Reductions between short vector problems and simultaneous approximation

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Lattice problems

Our goal is to reduce the first problem to the second:

Short vector problem (approx-SVP)

For $M \in \mathrm{M}_n(\mathbb{Z})$ and $\alpha \geq 1$, find $\mathbf{q_0} \in \mathbb{Z}^n$ with $0 \neq \|M\mathbf{q}_0\| \leq \alpha \min_{\mathbf{q}}^{\times} \|M\mathbf{q}\|$.

Simultaneous approximation problem

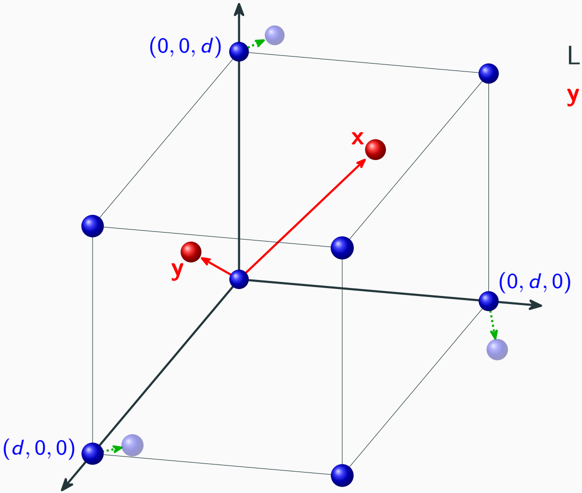
For $\mathbf{x} \in \mathbb{Q}^n$ and $\alpha' \geq 1$, find $q_0 \in \mathbb{Z}$ with $0 \neq \|\{q_0\mathbf{x}\}\| \leq \alpha' \min_q^{\times} \|\{q\mathbf{x}\}\|$.

Alternatively, we could ask that $q_0 \leq \alpha' N$ and $\|\{q_0\mathbf{x}\}\| \leq \alpha' \min_{q \leq N} \|\{q\mathbf{x}\}\|$.

Under a fixed ℓ_p-norm, the reduction is gap-preserving ($\alpha = \alpha'$). It requires $O(n^4 \log mn)$ operations on integers of length $O(n^4 \log mn)$, where m is the maximum input integer magnitude.

The goal

For short vector problems, we typically have lattices of this form:



Lattice generated by $\mathbf{x}$, $\mathbf{y}$, and $d\mathbb{Z}^3$.

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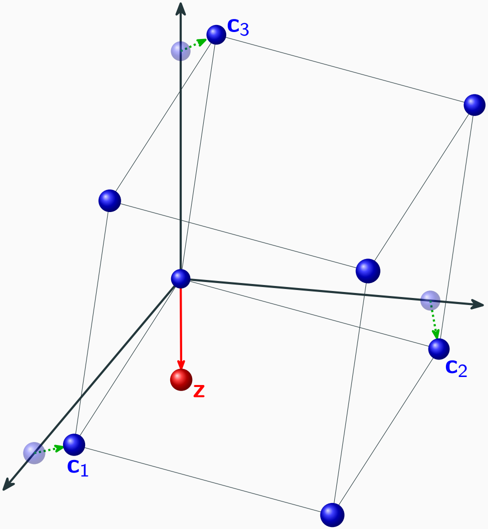

Lattice generated by $\mathbf{x}$, $\mathbf{y}$, and $d\mathbb{Z}^3$.

Another generating set is $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{z}\}$.

Do simultaneous approximation on the vector $[\mathbf{c}_1\,\mathbf{c}_2\,\mathbf{c}_3]^{-1}\mathbf{z}$.

Since $[\mathbf{c}_1\,\mathbf{c}_2\,\mathbf{c}_3]$ is nearly scaled orthonormal, it preserves shortness.

The goal

So the desired setup is

$$\underbrace{\begin{matrix}\text{scaled}\\ \text{orthonormal}\\ \text{sublattice}\end{matrix} + \begin{matrix}\text{small}\\ \text{lattice}\\ \text{vectors}\end{matrix}}_{n \text{ vectors}}, \qquad \begin{matrix}\text{extra}\\ \text{lattice}\\ \text{vector}\end{matrix}.$$

Let M be the input matrix (column vectors). Using a multiple of $\det M\,\mathbb{Z}^n$ as the sublattice, this becomes

$$M(c \operatorname{adj} M) \;+\; MA, \qquad M\mathbf{b},$$

where $A \in \mathrm{M}_n(\mathbb{Z})$, $\mathbf{b} \in \mathbb{Z}^n$, and $c \in \mathbb{Z}$. The goals are

1. the columns of $c \operatorname{adj} M + A$ and $\mathbf{b}$ generate $\mathbb{Z}^n$
2. MA is small relative to $c \det M$ (do 1, then make c bigger)

The goal

For 1, we'll make sure that replacing the last column of $c \operatorname{adj} M + A$ with $\mathbf{b}$ gives a matrix with determinant 1.

By Cramer's rule, we want the last entry in

$$\operatorname{adj}(c \operatorname{adj} M + A)\mathbf{b}$$

to be 1. We can choose A and c so that

$$\operatorname{adj}(c \operatorname{adj} M + A)\mathbf{b} = \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ \color{red}{*} & \color{red}{*} & \cdots & * \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ \color{red}{1} \end{bmatrix}.$$

coprime

Then b_1 and b_2 can be Bézout coefficients.

