Computing paramodular forms

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Definition

Let K be a field with $\mathrm{car}(K) \neq 2$, a K-quadratic space is a finite dimensional K-vector space with a function $\phi : V \to K$ such that

- $\phi(x\mathbf{v}) = x^2\phi(\mathbf{v})$, $\mathbf{v} \in V$ $x \in K$.
- $\phi(\mathbf{v}, \mathbf{v}') = \phi(\mathbf{v} + \mathbf{v}') - \phi(\mathbf{v}) - \phi(\mathbf{v}')$ is a symetric bilinear form.

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V, then

$$\begin{aligned} q(x_1, \ldots, x_n) &= \phi\left(\sum_j x_j \mathbf{v}_j\right) \\ &= \sum_{i,j} x_i x_j \phi(\mathbf{v}_i, \mathbf{v}_j) \end{aligned}$$

is a K-quadratic form. Different bases gives K-equivalent quadratic forms.
The discriminant of V is

$$\operatorname{disc}(V) := \tfrac{1}{2}\det(\phi(\mathbf{v}_i, \mathbf{v}_j)_{i,j}).$$

Definition

An autometry of the quadratic space (V, ϕ) is a linear map

$$\sigma : V \to V,$$

such that

$$\phi(\sigma(\mathbf{v})) = \phi(\mathbf{v}),\ \mathbf{v} \in V.$$

The autometries of (V, ϕ) form a group with the composition as product. This group is called the orthogonal group and denoted by $\mathrm{O}(V)$.

If $\sigma \in \mathrm{O}(V)$ then $\det \sigma = \pm 1$. If $\det \sigma = +1$, we say that the autometry is proper. The proper autometries form a subgroup of $\mathrm{O}(V)$ and we denote it by $\mathrm{O}^+(V)$.

A lattice in (V, ϕ) is free maximal $\mathbb{Z}$-module Λ ($\Lambda \simeq \mathbb{Z}^{\dim(V)}$). The lattice is integral if $\phi(\Lambda) \subset \mathbb{Z}$.
The lattices Λ and Γ are properly equivalent, and we denote it by $\Lambda \sim \Gamma$, if

$$\sigma\Lambda = \Gamma, \text{ for some } \sigma \in \mathrm{O}^+(V).$$

We denote the proper equivalence class of Λ by $[\Lambda]$.
The genus of Λ is

$$\mathrm{Gen}(\Lambda) := \{\Gamma \text{ lattice} : \Lambda_p \sim \Gamma_p \text{ for all } p\}.$$

The class set $\mathrm{Cl}(\Lambda)$ is the set of the proper equivalence classes in $\mathrm{Gen}(\Lambda)$, and $\# \mathrm{Cl}(\Lambda) < \infty$.

Definition

Let Λ, Γ be integral lattices in a positive definite quadratic space (V, ϕ), a prime p and $k \geq 1$. We say that Λ and Γ are p^k-neighbors if $\Lambda_q = \Gamma_q$ for all prime $q \neq p$ and

$$\Lambda/(\Lambda \cap \Gamma) \cong \Gamma/(\Lambda \cap \Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^k$$

Theorem

There is a bijection between the set of non-singular projective solutions of

$$\phi(\mathbf{v}) \equiv 0 \pmod{p}, \quad \mathbf{v} \in \Lambda$$

and the p-neighbor lattices of Λ.

If $p \nmid \operatorname{disc}(\Lambda)$, then the number of p^k-neighbors of Λ is $O(p^{k(4-k)})$, for the quinary case.

