Post-critically finite cubic polynomials

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Definitions

Let K be a number field and $f \in K(z)$ a rational function (morphism of $\mathbb{P}^1$). Define the orbit of a point $\alpha \in \mathbb{P}^1$ for f as

$$\mathcal{O}_f(\alpha) = \{f^n(\alpha) \colon n \geq 0\},$$

where

$$f^n = f \circ f \circ \ldots \circ f.$$

Definitions

We classify points based on their behavior under iteration of f.
Let $\alpha \in \mathbb{P}^1$

- We say α is *periodic of period m* for f if $f^m(\alpha) = \alpha$.
- We say α is *preperiodic of period* (k, m) for f if $f^k(\alpha) = f^{k+m}(\alpha)$.
- We say α is a *wandering point* for f if the orbit of α is not finite.

Example

$$f(z) = z^2 - 1$$

Definitions

We say that $f \in K$ is $\bar{K}$-conjugate to g if $g = f^{\phi}$ for some $\phi \in \mathrm{PGL}_2(\bar{K})$, where

$$f^{\phi} = \phi \circ f \circ \phi^{-1}.$$

If α is preperiodic of period (k, m) for f the $\phi(\alpha)$ is preperiodic of period (k, m) for f^{ϕ}.

$$[f] = \{f^{\phi} : \phi \in \mathrm{PGL}_2(\bar{K})\}$$

Definitions

Let $\mathrm{Crit}(f) = \{\text{critical points of } f\}$.

Definition

f is post-critically finite (PCF) if every element of $\mathrm{Crit}(f)$ has finite forward orbit.

Example

$$f(z) = 2z^3 - 3z^2 + \frac{1}{2}$$

Ingram, 2011

Theorem

The set of conjugacy classes of post-critically finite polynomials of degree d with coefficients of algebraic degree at most B is a finite and effectively computable set.

Application

If $f(z) = z^3 + Az + B$ has coefficients in $\mathbb{Q}$ and is post-critically finite, then

$$(A, B) \in \left\{(-3, 0), \left(-\frac{3}{2}, 0\right), \left(-\frac{3}{4}, \frac{3}{4}\right), \left(-\frac{3}{4}, -\frac{3}{4}\right), (0, 0), \left(\frac{3}{2}, 0\right), (3, 0)\right\}.$$

Missing some cubic PCF polynomials

Example

$$f(z) = 2z^3 - 3z^2 + \frac{1}{2}$$

Conjugate by $\phi(z) = \sqrt{2}z - \frac{1}{\sqrt{2}}$ so f^ϕ is in the monic-centered form

$$f^\phi(z) = z^3 - \frac{3}{2}z - \frac{1}{\sqrt{2}} \notin \mathbb{Q}$$

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Problem: Monic centered form $f(z) = z^3 + Az + B$ does not preserve field of definition.

Motivation

Question

Can we use Ingram's techniques and a different normal form to find **all** PCF cubic polynomials defined over $\mathbb{Q}$ (up to conjugacy over $\bar{\mathbb{Q}}$)?

All cubic PCF polynomials

Theorem

There are exactly fifteen $\bar{\mathbb{Q}}$ conjugacy classes of cubic PCF polynomials defined over $\mathbb{Q}$:

$$
\begin{array}{llllll}
(1) & z^3 & (2) & -z^3+1 & (3) & -2z^3+3z^2+\frac{1}{2} \\
(4) & -2z^3+3z^2 & (5) & -z^3+\frac{3}{2}z^2-1 & (6) & 2z^3-3z^2+1 \\
(7) & 2z^3-3z^2+\frac{1}{2} & (8) & z^3-\frac{3}{2}z^2 & (9) & -3z^3+\frac{9}{2}z^2 \\
(10) & -4z^3+6z^2-\frac{1}{2} & (11) & 4z^3-6z^2+\frac{3}{2} & (12) & 3z^3-\frac{9}{2}z^2+1 \\
(13) & -z^3+\frac{3}{2}z^2-1 & (14) & -\frac{1}{4}z^3+\frac{3}{2}z+2 & (15) & -\frac{1}{28}z^3-\frac{3}{4}z+\frac{7}{2}
\end{array}
$$

