Post-critically finite cubic polynomials

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Definitions

- Let K be a number field,
- $f \in K(z)$ a rational function (morphism of $\mathbb{P}^1$).
- Orbit of a point $\alpha \in \mathbb{P}^1$ is $\mathcal{O}_f(\alpha) = \{f^n(\alpha) \colon n \geq 0\}$.
- Conjugacy class: $[f] = \{\phi \circ f \circ \phi^{-1} : \phi \in \mathrm{PGL}_2(\bar{K})\}$
- $\mathrm{Crit}(f) = \{\alpha : f'(\alpha) = 0\}$

Definition

A map f is post-critically finite (PCF) if every critical point of f has finite forward orbit.

Motivation

Theorem (Ingram, 2011)

The set of conjugacy classes of post-critically finite polynomials of degree d with coefficients of algebraic degree at most B is a finite and effectively computable set.

Question

Can we use Ingram's techniques and a different normal form to find **all** PCF cubic polynomials defined over $\mathbb{Q}$ (up to conjugacy over $\bar{\mathbb{Q}}$)?

Main Result

Theorem

There are exactly fifteen $\bar{\mathbb{Q}}$ conjugacy classes of cubic PCF polynomials defined over $\mathbb{Q}$:

$$
\begin{array}{llllll}
(1) & z^3 & (2) & -z^3+1 & (3) & -2z^3+3z^2+\frac{1}{2} \\
(4) & -2z^3+3z^2 & (5) & -z^3+\frac{3}{2}z^2+1 & (6) & 2z^3-3z^2+1 \\
(7) & 2z^3-3z^2+\frac{1}{2} & (8) & z^3-\frac{3}{2}z^2 & (9) & -3z^3+\frac{9}{2}z^2 \\
(10) & -4z^3+6z^2-\frac{1}{2} & (11) & 4z^3-6z^2+\frac{3}{2} & (12) & 3z^3-\frac{9}{2}z^2+1 \\
(13) & -z^3+\frac{3}{2}z^2-1 & (14) & -\frac{1}{4}z^3+\frac{3}{2}z+2 & (15) & -\frac{1}{28}z^3-\frac{3}{4}z+\frac{7}{2}
\end{array}
$$

Strategy

1. Find normal forms that respect the field of definition.
2. For a map to be PCF, it must be post-critically bounded in each absolute value. Find archimedean and p-adic bounds on the coefficients for maps in the normal forms to be post-critically bounded.
3. Use the bounds in 2 to create a finite search space of possibly PCF maps.
4. For each map in the finite search space, test if it is PCF or not.

Normal Form for Unicritical Cubics

Let K be a number field, and let $f(z) \in K[z]$ be a cubic polynomial. Then there are 3 possibilities.

(i.) f has exactly one critical point, $\gamma \in K$.

(ii.) f has two distinct critical points, $\gamma_1 \neq \gamma_2$, both K-rational.

(iii.) f has two distinct critical points, $\gamma_1 \neq \gamma_2$ with $K(\gamma_1) = K(\gamma_2)$ a quadratic extension of K.

In case (i.) we say that f is unicritical, and in cases (ii.) and (iii.) we say f is bicritical.

Notation

From Ingram:

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \in K[z]$$

$$(2d)_\nu = \begin{cases} 1 & \nu \text{ is non-archimedean} \\ 2d & \nu \text{ is archimedean} \end{cases}$$

$$C_{f,\nu} = (2d)_\nu \max_{0 \le i < d} \left\{ 1, \left| \frac{a_i}{a_d} \right|_\nu^{\frac{1}{d-i}}, |a_d|_\nu^{-\frac{1}{d-1}} \right\}$$

Lemma

Let $f(z) \in \mathbb{Q}[z]$ be a polynomial of degree $d \geq 2$. For $\alpha \in \mathbb{Q}$, if there exists $\nu \in M_{\mathbb{Q}}$ and $n \in \mathbb{N}$ such that $\left| f^n(\alpha) \right|_\nu > C_{f,\nu}$, then α must be a wandering point for f.

Unicritical polynomials

Theorem

Let $f(z) = az^d + 1 \in \mathbb{Q}[z]$ and $d \geq 2$. For d even, f is PCF if and only if $a \in \{-2, -1\}$. For d odd, f is PCF if and only if $a = -1$.

