Genus 3 hyperelliptic curves with CM via Shimura Reciprocity

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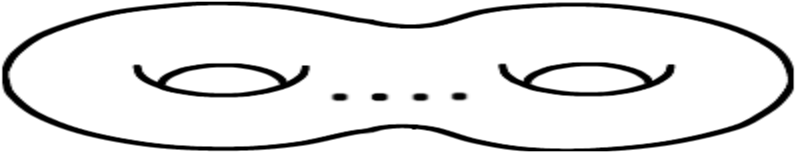

Problem (Main Problem)

Let X be a hyperelliptic curve of genus 3 over $\mathbb{C}$ with CM by K. Compute the Rosenhain class polynomials

$$H_k^R(t) = \prod_{\sigma \in \mathrm{Gal}(\mathcal{CM}(2)/K^r)} (t - \lambda_k^\sigma(X))$$

for $k = 1, \ldots, 5$.

Invariants of genus 3 hyperelliptic curves over $\mathbb{C}$.

- The Shiodas are invariants for $\mathcal{M}_3^{hyp}$.
- The Rosenhains are invariants for $\mathcal{M}_3^{hyp}[2]$.

Why the Rosenhains?

- By their definition, the Shioda invariants have larger denominators.
- In our examples the precision for the computation of H_k^R is by a factor ≈ 10 smaller than the precision for the Shioda class polynomials.
- Time for the computation for our examples: Rosenhains ≈ 1 min, Shiodas ≈ 60 min on a single core of a CPU.

Our main setup.

Let (K, Φ) be a primitive CM-pair.

- We have solved (computationally) the CM hyperelliptic Schottky Problem, i.e. for a p.p.a.v. (A, E) represented by a torus with $Z \in \mathbb{H}_3$,
  - $\exists$ X a hyperelliptic curve of genus 3 over $\mathbb{C}$ with
  - $(\mathrm{Jac}(X), E) = (A, E)$ is of type (K, Φ).
- From Poor's construction there is an azygetic system η for Z which determines
$$\nu_\eta : \mathrm{Jac}(X)[2] \longrightarrow (\mathbb{Z}/2\mathbb{Z})^6.$$
- We compute a Rosenhain model for $X \in \mathcal{M}_3^{hyp}[2]$ by using the formula of Takase (generalized by Vincent)[1].

[1] Theorem 2.5 in our paper

Shimura's CM Theory.

Let (K, Φ) be a CM-pair and denote by (K^r, Φ^r) the reflex CM-pair.

- By class field theory

$$\Phi_2 : Cl_2(K^r) = I_2(K^r)/P_2(K^r) \xrightarrow{\sim} \mathrm{Gal}(\mathcal{H}_2/K^r).$$

- By Shimura's Main Thm.2 of CM[2]

$$\mathcal{CM}(2) \subset \mathcal{H}_2,$$

with $\mathrm{Gal}(\mathcal{CM}(2)/K^r) \cong I_2(K^r)/H_2(K^r)$.

- $\mathrm{Princ}(K, \Phi, 2) =$ set of p.p.a.v. of type (K, Φ) + a proper 2-tp.

$\rightsquigarrow$ For $A = A(\mathfrak{a}, \xi, t) \in \mathrm{Princ}(K, \Phi, 2)$, and $[\mathfrak{c}_\sigma] \in I_2(K^r)/H_2(K^r)$,

$$A^{\mathfrak{c}_\sigma} = A(N_{\Phi^r}(\mathfrak{c}_\sigma)^{-1}\mathfrak{a}, N_{K^r/\mathbb{Q}}(\mathfrak{c}_\sigma)\xi, t \mod (N_{\Phi^r}(\mathfrak{c}_\sigma)^{-1}\mathfrak{a})).$$

[2] Theorem 3.2 in our paper.

An effective version of Shimura's Reciprocity Law (SRL).

Let $\mathcal{F}_2$ be the field of modular functions of level 2.

We choose the setup as above:

- Let $A(\mathfrak{a}, \xi)$ be a p.p.a.v. of type (K, Φ).
- Let $B = [B_1 | B_2]$ be a symplectic basis of $\mathfrak{a}$ with respect to E_ξ.

Theorem (Streng)

For any $[\mathfrak{c}_\sigma] \in I_2(K^r)/H_2(K^r)$ let $C = [C_1|C_2]$ be a symplectic basis for $N_{\Phi^r}(\mathfrak{c}_\sigma)^{-1}\mathfrak{a}$ with respect to $E_{N_{K^r/\mathbb{Q}}(\mathfrak{c}_\sigma)\xi}$. There are matrices $M \in \mathrm{GSp}_6(\mathbb{Q})^+$ and $U \in \mathrm{Sp}_6(\mathbb{Z}/2\mathbb{Z})$ with M invertible $\bmod 2$ and $U \equiv M^{-1} \pmod 2$ s.t. $C = BM^t$. For any $f \in \mathcal{F}_2$

$$f^{[\mathfrak{c}]}(Z) = f^{UM}(Z) = f^{U}(M.Z).$$

The statement of Theorem 4.4

- Let $[\mathfrak{c}_\sigma] \in I_2(K^r)/H_2(K^r)$ corresponding to $\sigma \in \mathrm{Gal}(\mathcal{CM}(2)/K^r)$.
- Let $Z \in \mathbb{H}_3$ be p.m. for a hyperelliptic p.p.a.v. $(\mathrm{Jac}(X), E_\xi, \nu_\eta)$, and η an a.s. for Z.
- Let $X \in \mathcal{M}_3^{hyp}[2]$ with Rosenhain coefficients λ_i computed via Takase's formula.

