theta}$ and $e^{-i\theta}$ denote the eigenvalues of a random matrix A in $\mathrm{SU}(2)$.
The Haar measure (on conjugacy classes) is $\mu = \frac{2}{\pi}\sin^2\theta\, d\theta$.
The nth moment of $\mathrm{tr}(A) = 2\cos\theta$ can then be computed as

$$M_n[\mathrm{tr}(A)] = \frac{2}{\pi}\int_0^\pi (2\cos\theta)^n \sin^2\theta\, d\theta = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{n+1}\binom{n}{n/2} & \text{if } n \text{ is even} \end{cases}$$

which yields the moment sequence $1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, \ldots$.

For the group $\mathrm{U}(1)$ embedded in $\mathrm{SU}(2)$ we have $\mu = \frac{1}{\pi}\, d\theta$ and

$$M_n[tr(A)] = \frac{1}{\pi}\int_0^\pi (2\cos\theta)^n\, d\theta = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{n}{n/2} & \text{if } n \text{ is even} \end{cases}$$

yielding the moment sequence $1, 0, 2, 0, 6, 0, 20, 0, 70, 0, 252, \ldots$.

Zeta functions and L-polynomials

For a smooth projective curve $C/\mathbb{Q}$ of genus g and a prime p define

$$Z(\overline{C}/\mathbb{F}_p;T) = \exp\left(\sum_{k=1}^{\infty} N_k T^k/k\right),$$

where $N_k = \#\overline{C}(\mathbb{F}_{p^k})$. This is a rational function of the form

$$Z(\overline{C}/\mathbb{F}_p;T) = \frac{L_p(T)}{(1-T)(1-pT)},$$

where $L_p(T)$ is an integer polynomial of degree $2g$.

For $g = 1$ we have $L_p(t) = pT^2 + c_1T + 1$, and for $g = 2$,

$$L_p(T) = p^2T^4 + c_1pT^3 + c_2T^2 + c_1T + 1.$$

Unitarized L-polynomials

The normalized polynomial

$$\bar{L}_p(T) = L_p(T/\sqrt{p}) = \sum_{i=0}^{2g} a_i T^i$$

has real coefficients satisfying $a_i = a_{2g-i}$ and $|a_i| \le \binom{2g}{i}$, and its roots lie on the unit circle.

We now consider the limiting distribution of $a_1, a_2, \ldots, a_g$ over all primes $p \le N$ of good reduction, as $N \to \infty$.

We will focus on curves of genus $g = 2$.

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The generalized Sato-Tate conjecture

Let C be a smooth projective curve of genus g.
The normalized L-polynomials $\bar{L}_p(T)$ of C are monic, of degree $2g$, and have roots on the unit circle occurring in conjugate pairs.

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In the exceptional cases, we can look for a compact $G \subseteq \mathrm{USp}(2g)$ whose distribution of charpolys matches the distribution of $\bar{L}_p(T)$.

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Conjecture [preliminary version]

Let $G \subseteq \mathrm{USp}(2g)$ be the Sato-Tate group of C.
The distribution of $\bar{L}_p(T)$ is governed by the Haar measure of G.

Sato-Tate groups in genus 2

Theorem 1 [FKRS 2012]

Up to conjugacy, there are exactly 52 subgroups of $\mathrm{USp}(4)$ that arise as the Sato-Tate group of a genus 2 curve, of which 34 arise for curves defined over $\mathbb{Q}$.

We give explicit constructions for each group G, along with explicit examples of genus 2 curves that have G as its Sato-Tate group.

We have compared the distribution of L-polynomial coefficients for each of examples with the distribution predicted by the generalized Sato-Tate conjecture and find very close agreement in every case.

