Faster computation of isogenies of large prime degree.

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ANTS 2020

Isogenies between montgomery elliptic curves

For Montgomery curves

$$E_{/k} : y^2 = x^3 + Ax^2 + x$$

the cyclic isogeny φ of odd prime degree n with kernel $\mathcal{G} = \langle P \rangle$ is the algebraic map defined by:

$$\begin{aligned} \varphi \quad : \quad & E \longrightarrow E/\mathcal{G} \\ & (x, y) \longmapsto \left(\frac{g(x)}{h(x)}, y \left(\frac{g(x)}{h(x)} \right)' \right) \end{aligned}$$

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where:

$$\frac{g(X)}{h(X)} = (-1)^{n-1} X^n \prod_{i=1}^{n-1} \frac{1/X - x([i]P)}{X - x([i]P)}$$

A summary of the result

Isogeny evaluation problem over a field k

Input: Generator $P \in E(k)$ of order n for cyclic kernel $\mathcal{G} = \langle P \rangle$, a point $Q \in E(k)$.

Output: The codomain $E/\mathcal{G}$ and the image point $\varphi(Q)$;

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Biquadratic expressions

The group law on the elliptic curve gives:

$$(X - x(P \oplus Q))(X - x(P \ominus Q)) = X^2 + \frac{F_1(x(P), x(Q))}{F_0(x(P), x(Q))} X + \frac{F_2(x(P), x(Q))}{F_0(x(P), x(Q))}$$

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where

$$\begin{aligned} F_0(X, Y) &= (X - Y)^2 \\ F_1(X, Y) &= -2((XY + 1)(X + Y) + 2AXY) \\ F_2(X, Y) &= (XY - 1)^2 \end{aligned}$$

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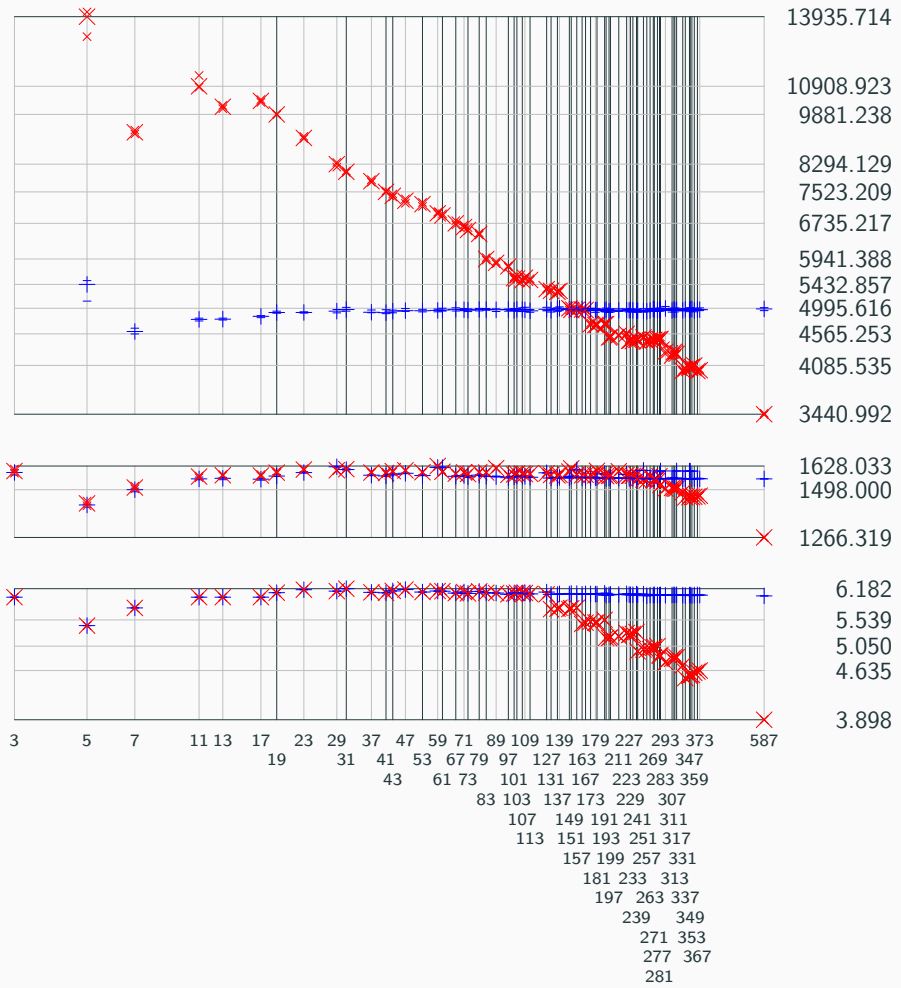
4. $h(\alpha) = Res_Y(B, G_2)/Res_Y(B, G_1)$

Questions?

https://velusqrt.isogeny.org

Concrete Performances (small degrees)

Performance of new vs. old algorithm. Time to eval. an isogeny.



