Forbidden isogenies

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14th Algorithmic Number Theory Symposium
Rump session
1 July 2020

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Graphs of Richelot isogenies

Richelot graphs

- *Vertices*: Principally polarized abelian surfaces over $\mathbb{F}_q$
- *Edges*: Richelot isogenies from one PPAS to another

One might choose to restrict to subgraphs:

- Supersingular abelian surfaces
- Super*special* abelian surfaces
- Jacobians
- $\ldots$

A few papers that discuss algorithms based on Richelot graphs

Wouter Castryck, Thomas Decru, Benjamin Smith:
Hash functions from superspecial genus-2 curves using Richelot isogenies

Craig Costello, Benjamin Smith:
The supersingular isogeny problem in genus 2 and beyond

E. V. Flynn, Yan Bo Ti:
Genus two isogeny cryptography

Toshiyuki Katsura, Katsuyuki Takashima:
Counting Richelot isogenies between superspecial abelian surfaces

Katsuyuki Takashima:
Efficient algorithms for isogeny sequences and their cryptographic applications

What do these graphs even *look* like?

Volcanos?

Mount Ngauruhoe, by Flickr user russellstreet

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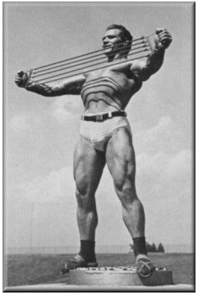
Expanders?



Olympic athlete John Grimek

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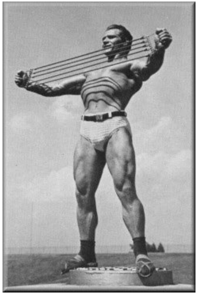

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Are they connected? Are there short paths? What's the diameter?

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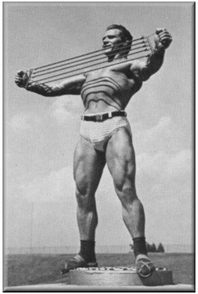

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Why are we stuck using *these* confusing graphs?

Humanistic mathematics

As the poet Mary Oliver writes in “The Summer Day”:

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If we want *meaning* and *hope* in our lives and in our math,
we need to find a better graph.

Where to look?

The answer is hidden in plain sight

Castryck/Decru/Smith: “Let K be a field of characteristic $p > 5$.”

Costello/Smith: “Throughout, p denotes a prime > 3, and ℓ a prime not equal to p.”

Flynn/Ti: “Let p and ℓ be distinct primes… We will use Richelot isogenies [$\ell = 2$].”

Katsura/Takashima: “Let k be an algebraically closed field of characteristic $p > 5$.”

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Wake up, sheeple!

My colleague Brad Brock and I:

Why not Richelot isogenies. . . *in characteristic two*

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Why not Richelot isogenies... *in characteristic two*

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We plunged straight into the belly of the beast:

We studied *purely inseparable Richelot isogenies.*

Supersingular genus-2 curves in characteristic 2

For every $t \in \overline{\mathbb{F}}_2$ let C_t be the curve

$$C_t\colon\, y^2 + y = \begin{cases} t(x^5 + x^3) & \text{if } t \neq 0; \\ x^5 & \text{if } t = 0. \end{cases}$$

These curves are supersingular, and every supersingular genus-2 curve over $\overline{\mathbb{F}}_2$ is isomorphic to exactly one of them.

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Let $\mathcal{G}$ be the graph of Richelot isogenies on the curves C_t.

Is the world *ready* for these results?

Theorem

The graph $\mathcal{G}$ *is connected.*

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Suppose $s \in \mathbb{F}_{2^m}$ and $t \in \mathbb{F}_{2^n}$. Then the shortest path in $\mathcal{G}$ connecting C_s and C_t has length bounded above by the following expression in m and n:

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We can classify the pairs (s, t) for which it is not sharp.

Further results

Let $R(s,t)$ denote the number of non-isomorphic Richelot isogenies from C_s to C_t.

Theorem

We have

$$R(s,t) = \begin{cases} 60 & \text{if } s \text{ and } t \text{ are both nonzero;} \\ 12 & \text{if exactly one of } s \text{ and } t \text{ is zero;} \\ 4 & \text{if } s = t = 0. \end{cases}$$

We give contructions that allow one to compute all of these isogenies.

If you dare think outside the box, why not use this graph for your next algorithm?

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You're welcome.

arXiv:2002.02122 [math.AG]