The space of orthogonal modular forms for Λ (with trivial weight) is

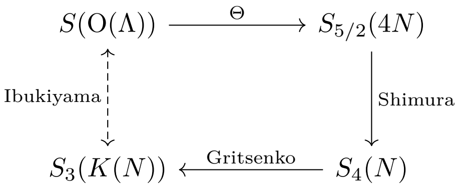
$$M(\mathrm{O}(\Lambda)) := \mathrm{Fun}(\mathrm{Cl}(\Lambda), \mathbb{Q}).$$

In the basis of characteristic functions for $\mathrm{Cl}(\Lambda)$ we have $M(\mathrm{O}(\Lambda)) \cong \mathbb{Q}^h$ with $h = \# \mathrm{Cl}(\Lambda)$.
For $p \nmid \mathrm{disc}(\Lambda)$, we define the Hecke operator

$$\begin{aligned} T_{p,k} : M(\mathrm{O}(\Lambda)) &\to M(\mathrm{O}(\Lambda)) \\ f &\mapsto T_{p,k}(f) \\ &\qquad T_{p,k}(f)([\Lambda']) := \sum_{\Gamma'} f([\Gamma']), \end{aligned}$$

Where we sum over all the p^k-neighbors of Λ'.
The operators $T_{p,k}$ commute and are self-adjoint with respect to some inner product.
We denote $S(\mathrm{O}(\Lambda))$ the orthogonal complement of the constant functions and we call it the cuspidal subspace.

Let Λ be a quinary lattice of discriminant N. We define the map $\Theta : S(\Lambda) \to S_{5/2}(4N)$, as $\Theta(\sum_i a_i[\Lambda_i]) = \sum_i a_i\Theta(\Lambda_i)$. If $f \in S(\Lambda)$ is a Hecke eigenform, with $\Theta(f) \neq 0$, then the Shimura lift of $\Theta(f)$ is a modular form of weight 4 whose Gristenko lift should correspond to f, as in the following diagram



Hein, Ladd and Tornaría conjectured that, if $\Theta(f) = 0$ and N is prime, then f corresponds to a paramodular form that it is not a Gritsenko lift.

Example

Let $V = \mathbb{Q}^5$, $\phi = x^2 + xy - xt + y^2 - yt + z^2 + 2w^2 - wt + 3t^2$ quadratic form of discriminant 61, and $\Lambda = \mathbb{Z}^5$. This is the first example of prime discriminant in $\mathrm{O}(5)$ for which the theta series map on the genus has a kernel. We have

$$\#\,\mathrm{Cl}(\Lambda) = 8, \; \dim S_4^-(\Gamma_0(61)) = 6, \; \text{and} \; \dim(\ker(\Theta)) = 1.$$

Let $f \in S(\mathrm{O}(\Lambda))$ such that $\Theta(f) = 0$, which is an eigenfunction for the Hecke operators.

Some eigenvalues are, $T_{p,k}(f) = c_{p,k} f$

p	$c_{p,1}$	p	$c_{p,1}$	p	$c_{p,1}$	p	$c_{p,1}$	p	$c_{p,1}$
2	-7	5	3	11	-4	17	37	23	10
3	-3	7	-9	13	-3	19	-75	29	212

p	$c_{p,2}$	p	$c_{p,2}$	p	$c_{p,2}$	p	$c_{p,2}$	p	$c_{p,2}$
2	7	5	-9	11	36	17	176	23	76
3	-9	7	-42	13	-57	19	32	29	-66

By Ibukiyama dimension formulas we have

$$\dim S_3(K(61)) = \dim S(\mathrm{O}(\Lambda)) = \dim S_4^-(61) + \dim \ker \Theta.$$

In this case the correspondence from $S(\mathrm{O}(\Lambda))$ to $S_3(K(61))$ should be a bijection.

Example

Let $V = \mathbb{Q}^5$ and

$$\phi = x^2 + xy + y^2 + z^2 + xt + zt + t^2 + tw + 34w^2,$$

quadratic form with discriminant 167. The genus of $\Lambda = \mathbb{Z}^5$ has 19 other classes, so $\dim(S(\mathrm{O}(\Lambda)) = 18$.
But $\dim S_3(K(167)) = 19$, so the correspondence from $S(\mathrm{O}(\Lambda))$ to $S_3(K(167))$ is not a bijection.