`https://hensel.mit.edu:8000/home/pub/6`

Sato-Tate groups in genus 2 with $G^0 = \mathrm{U}(1)$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
1	1	C_1	C_1	0	0, 0, 0, 0, 0	8, 96, 1280, 17920	4, 18, 88, 454
1	2	C_2	C_2	1	0, 0, 0, 0, 0	4, 48, 640, 8960	2, 10, 44, 230
1	3	C_3	C_3	0	0, 0, 0, 0, 0	4, 36, 440, 6020	2, 8, 34, 164
1	4	C_4	C_4	1	0, 0, 0, 0, 0	4, 36, 400, 5040	2, 8, 32, 150
1	6	C_6	C_6	1	0, 0, 0, 0, 0	4, 36, 400, 4900	2, 8, 32, 148
1	4	D_2	D_2	3	0, 0, 0, 0, 0	2, 24, 320, 4480	1, 6, 22, 118
1	6	D_3	D_3	3	0, 0, 0, 0, 0	2, 18, 220, 3010	1, 5, 17, 85
1	8	D_4	D_4	5	0, 0, 0, 0, 0	2, 18, 200, 2520	1, 5, 16, 78
1	12	D_6	D_6	7	0, 0, 0, 0, 0	2, 18, 200, 2450	1, 5, 16, 77
1	2	$J(C_1)$	C_2	1	1, 0, 0, 0, 0	4, 48, 640, 8960	1, 11, 40, 235
1	4	$J(C_2)$	D_2	3	1, 0, 0, 0, 1	2, 24, 320, 4480	1, 7, 22, 123
1	6	$J(C_3)$	C_6	3	1, 0, 0, 2, 0	2, 18, 220, 3010	1, 5, 16, 85
1	8	$J(C_4)$	$\mathrm{C}_4 \times \mathrm{C}_2$	5	1, 0, 2, 0, 1	2, 18, 200, 2520	1, 5, 16, 79
1	12	$J(C_6)$	$\mathrm{C}_6 \times \mathrm{C}_2$	7	1, 2, 0, 2, 1	2, 18, 200, 2450	1, 5, 16, 77
1	8	$J(D_2)$	$\mathrm{D}_2 \times \mathrm{C}_2$	7	1, 0, 0, 0, 3	1, 12, 160, 2240	1, 5, 13, 67
1	12	$J(D_3)$	D_6	9	1, 0, 0, 2, 3	1, 9, 110, 1505	1, 4, 10, 48
1	16	$J(D_4)$	$\mathrm{D}_4 \times \mathrm{C}_2$	13	1, 0, 2, 0, 5	1, 9, 100, 1260	1, 4, 10, 45
1	24	$J(D_6)$	$\mathrm{D}_6 \times \mathrm{C}_2$	19	1, 2, 0, 2, 7	1, 9, 100, 1225	1, 4, 10, 44
1	2	$C_{2,1}$	C_2	1	0, 0, 0, 0, 1	4, 48, 640, 8960	3, 11, 48, 235
1	4	$C_{4,1}$	C_4	3	0, 0, 2, 0, 0	2, 24, 320, 4480	1, 5, 22, 115
1	6	$C_{6,1}$	C_6	3	0, 2, 0, 0, 1	2, 18, 220, 3010	1, 5, 18, 85
1	4	$D_{2,1}$	D_2	3	0, 0, 0, 0, 2	2, 24, 320, 4480	2, 7, 26, 123
1	8	$D_{4,1}$	D_4	7	0, 0, 2, 0, 2	1, 12, 160, 2240	1, 4, 13, 63
1	12	$D_{6,1}$	D_6	9	0, 2, 0, 0, 4	1, 9, 110, 1505	1, 4, 11, 48
1	6	$D_{3,2}$	D_3	3	0, 0, 0, 0, 3	2, 18, 220, 3010	2, 6, 21, 90
1	8	$D_{4,2}$	D_4	5	0, 0, 0, 0, 4	2, 18, 200, 2520	2, 6, 20, 83
1	12	$D_{6,2}$	D_6	7	0, 0, 0, 0, 6	2, 18, 200, 2450	2, 6, 20, 82
1	12	T	A_4	3	0, 0, 0, 0, 0	2, 12, 120, 1540	1, 4, 12, 52
1	24	O	S_4	9	0, 0, 0, 0, 0	2, 12, 100, 1050	1, 4, 11, 45
1	24	O_1	S_4	15	0, 0, 6, 0, 6	1, 6, 60, 770	1, 3, 8, 30
1	24	$J(T)$	$\mathrm{A}_4 \times \mathrm{C}_2$	15	1, 0, 0, 8, 3	1, 6, 60, 770	1, 3, 7, 29
1	48	$J(O)$	$\mathrm{S}_4 \times \mathrm{C}_2$	33	1, 0, 6, 8, 9	1, 6, 50, 525	1, 3, 7, 26