For any prime p, $C_{f,p} = \left\{1, |a|_p^{-1/(d-1)}\right\}$.
Note $f^2(0) = a + 1$, so require

$$|a+1|_p \leq \max\{|a|_p, 1\} \leq C_{f,p} \text{ for all primes } p.$$

This gives that $|a|_p \leq 1$ for all primes p. In the archimedean place we see that $|a| \leq 2$, so $a = \pm 1, \pm 2$. We can check to see if f is PCF in each of those cases.

Rational critical points

Theorem

If $f(z) \in \mathbb{Q}[z]$ is a cubic bicritical PCF polynomial with rational critical points, then $f(z)$ is conjugate to $f_{a,c}(z) = a(-2z^3 + 3z^2) + c$ where

$$(a, c) \in \left\{ (1, 0), \left(\pm 1, \frac{1}{2}\right), \left(\frac{1}{2}, \pm 1\right), \left(2, -\frac{1}{2}\right), \left(\frac{3}{2}, 0\right), (-1, 1), \left(-2, \frac{3}{2}\right), \left(-\frac{3}{2}, 1\right), \left(-\frac{1}{2}, 0\right) \right\}.$$

Sketch of Proof

Let $f(z) = a(-2z^3 + 3z^2) + c \in \mathbb{Q}[z]$.
For any prime p, if f is PCF,

$$\left|f(1)\right| = |a + c|_p \leq C_{f,p} \quad \text{and} \quad \left|f(0)\right| = |c|_p \leq C_{f,p}.$$

So, $\max\{|a|_p, |c|_p\} \leq C_{f,p}$.
For the archimedean place, $|a| < 4$ and $|c| \leq 2$.

Proposition

If $f_{a,c}(z) = a(-2z^3 + 3z^2) + c \in \mathbb{Q}[z]$ is PCF, then

$$\pm a \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\right\} \text{ and } \pm c \in \left\{0, 1, \frac{1}{2}, \frac{3}{2}, 2\right\}.$$

This gives us 126 possible PCF maps. We can test each map using Sage.

Irrational critical points

Theorem

If $f(z) \in \mathbb{Q}[z]$ is a cubic bicritical PCF polynomial that is not conjugate to a polynomial with rational critical points, then $f(z)$ is conjugate to $f_{D,a,c}(z) = a(\frac{z^3}{3} - Dz) + c$ *where*

$$(D, a, c) \in \left\{ \left(2, -\frac{3}{4}, 2\right), \left(-7, -\frac{3}{28}, \frac{7}{2}\right) \right\}.$$

Irrational critical points

Lemma

Let $f(z) = a(z^3/3 - Dz) + c \in \mathbb{Q}[z]$*. If* f *is PCF, then*

$$\pm aD \in \left\{\frac{3}{4}, \frac{3}{2}, \frac{9}{4}, 3, \frac{15}{4}, \frac{9}{2}, \frac{21}{4}\right\}.$$

Lemma

Let $f(z) = a(z^3/3 - Dz) + c \in \mathbb{Q}[z]$*. If* f *is PCF then* $|c|^2 < 11|D|$*.*

Lemma

Let $f(z) = a(z^3/3 - Dz) + c \in \mathbb{Q}[z]$*. If* f *is p-adically PCB, then*

$$\left|c\sqrt{a}\right|_p \leq \begin{cases} 1 & \text{if } p \geq 5 \\ 3^{-1/2} & \text{if } p = 3 \\ 2^3 & \text{if } p = 2. \end{cases}$$

Algorithm

Write

$$f_{D,a,c}(z) = a(z^3/3 - Dz) + c.$$

Then $f_{D,a,c}$ is conjugate to $f_{D,a,-c}$ and furthermore, if $c = 0$ then $f_{D,a,c}$ is conjugate to a map with rational critical points, so we need only to consider triples with $c > 0$.

We will build finite list of positive triples (D, a, c) that satisfy the conditions given. Each triple will correspond to four possible PCF polynomials allowing for negative values of a and D.

Algorithm

We split the algorithm into two cases corresponding to the parity of D.