Finding A

Suppose we have the following value of $c \operatorname{adj} M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ \color{red}{4} & \color{red}{1} & \color{red}{5} & -5 \\ \color{red}{4} & \color{red}{1} & \color{red}{-1} & 1 \\ \color{red}{2} & \color{red}{-1} & \color{red}{1} & -7 \end{bmatrix}$$

determinant: -36

$$\begin{bmatrix} \color{red}{2} & \color{red}{-1} & \color{red}{1} & -1 \\ 4 & 1 & 5 & -5 \\ \color{red}{4} & \color{red}{1} & \color{red}{-1} & 1 \\ \color{red}{2} & \color{red}{-1} & \color{red}{1} & -7 \end{bmatrix}$$

determinant: 0

Finding A

Suppose we have the following value of $c \operatorname{adj} M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ 4 & 1 & 5 & -5 \\ 4 & 1 & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: $-36 + 0x$

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ 4 & 1 & 5 & -5 \\ 4 & 1 & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: $0 + 6x$

Finding A

Suppose we have the following value of $c \operatorname{adj} M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ \color{red}{4} & \color{red}{1} & 5 & -5 \\ \color{red}{4} & \color{red}{1} & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: 0

$$\begin{bmatrix} \color{red}{2} & \color{red}{-1} & 1 & -1 \\ 4 & 1 & 5 & -5 \\ \color{red}{4} & \color{red}{1} & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: 6

Finding A

Suppose we have the following value of $c \operatorname{adj} M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ \color{red}{4} & \color{red}{1} & 5 & -5 \\ \color{red}{4} & \color{red}{1+y} & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: $0 + 4y$

$$\begin{bmatrix} \color{red}{2} & \color{red}{-1} & 1 & -1 \\ 4 & 1 & 5 & -5 \\ \color{red}{4} & \color{red}{1+y} & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: $6 + 2y$

Finding A

Suppose we have the following value of $c \operatorname{adj} M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ \color{red}{4} & 1 & 5 & -5 \\ 4 & 1+y & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: 4

$$\begin{bmatrix} \color{red}{2} & -1 & 1 & -1 \\ 4 & 1 & 5 & -5 \\ 4 & 1+y & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: 2

Finding A

Suppose we have the following value of $c \operatorname{adj} M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ \color{red}{4+1} & 1 & 5 & -5 \\ 4 & 1+y & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: 5

$$\begin{bmatrix} \color{red}{2} & -1 & 1 & -1 \\ 4+1 & 1 & 5 & -5 \\ 4 & 1+y & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: 2

Finding A

Suppose we have the following value of $c \operatorname{adj} M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ \color{red}{4+1} & \color{red}{1} & 5 & -5 \\ \color{red}{4} & \color{red}{1+y} & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix} \qquad \begin{bmatrix} \color{red}{2} & \color{red}{-1} & 1 & -1 \\ 4+1 & 1 & 5 & -5 \\ \color{red}{4} & \color{red}{1+y} & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: $1 + 5y$ determinant: $6 + 2y$

Finding A

Suppose we have the following value of $c \operatorname{adj} M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ 4+1 & 1 & 5 & -5 \\ 4 & 1+0 & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: 1

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ 4+1 & 1 & 5 & -5 \\ 4 & 1+0 & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: 6

Finding A

Suppose we have the following value of $c \operatorname{adj} M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ 4+1 & 1 & 5 & -5 \\ 4 & 1+0 & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: $-36 + 1x$

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ 4+1 & 1 & 5 & -5 \\ 4 & 1+0 & -1 & 1 \\ 2 & -1 & 1+x & -7 \end{bmatrix}$$

determinant: $0 + 6x$

Finding A

Suppose we have the following value of $c\,\mathrm{adj}\,M$ (here $c = 1$).

$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ 4+1 & 1 & 5 & -5 \\ 4 & 1+0 & -1 & 1 \\ 2 & -1 & 1+1 & -7 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 1 & -1 \\ 4+1 & 1 & 5 & -5 \\ 4 & 1+0 & -1 & 1 \\ 2 & -1 & 1+1 & -7 \end{bmatrix}$$

determinant: -35

determinant: 6

This gives

$$\mathrm{adj}(c\,\mathrm{adj}\,M + A)\mathbf{b} = \begin{bmatrix} 30 & 0 & 30 & 0 \\ -125 & 30 & 25 & 0 \\ 30 & 36 & -42 & -36 \\ 35 & 6 & -7 & -36 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -30 \\ 305 \\ 180 \\ 1 \end{bmatrix}.$$

Changing to simultaneous approximation

The columns of

$$M\operatorname{adj}(c\operatorname{adj}M + A) = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 0 & -1 & 6 \end{bmatrix} \qquad \text{and} \qquad M\mathbf{b} = \begin{bmatrix} -1 \\ 10 \\ 5 \\ -1 \end{bmatrix}$$

generate the same lattice as the columns of M. The matrix above is roughly scaled orthonormal, so do simultaneous approximation on

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 0 & -1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 10 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 61/36 \\ 31/30 \\ 1/180 \end{bmatrix}.$$

Avoiding Jacobsthal

For a pair of integers r, s, the previous algorithm finds a small t so that r and $s + t$ are coprime.

The maximum "smallest t" needed as s varies is called *Jacobsthal's function*, $J(r)$. We know

$$J(r) < 2\omega(r)^{2+2e\log\omega(r)} \qquad \text{(Stevens)},$$

$$J(r) \ll (\omega(r)\log\omega(r))^2 \qquad \text{(Iwaniec)},$$

where ω counts distinct prime factors.

These bounds make for difficult worst-case analysis, so leave "c" a variable. Then $r(c)$ and $s(c)$ are polynomials. And if $r(c) \neq 0$ there are at most $\deg r(c)$ integers t for which $r(c)$ and $s(c) + t$ are not coprime over $\mathbb{Q}(c)$.

This is the version presented in the paper.

References

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