All cubic PCF polynomials

Theorem

There are exactly fifteen $\bar{\mathbb{Q}}$ conjugacy classes of cubic PCF polynomials defined over $\mathbb{Q}$:

(1) z^3	(2) $-z^3+1$	(3) $-2z^3+3z^2+\frac{1}{2}$
(4) $-2z^3+3z^2$	(5) $-z^3+\frac{3}{2}z^2-1$	(6) $2z^3-3z^2+1$
(7) $2z^3-3z^2+\frac{1}{2}$	(8) $z^3-\frac{3}{2}z^2$	(9) $-3z^3+\frac{9}{2}z^2$
(10) $-4z^3+6z^2-\frac{1}{2}$	(11) $4z^3-6z^2+\frac{3}{2}$	(12) $3z^3-\frac{9}{2}z^2+1$
(13) $-z^3+\frac{3}{2}z^2-1$	(14) $-\frac{1}{4}z^3+\frac{3}{2}z+2$	(15) $-\frac{1}{28}z^3-\frac{3}{4}z+\frac{7}{2}$

Strategy

1. Find normal forms that respect the field of definition.
2. For a map to be PCF, it must be post-critically bounded in each absolute value. Find archimedean and p-adic bounds on the coefficients for maps in the normal forms to be post-critically bounded.
3. Use the bounds in 2 to create a finite search space of possibly PCF maps.
4. For each map in the finite search space, test if it is PCF or not.

Normal Form for Unicritical Cubics

Let K be a number field, and let $f(z) \in K[z]$ be a cubic polynomial. Then there are 3 possibilities.

i. f has exactly one critical point, $\gamma \in K$.

ii. f has two distinct critical points, $\gamma_1 \neq \gamma_2$, both K-rational.

iii. f has two distinct critical points, $\gamma_1 \neq \gamma_2$ with $K(\gamma_1) = K(\gamma_2)$ a quadratic extension of K.

Unicritical polynomials

Theorem

Let $f(z) \in K[z]$ be a degree 3 unicritical polynomial. Then either $f(z)$ is $\bar{K}$-conjugate to z^3, or f is conjugate to a unique polynomial of the form

$$az^3 + 1 \in K[z].$$

Unicritical polynomials

Theorem

Let $f(z) \in K[z]$ be a degree d unicritical polynomial. Then either $f(z)$ is $\bar{K}$-conjugate to z^d, or f is conjugate to a unique polynomial of the form

$$az^d + 1 \in K[z].$$

Unicritical polynomials

Theorem

Let $f(z) = az^d + 1 \in \mathbb{Q}[z]$ and $d \geq 2$. For d even, f is PCF if and only if $a \in \{-2, -1\}$. For d odd, f is PCF if and only if $a = -1$.

Rational critical points

Proposition

Let $f \in K[z]$ be a cubic polynomial with two K-rational critical points, γ_1 and γ_2. Then there exists $\phi \in PGL_2$ such that $f^{\phi}(z) = a(-2z^3 + 3z^2) + c$ for some $a, c \in K$.

Dynamical Belyi Polynomials

Dynamical Belyi polynomials

A dynamical Belyi polynomial of degree d is a map on $\mathbb{P}^1$ with fixed critical points 0 and 1. Let $\mathcal{B}_{d,k}$ denote a dynamical Belyi polynomial of degree d with ramification index $d-k$ at 1.

$$\mathcal{B}_{d,k}(z) = \left(\frac{1}{k!}\prod_{j=0}^{k}(d-j)\right) x^{d-k} \sum_{i=1}^{k} \frac{(-1)^i}{(d-k+i)} \binom{k}{i} x^i.$$