Step 1 **Loop over possible aD values.**

$$\pm aD \in \left\{ \frac{3}{4}, \frac{3}{2}, \frac{9}{4}, 3, \frac{15}{4}, \frac{9}{2}, \frac{21}{4} \right\}.$$

Step 2 **Compute $|a|_2$.**

Step 3 **Find an upper bound for $|c|_p$ for each prime p.**

$$\left| c\sqrt{a} \right|_p \leq \begin{cases} 1 & \text{if } p \geq 5 \\ 3^{-1/2} & \text{if } p = 3 \\ 2^3 & \text{if } p = 2. \end{cases}$$

So we can find $e \leq 3$ such that $|c|_2 \leq 2^e$, and $|c|_p \leq 1$ for each prime $p \geq 3$.

Algorithm

Step 4 **Factor D and c.** Write $D = mP$ or $D = 2mP$, where m and P are relatively prime odd squarefree integers, m divides numerator of aD and P divides denominator of a. Then P must also divide the numerator of c, so $c = \frac{Pk}{2^e}$ for some positive integer k.

Step 5 **Bound the factors of D and c.** Use $|c|^2 < 11|D|$, so $\frac{P^2k^2}{2^{2e}} < 11mP$ or $\frac{P^2k^2}{2^{2e}} < 22mP$. Therefore $Pk^2 < B$ where
$B = 11m \cdot 2^{2e}$ for D odd,
or $B = 11m \cdot 2^{2e+1}$ for D even.

Step 6 **Loop over P values.** For all odd, squarefree integers $P < B$, determine the set of possible k values such that $Pk^2 < B$.

Step 7 **Create the triple.** Each triple (m, P, k) yields a triple

$$(D, a, c) = \left(mP, \frac{aD}{mP}, \frac{Pk}{2^e}\right)$$

or

$$(D, a, c) = \left(2mP, \frac{aD}{2mP}, \frac{Pk}{2^e}\right)$$

Finally, check that $3|ac$ to verify that the triple satisfies the 3-adic condition. If so, add (D, a, c) to the list of possible PCF triples.

This yields a list of 5,957 triples corresponding to 23,828 possibly PCF polynomials.

Only the two listed in the theorem statement are PCF and are not conjugate to a polynomial already with rational critical points.

Thank you!

Dynamical Belyi Polynomials

Dynamical Belyi polynomials

A dynamical Belyi polynomial of degree d is a map on $\mathbb{P}^1$ with fixed critical points 0 and 1. Let $\mathcal{B}_{d,k}$ denote a dynamical Belyi polynomial of degree d with ramification index $d-k$ at 1.

$$\mathcal{B}_{d,k}(z) = \left(\frac{1}{k!} \prod_{j=0}^{k} (d-j) \right) x^{d-k} \sum_{i=1}^{k} \frac{(-1)^i}{(d-k+i)} \binom{k}{i} x^i.$$

Proposition

Let $g \in K[z]$ be a bicritical polynomial of degree $d \geq 3$ with $\mathrm{Crit}(g) = \{\gamma_1, \gamma_2\} \subseteq K$. There exists an element $\phi \in \mathrm{PGL}_2(K)$ such that $g^\phi = a\mathcal{B}_{d,k} + c$ for some $k \in \mathbb{N}$ and some $a, c \in K$.

Irrational critical points

Let $\mathcal{P}$ be a map of degree d with a fixed point at 0 and critical points at $\pm\sqrt{D}$. Then $\mathcal{P}$ is of the form

$$\mathcal{P}_{d,D}(z) = \sum_{j=0}^{\frac{d-1}{2}} (-D)^{\frac{d-1}{2}-j} \binom{\frac{d-1}{2}}{j} \frac{z^{2j+1}}{2j+1}.$$

Proposition

Let $g(z) \in K[z]$ be a bicritical polynomial of degree $d \geq 3$. Suppose that $\mathrm{Crit}(g) = \{\gamma_1, \gamma_2\} \not\subset K$. Then g is conjugate to a map of the form $a\mathcal{P}_{d,D}(z) + c$ for some $a, c \in K$ and some $D \in \mathcal{O}_K^\times / \mathcal{O}_K^2$.