Let M, U be the matrices from the S.R.L. for $[\mathfrak{c}_\sigma]$. Then

$$\lambda_k^\sigma(Z) = \exp(4\pi i(\eta_k + \eta_7)_1(\eta_6)_2) \cdot \zeta_4(\tilde{U}, \eta) \cdot \lambda_k'(Z'),$$

with $Z' = M.Z \in \mathbb{H}_3$, and $\tilde{U} \in \mathrm{Sp}_6(\mathbb{Z})$ a lift of U.

Problem

In general, $(\lambda_k'(Z'))_{1 \le k \le 5}$ does <u>not</u> represent the conjugate moduli point as a point in $\mathcal{M}_3^{hyp}[2]$.

How to solve this problem?

- Note that the action of $\mathfrak{c}_\sigma$ corresponds to an isogeny

$$I_{\mathfrak{c}_\sigma} : \mathrm{Jac}(X) \longrightarrow \mathrm{Jac}(X^\sigma),$$

  with $\ker I_{\mathfrak{c}_\sigma} \cap \mathrm{Jac}(X)[2] = \emptyset$ which induces a level 2 structure on $Jac(X^\sigma)[2]$.
- The action of M on the p.m. Z yields an action on the a.s. η of the form

$$M \circlearrowright (Z, \eta) = (Z', \tilde{U}^t \eta + (1/2)\delta_0).$$

- By constructing another tuple of matrices (M', U') with
  - $M'.Z \sim Z'$, $U' = Id_3 \quad \rightsquigarrow \quad (M'.Z, \eta)$, and
  - By SRL

$$\lambda_k^\sigma(Z) = \lambda_k(M'.Z),$$

    with $(\lambda_k(M'.Z))_{1 \le k \le 5}$ representing the conjugate moduli point in $\mathcal{M}_3^{hyp}[2]$.

The Rosenhain class polynomials.

- $K = \mathbb{Q}(\beta) = \mathbb{Q}[X]/(x^6 + 56x^4 + 592x^2 + 576)$.
- $K^r \cong K$.
- $K_0^r = \mathbb{Q}(\alpha) = \mathbb{Q}[X]/(x^3 - 56x^2 + 592x - 576)$.

$$H_1^R(t) = t^3 + (-(2577/26908)\alpha^2 + (114759/26908)\alpha - 69100/6727)t^2 + ((964/6727)\alpha^2 - (173963/26908)\alpha + 111913/6727)t + (-(7263/107632)\alpha^2 + (82251/26908)\alpha - 56934/6727).$$

$$H_2^R(t) = t^3 + ((1/784)\alpha^2 + (1/49)\alpha + -241/49)t^2 + ((1/784)\alpha^2 + (1/49)\alpha - 241/49)t + ((19/9604)\alpha^2 + (13/2401)\alpha - 9657/2401).$$

The Rosenhain class polynomials.

$$H_3^R(t) = t^3 + ((2577/26908)\alpha^2 - (114759/26908)\alpha + 28738/6727)t^2 + (-(1613/6727)\alpha^2 + (285073/26908)\alpha - 83763/6727)t + ((2521/15376)\alpha^2 - (27623/3844)\alpha + 7956/961).$$

$$H_4^R(t) = t^3 + (-(1/784)\alpha^2 - (1/49)\alpha - 53/49)t^2 + (-(3/9604)\alpha^2 + (1779/9604)\alpha - 2862/2401)t + ((9/2401)\alpha^2 - (1413/4802)\alpha + 6561/2401).$$

$$H_5^R(t) = t^3 - 6t^2 + 12t - 8.$$

The first Shioda class polynomial.

$$
\begin{aligned}
H_1^S(t) = t^3 + \\
&\Bigg(\Big(\frac{51118350147041007075289102892382502539160241}{56600637941163154415409455778366974323203716 78} \ldots \Big) \alpha^2 + \\
&\Big(-\frac{14322880374552213474542083923644095961299243}{28300318970581577207704727889183487161601858} \ldots \Big) \alpha + \\
&\Big(\frac{46001264276271986524710208895686020046 82}{45280510352930523532327564622693579458562 9} \ldots \Big) \Bigg) t^2 \\
&\Bigg(\Big(-\frac{45786629384434931967512304711283508528 24}{31062430102110339143176709331167795508574 1} \ldots \Big) \alpha^2 + \\
&\Big(-\frac{13084620317595913189365728314348783027469}{15531215051055169571588354665583897754287 0} \ldots \Big) \alpha \\
&\Big(-\frac{42611564467691430950342882530555 3}{14909966449012962788724820478960541} \ldots \Big) \Bigg) t \;+ \ldots
\end{aligned}
$$

The first Shioda class polynomial.

$$
\dots + \Bigg(\Big(\frac{36545221560643723862700022172632423619}{19879955265350617051633093971947389125 48} \dots\Big)\alpha^2
$$

$$
\Big(-\frac{126506167524141833514594276001386788 10}{993997763267530852581654698597369456274} \dots\Big)\alpha
$$

$$
-\frac{1153821644015682350199228400182601145320193}{8763408851664761802352547546817624594092443}\Bigg).
$$

Thank you for listening!