Sato-Tate groups in genus 2 with $G^0 \neq \mathrm{U}(1)$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
3	1	E_1	C_1	0	0, 0, 0, 0, 0	4, 32, 320, 3584	3, 10, 37, 150
3	2	E_2	C_2	1	0, 0, 0, 0, 0	2, 16, 160, 1792	1, 6, 17, 78
3	3	E_3	C_3	0	0, 0, 0, 0, 0	2, 12, 110, 1204	1, 4, 13, 52
3	4	E_4	C_4	1	0, 0, 0, 0, 0	2, 12, 100, 1008	1, 4, 11, 46
3	6	E_6	C_6	1	0, 0, 0, 0, 0	2, 12, 100, 980	1, 4, 11, 44
3	2	$J(E_1)$	C_2	1	0, 0, 0, 0, 0	2, 16, 160, 1792	2, 6, 20, 78
3	4	$J(E_2)$	D_2	3	0, 0, 0, 0, 0	1, 8, 80, 896	1, 4, 10, 42
3	6	$J(E_3)$	D_3	3	0, 0, 0, 0, 0	1, 6, 55, 602	1, 3, 8, 29
3	8	$J(E_4)$	D_4	5	0, 0, 0, 0, 0	1, 6, 50, 504	1, 3, 7, 26
3	12	$J(E_6)$	D_6	7	0, 0, 0, 0, 0	1, 6, 50, 490	1, 3, 7, 25
2	1	F	C_1	0	0, 0, 0, 0, 0	4, 36, 400, 4900	2, 8, 32, 148
2	2	F_a	C_2	0	0, 0, 0, 0, 1	3, 21, 210, 2485	2, 6, 20, 82
2	2	F_c	C_2	1	0, 0, 0, 0, 0	2, 18, 200, 2450	1, 5, 16, 77
2	2	F_{ab}	C_2	1	0, 0, 0, 0, 1	2, 18, 200, 2450	2, 6, 20, 82
2	4	F_{ac}	C_4	3	0, 0, 2, 0, 1	1, 9, 100, 1225	1, 3, 10, 41
2	4	$F_{a,b}$	D_2	1	0, 0, 0, 0, 3	2, 12, 110, 1260	2, 5, 14, 49
2	4	$F_{ab,c}$	D_2	3	0, 0, 0, 0, 1	1, 9, 100, 1225	1, 4, 10, 44
2	8	$F_{a,b,c}$	D_4	5	0, 0, 2, 0, 3	1, 6, 55, 630	1, 3, 7, 26
4	1	G_4	C_1	0	0, 0, 0, 0, 0	3, 20, 175, 1764	2, 6, 20, 76
4	2	$N(G_4)$	C_2	0	0, 0, 0, 0, 1	2, 11, 90, 889	2, 5, 14, 46
6	1	G_6	C_1	0	0, 0, 0, 0, 0	2, 10, 70, 588	2, 5, 14, 44
6	2	$N(G_6)$	C_2	1	0, 0, 0, 0, 0	1, 5, 35, 294	1, 3, 7, 23
10	1	$\mathrm{USp}(4)$	C_1	0	0, 0, 0, 0, 0	1, 3, 14, 84	1, 2, 4, 10

Genus 2 curves realizing Sato-Tate groups with $G^0 = \mathrm{U}(1)$

Group	Curve $y^2 = f(x)$	k	K
C_1	$x^6 + 1$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3})$
C_2	$x^5 - x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2})$
C_3	$x^6 + 4$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
C_4	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a); a^4 + 17a^2 + 68 = 0$
C_6	$x^6 + 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
D_2	$x^5 + 9x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
D_3	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
D_4	$x^5 + 3x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
D_6	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3}, a); a^3 + 3a - 2 = 0$
T	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a, b);$
			$a^3 - 7a + 7 = b^4 + 4b^2 + 8b + 8 = 0$
O	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-11}, a, b);$
			$a^3 - 4a + 4 = b^4 + 22b + 22 = 0$
$J(C_1)$	$x^5 - x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_2)$	$x^5 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(C_4)$	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}$	see entry for C_4
$J(C_6)$	$x^6 - 15x^4 - 20x^3 + 6x + 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{3}, a); a^3 + 3a^2 - 1 = 0$
$J(D_2)$	$x^5 + 9x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
$J(D_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(D_4)$	$x^5 + 3x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
$J(D_6)$	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}$	see entry for D_6
$J(T)$	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}$	see entry for T
$J(O)$	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}$	see entry for O
$C_{2,1}$	$x^6 + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3})$
$C_{4,1}$	$x^5 + 2x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt[4]{2})$
$C_{6,1}$	$x^6 + 6x^5 - 30x^4 + 20x^3 + 15x^2 - 12x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, a); a^3 - 3a + 1 = 0$
$D_{2,1}$	$x^5 + x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$D_{4,1}$	$x^5 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$D_{6,1}$	$x^6 + 6x^5 - 30x^4 - 40x^3 + 60x^2 + 24x - 8$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-3}, a); a^3 - 9a + 6 = 0$
$D_{3,2}$	$x^6 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$D_{4,2}$	$x^6 + x^5 + 10x^3 + 5x^2 + x - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a); a^4 - 14a^2 + 28a - 14 = 0$
$D_{6,2}$	$x^6 + 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
O_1	$x^6 + 7x^5 + 10x^4 + 10x^3 + 15x^2 + 17x + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a, b);$
			$a^3 + 5a + 10 = b^4 + 4b^2 + 8b + 2 = 0$