Proposition

Let $g \in K[z]$ be a bicritical polynomial of degree $d \geq 3$ with $\mathrm{Crit}(g) = \{\gamma_1, \gamma_2\} \subseteq K$*. There exists an element $\phi \in \mathrm{PGL}_2(K)$ such that $g^\phi = a\mathcal{B}_{d,k} + c$ for some $k \in \mathbb{N}$ and some $a, c \in K$.*

Rational critical points

Theorem

If $f(z) \in \mathbb{Q}[z]$ is a cubic bicritical PCF polynomial with rational critical points, then $f(z)$ is conjugate to $f_{a,c}(z) = a(-2z^3 + 3z^2) + c$ where

$$(a, c) \in \left\{(1, 0), \left(\pm 1, \frac{1}{2}\right), \left(\frac{1}{2}, \pm 1\right), \left(2, -\frac{1}{2}\right), \left(\frac{3}{2}, 0\right), (-1, 1), \left(-2, \frac{3}{2}\right), \left(-\frac{3}{2}, 1\right), \left(-\frac{1}{2}, 0\right)\right\}.$$

Irrational critical points

Proposition

If $f \in K[z]$ is a bicritical cubic polynomials with critical points γ_1, γ_2 such that $K(\gamma_1) = K(\gamma_2)$ are quadratic extensions of K then there exists $\phi \in \mathrm{PGL}_2$ such that

$$f^{\phi}(z) = a\left(\frac{z^3}{3} - Dz\right) + c$$

where $a, c \in K$ and $D \in \mathcal{O}_K^{\times}/\mathcal{O}_K^2$.

Irrational critical points

Irrational critical points

Let $\mathcal{P}$ be a map of degree d with a fixed point at 0 and critical points at $\pm\sqrt{D}$. Then $\mathcal{P}$ is of the form

$$\mathcal{P}_{d,D}(z) = \sum_{j=0}^{\frac{d-1}{2}} (-D)^{\frac{d-1}{2}-j} \binom{\frac{d-1}{2}}{j} \frac{z^{2j+1}}{2j+1}.$$

Proposition

Let $g(z) \in K[z]$ be a bicritical polynomial of degree $d \geq 3$. Suppose that $\mathrm{Crit}(g) = \{\gamma_1, \gamma_2\} \not\subset K$. Then g is conjugate to a map of the form $a\mathcal{P}_{d,D}(z) + c$ for some $a, c \in K$ and some $D \in \mathcal{O}_K^\times / \mathcal{O}_K^2$.

Irrational critical points

Theorem

If $f(z) \in \mathbb{Q}[z]$ is a cubic bicritical PCF polynomial that is not conjugate to a polynomial with rational critical points, then $f(z)$ is conjugate to $f_{D,a,c}(z) = a(\frac{z^3}{3} - Dz) + c$ *where*

$$(D, a, c) \in \left\{ \left(2, -\frac{3}{4}, 2\right), \left(-7, -\frac{3}{28}, \frac{7}{2}\right) \right\}.$$

$$a\left(\frac{z^3}{3} - Dz\right) + c.$$

All cubic PCF polynomials

Theorem

There are exactly fifteen $\bar{\mathbb{Q}}$ conjugacy classes of cubic PCF polynomials defined over $\mathbb{Q}$:

(1)	z^3	(2)	$-z^3+1$	(3)	$-2z^3+3z^2+\frac{1}{2}$
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(10)	$-4z^3+6z^2-\frac{1}{2}$	(11)	$4z^3-6z^2+\frac{3}{2}$	(12)	$3z^3-\frac{9}{2}z^2+1$
(13)	$-z^3+\frac{3}{2}z^2-1$	(14)	$-\frac{1}{4}z^3+\frac{3}{2}z+2$	(15)	$-\frac{1}{28}z^3-\frac{3}{4}z+\frac{7}{2}$

The details will be presented at UTC 18:00 on 1 July, 2020.

Thank you!