Let $\rho : \mathrm{O}^+(V) \to \mathrm{GL}(W)$ be a representation, with W a finite dimensional $\mathbb{Q}$-vector space.
Let $\Lambda = \Lambda_1, \dots, \Lambda_h$ be representatives of $\mathrm{Cl}(\Lambda)$.
We define the space of orthogonal modular forms for Λ with weight ρ as

$$M(\mathrm{O}(\Lambda), \rho) := \{f : \mathrm{Cl}(\Lambda) \to W : f([\Lambda_i]) \in W^{\mathrm{O}(\Lambda_i))}\} \cong \bigoplus_{i=1}^{h} W^{\mathrm{O}(\Lambda_i)}.$$

Given $p \nmid \mathrm{disc}(\Lambda)$, for Γ a p^k-neighbor of Λ, we have $\Gamma = \gamma\Lambda_j$ for a unique j and $\gamma \in \mathrm{O}^+(V)$, unique modulo multiplication by $\mathrm{O}^+(\Lambda_j)$.
We define the Hecke operator

$$T_{p,k} : M(\mathrm{O}(\Lambda), \rho) \to M(\mathrm{O}(\Lambda, \rho)$$
$$T_{p,k}(f)([\Lambda']) := \sum_{\Gamma'} \rho(\gamma')(f([\Gamma'])),$$

where the sum is over all $\gamma\Lambda_j = \Gamma'$ p^k-neighbors ofΛ'.

If $d \mid D$ we define the character $\nu_d : \mathbb{Q}_{>0}^{\times}/(\mathbb{Q}_{>0}^{\times})^2 \to \{\pm 1\}$ defined in primes as

$$\nu_d(p) := \begin{cases} -1 & \text{si } p \mid d \\ 1 & \text{si } p \nmid d \end{cases},$$

and the one dimensional representation
$\rho_d : \mathrm{O}^+(V) \to \{\pm 1\} \subset \mathbb{Q}^\times \cong \mathrm{GL}(\mathbb{Q})$

$$\rho_d(\sigma) := \nu_d(\theta(\sigma)).$$

We return to the quinary example of discriminant 167, in this case, all the classes but one, have an autometry with spinor norm 167. Then $S(\mathrm{O}(\Lambda), \rho_{167}) \simeq \mathbb{Q}$.
Let $f \in S(\mathrm{O}(\Lambda), \rho_{167})$, $f \neq 0$. Some eigenvalues are,
$T_{p,k}(f) = c_{p,k} f$,

p	$c_{p,1}$	p	$c_{p,1}$	p	$c_{p,1}$	p	$c_{p,1}$	p	$c_{p,1}$
2	-8	5	-4	11	-22	17	-47	23	41
3	-10	7	-14	13	-4	19	-12	29	50

p	$c_{p,2}$	p	$c_{p,2}$	p	$c_{p,2}$	p	$c_{p,2}$	p	$c_{p,2}$
2	10	5	-44	11	-67	17	260	23	-198
3	11	7	-9	13	-158	19	41	29	-187

For p prime, let Λ_p be a lattice in the unique genus of discriminant p.

Theorem

For $p < 7000$ we have

$$\dim(S_3(K(p)) = \dim S(\mathrm{O}(\Lambda_p)) + \dim S(\mathrm{O}(\Lambda_p), \rho_p).$$

Conjecture

For p prime,

$$S_3(K(p)) \simeq S(\mathrm{O}(\Lambda_p)) \oplus S(\mathrm{O}(\Lambda_p), \rho_p).$$

Also, $S(\mathrm{O}(\Lambda_p))$ corresponds to the forms in $S_3(K(p))$ with $+$ sign of the functional equation of its associated L-function, and $S(\mathrm{O}(\Lambda_p), \rho_p)$ corresponds to the forms in $S_3(K(p))$ with $-$ sign of the functional equation of its associated L-function.