Genus 2 curves realizing Sato-Tate groups with $G^0 \neq \mathrm{U}(1)$

Group	Curve $y^2 = f(x)$	k	K
F	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i, \sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_a	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
F_{ab}	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_{ac}	$x^5 + 1$	$\mathbb{Q}$	$\mathbb{Q}(a); a^4 + 5a^2 + 5 = 0$
$F_{a,b}$	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
E_1	$x^6 + x^4 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
E_2	$x^6 + x^5 + 3x^4 + 3x^2 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$
E_3	$x^5 + x^4 - 3x^3 - 4x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a); a^3 - 3a + 1 = 0$
E_4	$x^5 + x^4 + x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a); a^4 - 5a^2 + 5 = 0$
E_6	$x^5 + 2x^4 - x^3 - 3x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{7}, a); a^3 - 7a - 7 = 0$
$J(E_1)$	$x^5 + x^3 + x$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$J(E_2)$	$x^5 + x^3 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(E_3)$	$x^6 + x^3 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$J(E_4)$	$x^5 + x^3 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$J(E_6)$	$x^6 + x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$G_{1,3}$	$x^6 + 3x^4 - 2$	$\mathbb{Q}(i)$	$\mathbb{Q}(i)$
$N(G_{1,3})$	$x^6 + 3x^4 - 2$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$G_{3,3}$	$x^6 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
$N(G_{3,3})$	$x^6 + x^5 + x - 1$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$\mathrm{USp}(4)$	$x^5 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}$

Part II

The Sato-Tate problem for an abelian variety

Let A be an abelian variety of dimension $g > 0$ over a number field k.

For every prime $\mathfrak{p}$ of good reduction for A, there exists a unique integer polynomial $L_{\mathfrak{p}}(T) = \prod(1 - \alpha_i T)$ of degree $2g$ for which

$$\#A(\mathbb{F}_{q^n}) = \prod_{i=1}^{2g}(1 - \alpha_i^n)$$

holds for all positive integers n, with $q = \|\mathfrak{p}\|$ [Weil].

Each *normalized* L*-polynomial* $\bar{L}_{\mathfrak{p}}(T) = L_{\mathfrak{p}}(T/\sqrt{q})$ determines a unique conjugacy class in the unitary symplectic group $\mathrm{USp}(2g)$.

The *Sato-Tate problem*, in its simplest form, is to find a measure for which these classes are equidistributed.

The Sato-Tate group ST_A

Let $\rho_\ell\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ be the ℓ-adic Galois representation arising from the action of G_k on $V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}$.

Let G_k^1 be the kernel of the cyclotomic character $\chi_\ell\colon G_k \to \mathbb{Q}_\ell^\times$.
Let $G_\ell^{1,\mathrm{Zar}}$ be the Zariski closure of $\rho_\ell(G_k^1)$ in $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$.
Choose an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ and let $G^1 = G_\ell^{1,\mathrm{Zar}} \otimes_\iota \mathbb{C}$.

Definition [Serre]

$\mathrm{ST}_A \subseteq \mathrm{USp}(2g)$ is a maximal compact subgroup of $G^1 \subseteq \mathrm{Sp}_{2g}(\mathbb{C})$.
For each prime $\mathfrak{p}$ of good reduction for A, let $s(\mathfrak{p})$ denote the conjugacy class of $\|\mathfrak{p}\|^{-1/2}\rho_\ell(\mathrm{Frob}_\mathfrak{p}) \in G^1$ in ST_A.