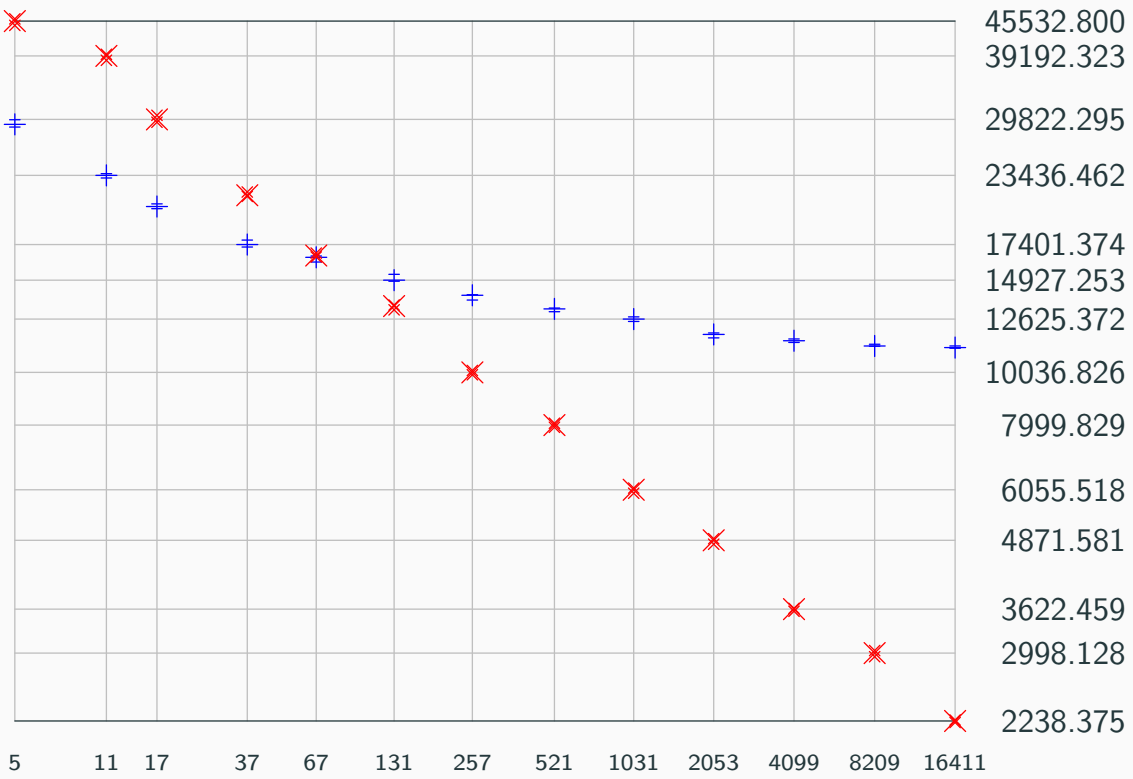
x-axis: isogeny degree n,
y-axis (divided by $n+2$):

Top: Cycle counts of pure C implem. on Flint.

Middle: Cycle counts of assembly optim. implem. based on original CSIDH-512.

Bottom: $\mathbb{F}_p$ mul. counts of the assembly optim. implem.

Concrete Performances (large degree)



Performance comparison of new vs. old algorithm in a Julia/Nemo implementation on a 256-bits base field.

x-axis: isogeny degree. y-axis: cycle counts.

Application to isogeny-based cryptography

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$1\ \%$ improvement for CSIDH-512 ($10\ \%$ for CSIDH-1024).

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First secure implementation: from minutes to seconds for key exchange.

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others: Galbraith, Petit, Silva '17,

Delpech de Saint Guilhem, Kutas, Petit, Silva '19,

. . . (to be assessed).