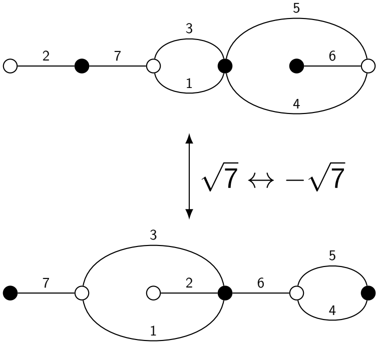
A Database of Belyĭ maps



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What is a Belyĭ map?

A *Belyĭ map* over $\mathbb{C}$ is a nonconstant morphism of algebraic curves

$$\phi : X \to \mathbb{P}^1_{\mathbb{C}}$$

that is unramified outside $\{0, 1, \infty\}$.

Motivation:

- Belyĭ's Theorem
- Inverse Galois theory
- Dessins d'enfants
- Constructing number fields with few ramified primes

Overview of the method

- **Input:** A permutation triple $\sigma = (\sigma_0, \sigma_1, \sigma_\infty)$
- **Output:** Equations for the corresponding curve X and Belyĭ map $\phi : X \to \mathbb{P}^1$ with monodromy representation $\langle \sigma \rangle$

1. Compute the triangle group $\Gamma \leq \Delta(a, b, c)$
2. Compute the modular forms in $S_k(\Gamma)$ numerically for appropriate k
3. Find numerical relations among the elements of $S_k(\Gamma)$ and hence numerical equations for X and ϕ
4. Rescale and recognize coefficients in a number field
5. Verify that ϕ has the correct ramification and monodromy

Example 7T5-[3,4,4]-331-421-421-g0

- **Input:** $\sigma = ((1\,3\,7)(4\,5\,6), (1\,4\,5\,3)(2\,7), (1\,2\,7\,5)(4\,6))$

1. Compute Γ an index 7 subgroup of $\Delta(3,4,4)$
2. Numerically compute power series for $g, h \in S_4(\Gamma)$:
   $g(w) = 1 + (1.51028 + 2.61589\sqrt{-1})w^2 + O(w^3)$
   $h(w) = w + (1.03681 - 1.79580\sqrt{-1})w^2 + O(w^3)$
3. With $x = h(w)/(g(w) + 2.63937h(w))$ we obtain the relation

$$\frac{x^7-(7.80111+13.51192\sqrt{-1})x^6-(39.72763-68.81028\sqrt{-1})x^5+122.93476x^4+(23.29066+40.34061\sqrt{-1})x^3}{x^7-(2.40086+4.15842\sqrt{-1})x^6+(0.14942-0.25881\sqrt{-1})x^5-13.44992x^4-(6.05994+10.49612\sqrt{-1})x^3+(1.35878-2.35349\sqrt{-1})x^2-0.56533x-(0.10710-0.18551\sqrt{-1})}$$

4. Rescaling yields

$$\frac{x^7+66.42060x^6+1439.97776x^5+9484.71576x^4+15299.52755x^3}{x^7+20.44157x^6-5.41618x^5-1037.69445x^4+3980.74651x^3-3799.83640x^2-3365.16570x+5428.25391}$$

5. Recognizing coefficients in $\mathbb{Q}(\sqrt{7})$ and cleaning up we get

$$\phi = \lambda \cdot \frac{x^3\left(x-\frac{1}{729}\left(68\sqrt{7}+236\right)\right)^1\left(x-\frac{1}{9}\left(20-4\sqrt{7}\right)\right)^3}{\left(x-\frac{4}{21}\left(\sqrt{7}+3\right)\right)^2\left(x-\frac{4}{21}\left(\sqrt{7}+1\right)\right)^4}, \quad \text{where } \lambda = \frac{1}{3087}\left(173\sqrt{7}+343\right)$$

Completeness of Computation

d \ g	0	1	2	3	≥ 4	total
1	1/1	0	0	0	0	1/1
2	1/1	0	0	0	0	1/1
3	2/2	1/1	0	0	0	3/3
4	6/6	2/2	0	0	0	8/8
5	12/12	6/6	2/2	0	0	20/20
6	38/38	29/29	7/7	0	0	74/74
7	89/89	50/50	7/13	2/3	0	148/155
8	81/261	83/217	0/84	0/11	0	164/573
9	97/583	33/427	0/163	0/28	0/6	130/1207

`http://beta.lmfdb.org/Belyi/`
`https://github.com/michaelmusty/BelyiDB`
`https://arxiv.org/abs/1805.07751`