The generalized Sato-Tate conjecture

Conjecturally, ST_A does not depend on ℓ; this is known for $g \leq 3$.
In any case, the characteristic polynomial of $s(\mathfrak{p})$ is always $\bar{L}_\mathfrak{p}(T)$.

Let μ_{ST_A} denote the image of the Haar measure on $\mathrm{Conj}(\mathrm{ST}_A)$.

Conjecture [refined version]

The conjugacy classes $s(\mathfrak{p})$ are equidistributed with respect to μ_{ST_A}.

In particular, the distribution of $\bar{L}_\mathfrak{p}(T)$ matches the distribution of characteristic polynomials of random matrices in ST_A.

The Sato-Tate axioms (weight 1)

A subgroup G of $\mathrm{USp}(2g)$ satisfies the *Sato-Tate axioms* if:

1. G is closed.
2. (Hodge circles) There is a subgroup H that is the image of a homomorphism $\theta\colon \mathrm{U}(1) \to G^0$ such that $\theta(u)$ has eigenvalues u and u^{-1} with multiplicity g, and H can be chosen so that its conjugates generate a dense subset of G^0.
3. (Rationality) For each component H of G and each irreducible character χ of $\mathrm{GL}_{2g}(\mathbb{C})$ we have $\mathrm{E}[\chi(\gamma) : \gamma \in H] \in \mathbb{Z}$.

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For any fixed g, the set of subgroups of $\mathrm{USp}(2g)$ that satisfy the Sato-Tate axioms is **finite** up to conjugacy.

Theorem

For $g \leq 3$, the group ST_A satisfies the Sato-Tate axioms.

Conjecturally, this holds for all g.

Sato-Tate groups in genus 2

Theorem 1 [FKRS 2012]

Up to conjugacy, 55 subgroups of $\mathrm{USp}(4)$ satisfy the Sato-Tate axioms:

$$\begin{aligned}
\mathrm{U}(1):\quad & C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O,\\
& J(C_1), J(C_2), J(C_3), J(C_4), J(C_6),\\
& J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O),\\
& C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1\\
\mathrm{SU}(2):\quad & E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6)\\
\mathrm{U}(1)\times\mathrm{U}(1):\quad & F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c}\\
\mathrm{U}(1)\times\mathrm{SU}(2):\quad & \mathrm{U}(1)\times\mathrm{SU}(2),\ N(\mathrm{U}(1)\times\mathrm{SU}(2))\\
\mathrm{SU}(2)\times\mathrm{SU}(2):\quad & \mathrm{SU}(2)\times\mathrm{SU}(2),\ N(\mathrm{SU}(2)\times\mathrm{SU}(2))\\
\mathrm{USp}(4):\quad & \mathrm{USp}(4)
\end{aligned}$$

Of these, exactly 52 arise as ST_A for an abelian surface A (34 over $\mathbb{Q}$).

Galois types

Let A be an abelian surface defined over a number field k.
Let K be the minimal extension of k for which $\mathrm{End}(A_K) = \mathrm{End}(A_{\bar{\mathbb{Q}}})$.
The group $\mathrm{Gal}(K/k)$ acts on the $\mathbb{R}$-algebra $\mathrm{End}(A_K)_{\mathbb{R}} = \mathrm{End}(A_K) \otimes_{\mathbb{Z}} \mathbb{R}$.

The *Galois type* of A is the isomorphism class of $[\mathrm{Gal}(K/k), \mathrm{End}(A_K)_{\mathbb{R}}]$.

An isomorphism $[G, E] \simeq [G', E']$ is an isomorphism $G \simeq G'$ of groups and an equivariant isomorphism $E \simeq E'$ of $\mathbb{R}$-algebras.

One may have $G \simeq G'$ and $E \simeq E'$ but $[G, E] \not\simeq [G', E']$.

Galois types and Sato-Tate groups

Theorem 2 [FKRS 2012]

Up to conjugacy, the Sato-Tate group G of an abelian surface A is uniquely determined by its Galois type, and vice versa.

We also have $G/G^0 \simeq \mathrm{Gal}(K/k)$, and G^0 is uniquely determined by the isomorphism class of $\mathrm{End}(A_K)_{\mathbb{R}}$, and vice versa:

$\mathrm{U}(1)$	$\mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	$\mathbb{C}\times\mathbb{R}$
$\mathrm{SU}(2)$	$\mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2)\times\mathrm{SU}(2)$	$\mathbb{R}\times\mathbb{R}$
$\mathrm{U}(1)\times\mathrm{U}(1)$	$\mathbb{C}\times\mathbb{C}$	$\mathrm{USp}(4)$	$\mathbb{R}$

There are 52 distinct Galois types in genus 2.

The proof uses the *algebraic Sato-Tate group* of Banaszak and Kedlaya, which, in genus $g \leq 3$, uniquely determines ST_A.

Exhibiting Sato-Tate groups in genus 2

The 34 Sato-Tate groups that can arise over $\mathbb{Q}$ can all be realized as the Sato-Tate group of the Jacobian of a hyperelliptic curve that we are able to exhibit explicitly, following an extensive computational search.

Exhibiting Sato-Tate groups in genus 2

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The remaining 18 groups all arise as subgroups of these 34.
These we obtain by simply extending fields of definition.

In fact, one can realize all 52 groups using just 9 curves.

Searching for curves

We surveyed the $\bar{L}$-polynomial distributions of genus 2 curves

$$y^2 = x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

$$y^2 = x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

with integer coefficients $|c_i| \leq 128$, over 2^{48} curves.

We specifically searched for cases not already addressed in [KS09].

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We specifically searched for cases not already addressed in [KS09].

We found over 10 million non-isogenous curves with exceptional distributions, including at least 3 apparent matches for all of our target Sato-Tate groups.

Representative examples were computed to high precision $N = 2^{30}$.

For each example, the field K was then determined, allowing the Galois type, and hence the Sato-Tate group, to be **provably** identified.

Computational methods

There are four standard ways to compute $L_p(T)$ for a genus 2 curve:

1. point counting: $O(p^2 \log^{1+\epsilon} p)$.
2. group computation: $O(p^{3/4} \log^{1+\epsilon} p)$.
3. p-adic methods: $O(p^{1/2} \log^{2+\epsilon} p)$.
4. CRT approach: $O(\log^{8+\epsilon} p)$.

For the feasible range of $p \leq N$, we found (2) to be the best [KS08].
We can accelerate the computation with partial use of (1) and (4).

The `smalljac` software package provides an open source implementation of this approach (soon to be available in `Sage`).

A very recent breakthrough

All of the methods above perform separate computations for each p. But we want to compute $L_p(T)$ for all good $p \leq N$ using reductions of *the same curve* in each case.

Is their a way to take advantage of this?

A very recent breakthrough

All of the methods above perform separate computations for each p. But we want to compute $L_p(T)$ for all good $p \le N$ using reductions of *the same curve* in each case.

Is their a way to take advantage of this?

Theorem (Harvey, 2012)

Let $y^2 = f(x)$ *be a hyperelliptic curve of genus* g *with* $\log \|f\| = O(\log N)$. *One can compute* $L_p(T)$ *for all odd* $p \le N$ *with* $p \nmid \operatorname{disc}(f)$ *in time*

$$O\big(g^{8+\epsilon} N \log^{3+\epsilon} N\big).$$

This yields an average time of $O\big(g^{8+\epsilon} \log^{4+\epsilon} N\big)$ per prime.

This is the first algorithm to achieve an average running time that is polynomial in both g and $\log p$.

Some preliminary implementation results

With suitable optimizations, this algorithm can be made quite practical.

In genus 2 we are already able to surpass the performance of `smalljac` for $N \geq 2^{22}$, and further improvements are under way.

When combined with group computations in genus 3, we expect to obtain a dramatic improvement over existing methods.

We are also looking at adapting the algorithm to handle certain families of non-hyperelliptic curves, including Picard curves.

[Harvey-S, work in progress]

Sato-Tate in genus 3

For $g = 3$ there are 14 possibilities for the connected part of ST_A.
There are at least 400 groups that satisfy the Sato-Tate axioms.

In order to realize cases with large component groups, one needs abelian threefolds with many endomorphisms. An obvious place to start is with Jacobians of curves with large automorphism groups (and their twists). Some notable cases enumerated by Wolfart:

$$y^2 = x^8 - x, \quad y^2 = x^7 - x, \quad y^2 = x^8 - 1$$
$$y^2 = x^8 - 14x^4 + 1, \quad y^3 = x^4 - x, \quad y^3 = x^4 - 1$$
$$x^4 + y^4 = 1, \quad x^3y + y^3z + z^3x = 0.$$

However, Jacobians may not be enough!